TRUNCATED SIMPLICIAL RESOLUTIONS AND SPACES OF RATIONAL MAPS

JACOB MOSTOVOY

ABSTRACT. We show that whenever $m \leq n$, the space of all continuous rational maps from $\mathbf{CP}^m$ to $\mathbf{CP}^n$ has the same homology as the space of all continuous maps between these spaces in dimensions smaller than $d(2n-2m+1)-1$. This improves the result of the paper "Spaces of Rational Maps and the Stone-Weierstrass Theorem" (*Topology* 45 (2006), 281–293).

1. Introduction

The purpose of this note is to correct the proof of the main theorem of [2] and improve the bound it gives by approximately a factor of 2.

Let $\mathbf{Rat}_f^{m,n}$ be the set of all continuous rational maps of degree d from $\mathbf{CP}^m$ to $\mathbf{CP}^n$ that restrict to a given map

$$f : \mathbf{CP}^{m-1} \to \mathbf{CP}^n$$

on a fixed hyperplane $\mathbf{CP}^{m-1} \subset \mathbf{CP}^m$. The space of all continuous maps $\mathbf{CP}^m \to \mathbf{CP}^n$ that restrict to f on $\mathbf{CP}^{m-1}$ is homotopy equivalent to $\Omega^{2m}\mathbf{CP}^n$. The following theorem generalises a well-known result of G. Segal [3]:

Theorem 1.1. *For $m \leq n$ the map*

$$\mathbf{Rat}_f^{m,n} \to \Omega^{2m}\mathbf{CP}^n$$

given by the inclusion of rational maps into the space of all continuous maps from $\mathbf{CP}^m$ to $\mathbf{CP}^n$ that restrict to a given map f of degree d on a fixed hyperplane, induces an isomorphism in homology groups in all dimensions smaller than

$$d(2n-2m+1)-1.$$

If $m < n$ it is also induces isomorphisms of homotopy groups in these dimensions.

As explained in [2], this result implies a similar statement for the spaces of "free" maps. Namely, the above theorem holds with the space $\mathbf{Rat}_f^{m,n}$ replaced by the space of all continuous rational maps of degree d from $\mathbf{CP}^m$ to $\mathbf{CP}^n$ and $\Omega^{2m}\mathbf{CP}^n$ by the space of all continuos maps between these projective spaces. In what follows, we shall use the definitions and the notations of [2] without further reference.

In fact, in [2] this theorem was claimed with the generally weaker bound of

$$(2n-2m+1)\left(\left[\frac{d+1}{2}\right]+1\right),$$

where the $[x]$ stands for the integer part of x. Moreover, after [2] was published, two mistakes were found in the proof of Theorem 1.1 by A. Kozlowski and M. Adamaszek. The new ingredient in the present note is the truncated simplicial resolution which we use instead of the degenerate simplicial resolution of [2]. The truncated resolutions are, at the same time, simpler and more effective; we describe them in the next section. The rest of the paper is dedicated to the proof of Theorem 1.1.

2. Truncated simplicial resolutions.

Let $h : X \to Y$ be a finite surjective map of topological spaces and let i be an embedding of X into $\mathbf{R}^N$ for some N, with the property that for each $y \in Y$ any k points of the set $i \circ h^{-1}(y)$ span a $k-1$-dimensional affine subspace of $\mathbf{R}^N$. A *(non-degenerate) simplicial resolution* associated with the map h is the projection map

$$h^\Delta : X^\Delta \to Y,$$

where X^Δ is the space of pairs $(t, y) \in \mathbf{R}^N \times Y$ with $y \in Y$ and t belonging to the convex hull of the set $i \circ h^{-1}(y)$ in $\mathbf{R}^N$.

Assume from now on that X, Y are semi-algebraic sets and h is a semi-algebraic map. Then the simplicial resolution is a quasifibration with contractible fibres (simplices, to be precise) and, hence, a homotopy equivalence. There is an increasing filtration

$$X_1 \subset X_2 \subset \ldots \subset X^\Delta$$

on any simplicial resolution associated with a map $h : X \to Y$. For any $y \in Y$ the set $(h^\Delta)^{-1}(y)$ is a simplex; the subspace $X_k \subset X^\Delta$ is defined as the union of the $k-1$-skeleta of these simplices over all $y \in Y$. In particular, $X_1 = X$.

If the map h is not finite, the embedding i of X into $\mathbf{R}^N$ with the required properties may not exist. Still, the non-degenerate simplical resolution may be constructed by choosing a sequence of embeddings $i_k : X \to \mathbf{R}^{N_k}$ such that any $2k$ points of the set $i_k \circ h^{-1}(y)$ span a $2k-1$-dimensional affine subspace of $\mathbf{R}^N$. Then we define X_k to be the space of pairs $(t, y) \in \mathbf{R}^N \times Y$ with $y \in Y$ and t belonging to a closed $k-1$-simplex whose vertices are in the set $i_k \circ h^{-1}(y)$ in $\mathbf{R}^N_k$. It can be seen that X_k is naturally identified with a subspace of X_{k+1} and the space X^Δ is then defined as the direct limit of the X_k. For X, Y semi-algebraic sets and h a semi-algebraic map the simplicial resolution exists and is a homotopy equivalence.

In some concrete examples it happens that only the first several terms of the filtration X_k are easy to describe. The following construction is tailored to produce a more manageable replacement for the simplicial resolution in such a case.

For a positive integer d denote by $h_d : X_d \to Y$ be the restriction of h^Δ to X_d. The fibres of the map h_d are the $d-1$-skeleta of the fibres of h^Δ; they fail to be simplices over the subspace

$$\{y \in Y \mid h^{-1}(y) \text{ consists of more than } d \text{ points}\}.$$

Write $Y(d)$ for the closure of this subspace. We modify X_d so as to make all the fibres of h_d contractible by adding to each fibre over $Y(d)$ a cone whose base is this fibre. We denote the resulting space by $X^\Delta(d)$ and the natural extension of h_d to $X^\Delta(d)$ by $h^{\Delta(d)}$; this map is the *truncated (after the dth term) simplicial resolution of* Y. In the assumptions of semi-algebraicity of all the ingredients $h^{\Delta(d)}$ is a quasifibration and, hence, a homotopy equivalence.

More formally, if h is a finite map, $X^\Delta(d)$ is a subspace in $\mathbf{R} \times \mathbf{R}^N \times Y$ defined as the union of two pieces, namely $\{0\} \times X_d$ and the subspace consisting of all the intervals that join the points $(1, 0, y)$ and $(0, t, y)$ where $y \in Y(d)$ and $(t, y) \in X_d$. On $X^\Delta(d)$ there is a filtration $X_k(d)$ whose terms coincide with $\{0\} \times X_k$ for

$1 \leq k \leq d$ and are equal to the whole space $X^{\Delta}(d)$ for $k > d$. If h is not finite this description requires only a minor change in the notation: one has to write $\mathbf{R}^{N_d}$ instead of $\mathbf{R}^N$.

The most important property of truncated simplicial resolutions is the following obvious lemma.

Lemma 2.1. $\dim\left(X_{d+1}(d) - X_d(d)\right) = \dim\left(X_d - X_{d-1}\right) + 1.$

Indeed, $X_{d+1}(d) - X_d(d)$ is a fibrewise open cone on the closure of the space $X_d(d) - X_{d-1}(d) = X_d - X_{d-1}$.

3. Application to rational maps

First, recall some basics from [2]. Let $f_0, \ldots, f_n$ be complex homogeneous polynomials of degree d in in m variables $z_0, \ldots, z_{m-1}$ with no common zero. Consider the space $W_{p,q}$ of all $n+1$-tuples of polynomials in $m+1$ variables $z_0, \ldots, z_m$ and their conjugates, of degree p in the z_i and $q = p - d$ in the $\overline{z}_i$, whose restriction to the hyperplane $z_m = 0$ coincides with the $n+1$-tuple $(f_0, \ldots, f_n) \cdot |z|^{2q}$ where $|z|^2 = \sum z_i \overline{z}_i$. In $W_{p,q}$ there is a discriminant Σ which consists of $(n+1)$-tuples of polynomials that all have a common zero. Its complement is denoted by $\overline{\mathbf{Rat}}_f(p,q)$; for $q = 0$ it is precisely the space $\mathbf{Rat}_f^{m,n}$ of all continuous rational maps of degree d from $\mathbf{CP}^m$ to $\mathbf{CP}^n$ which restrict to the given map $f = (f_0, \ldots f_n)$ on the hyperplane $z_m = 0$. Our main interest is to understand the stabilization map

$$\overline{\mathbf{Rat}}_f(p,q) \to \overline{\mathbf{Rat}}_f(p+1, q+1)$$

given by multiplying all the polynomials by $|z|^2$. We shall prove here the following statement:

Proposition 3.1. *The stabilization map* $\overline{\mathbf{Rat}}_f(p,q) \to \overline{\mathbf{Rat}}_f(p+1,q+1)$ *is a homology equivalence in dimensions smaller than* $p(2n - 2m + 1) - 1$.

Proof. Define Z to be the space

$$\{(F_0, \ldots, F_n, x) \mid F_i(x) = 0 \text{ for all } 0 \leq i \leq n,\ (F_0, \ldots, F_n) \in W_{p,q},\ x \in \mathbf{C}^m\},$$

where $\mathbf{C}^m$ is the affine chart $z_m = 1$ in $\mathbf{CP}^m$. There is a map $Z \to \Sigma$ given by forgetting x, and we consider the non-degenerate simplicial resolution

$$Z^{\Delta} \to \Sigma$$

associated with it (in [2] the space Z^{Δ} was denoted by $\widetilde{Z}^{\Delta}$). The first p terms of the filtration on this resolution are easy to describe. Indeed, the condition that the polynomials F_i vanish at a given point gives a linear condition on the coefficients of each F_i and r distinct points in $\mathbf{C}^m$ produce linearly independent conditions whenever $r \leq p$. It follows that for these values of r the space $Z_r - Z_{r-1}$ is an open disk bundle of rank $2N_{p,q} - 2r(n+1) + r - 1$ over the configuration space $C_r(\mathbf{C}^m)$ of r distinct unordered points in $\mathbf{C}^m$.

For greater values of r the condition produced by r points on the coefficients of the F_i cease to be linearly independent and we have no effective way of describing the corresponding terms of the filtration. Therefore, we consider the corresponding truncated (after the pth term) simplicial resoltion $Z^{\Delta}(p)$.

This truncated simplicial resolution extends to a map of one-point compactifications $\widehat{Z}^{\Delta}(p) \to \widehat{\Sigma}$ which is also a homotopy equivalence, being a quasifibration with contractible fibres. The space $\widehat{Z}^{\Delta}(p)$ is filtered by one-point compactifications

$\widehat{Z}_k(p)$, togehter with the additional term $\widehat{Z}_0(p)$ which is the added point. The spectral sequence that arises from this filtration converges to the reduced cohomology of the one-point compactification of Σ:

$$E_1^{r,s} = \widetilde{H}^{r+s}(\widehat{Z}_r(p)/\widehat{Z}_{r-1}(p), \mathbf{Z}),$$

where $\widehat{Z}_{-1}(p)$ is the empty set. The cohomology of $\widehat{\Sigma}$ is related by the Alexander duality to the homology of $\overline{\mathbf{Rat}}_f(p,q)$:

$$\widetilde{H}^r(\widehat{\Sigma}, \mathbf{Z}) \simeq \widetilde{H}_{2N_{p,q}-r-1}(\overline{\mathbf{Rat}}_f(p,q))$$

where $N_{p,q}$ is the complex dimension of $W_{p,q}$. The following spectral sequence converges to the reduced homology of $\overline{\mathbf{Rat}}_f(p,q)$:

$$E^1_{-r,s} = \widetilde{H}^{2N_{p,q}+r-s-1}(\widehat{Z}_r(p)/\widehat{Z}_{r-1}(p), \mathbf{Z}) \tag{1}$$

with $r, s \geq 0$.

For $r > 0$ the space $\widehat{Z}_r(p)/\widehat{Z}_{r-1}(p)$ is the one-point compactification of the space $Z_r(p) - Z_{r-1}(p)$. As we have seen, for $r \leq p$ this space is an open disk bundle of rank $2N_{p,q} - 2r(n+1) + r - 1$ over the configuration space $C_r(\mathbf{C}^m)$ of r distinct unordered points in $\mathbf{C}^m$. More precisely, it is the Whitney sum of an orientable vector bundle of rank $2N_{p,q} - 2r(n+1)$, and the bundle of rank $r-1$ induced by the $r-1$-dimensional irreducible representation of the symmetric group on r letters (namely, the non-trivial summand of the permutation representation). As a consequence, for these values of r we have

$$H^i(\widehat{C}_r(\mathbf{C}^m), \pm\mathbf{Z}) = \widetilde{H}^{2N_{p,q}-2(n+1)r+r-1+i}(\widehat{Z}_r(p)/\widehat{Z}_{r-1}(p), \mathbf{Z})$$

for all i, and, hence,

$$E^1_{-r,s} = H^{2(n+1)r-s}(\widehat{C}_r(\mathbf{C}^m), \pm\mathbf{Z})$$

for all $0 < r \leq p$ and all s. Here $\pm\mathbf{Z}$ is the local system on the configuration space induced by the sign representation of the symmetric group. Note that this expression does not depend on p and q, and vanishes whenever $s < 2r(n-m+1)$.

Since the filtration $\widehat{Z}_r(p)$ stabilises at $r = p+1$, we have $E^1_{-r,s} = 0$ whenever $r > p+1$. The only terms over which we do not have complete control are $E^1_{-(p+1),s}$. The most we can say is that since by Lemma 2.1

$$\dim \widehat{Z}_{p+1}(p)/\widehat{Z}_p(p) = \dim \widehat{Z}_p(p)/\widehat{Z}_{p-1}(p) + 1 = 2N_{p,q} - p(2n-2m+1),$$

we have that

$$E^1_{-(p+1),s} = 0$$

for $s < 2p(n-m+1)$.

The terms $E^1_{-r,s}$ and $E^\infty_{-r,s}$ of the spectral sequence for the homology of $\overline{\mathbf{Rat}}_f(p,q)$ are shown in the figure. The (possibly) non-zero entries are all situated above the staircase shaped curve. The shaded entries are stable in the sense that they do not depend on p and q. These entries in the E^1 come from the cohomology of the configuration spaces; in the E^∞ these are the entries that cannot be hit by any non-trivial differential coming from any of the groups $E^k_{-(p+1),s}$.

Now, consider the stabilization map $\overline{\mathbf{Rat}}_f(p,q) \to \overline{\mathbf{Rat}}_f(p+1,q+1)$ given by multiplying all the polynomials by $|z|^2$. The induced map in homology via the Alexander duality gives rise to a map

$$H^i(\widehat{\Sigma}_{p,q}, \mathbf{Z}) \to H^{i+2(N_{p+1,q+1}-N_{p,q})}(\widehat{\Sigma}_{p+1,q+1}, \mathbf{Z}),$$

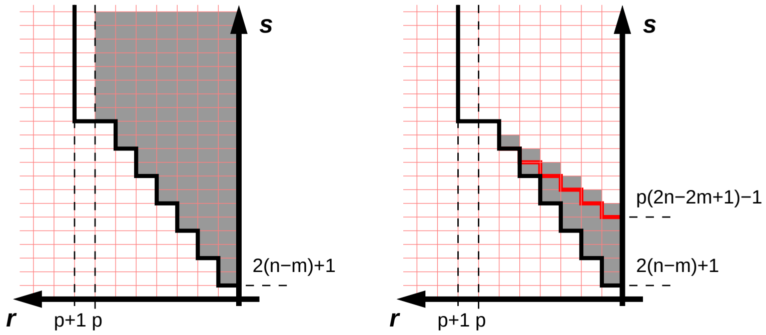

FIGURE 1. The terms E^1 and E^∞ of the spectral sequence converging to the reduced homology of $\overline{\mathbf{Rat}}_f(p,q)$. The group $E^*_{a,b}$ is placed in the square $a < x < a+1$, $b-1 < y < b$. The shaded entries do not depend on p and q.

where $\Sigma_{p,q} = \Sigma$ is the discriminant in $W_{p,q}$. This map sends a cohomology class defined as the linking number with a cycle in $\overline{\mathbf{Rat}}_f(p,q)$ to the class defined as the linking number with the same cycle in $\overline{\mathbf{Rat}}_f(p+1,q+1)$. On any cocycle coming from the rth term of the filtration of the simplicial resolution of $\Sigma_{p,q}$ this map coincides with the isomorphism of the stable parts of the corresponding spectral sequences, since these are explicitly given as suspension-like isomorphisms. Comparing the spectral sequences we get the homology isomorphisms in the required range. □

Proposition 3.1 implies Theorem 1.1 since the limit of the sequence of inclusions $\overline{\mathbf{Rat}}_f(p,q) \to \overline{\mathbf{Rat}}_f(p+1,q+1)$ is weakly homotopy equivalent to the space of all continuous maps with the fixed behaviour on a hyperplane, see Proposition 3 in [2] and the discussion in the next section. Moreover, when $m < n$ the spaces of maps in question are simply-connected and this implies that the isomorphisms in homology are also isomorphisms in homotopy.

4. CORRECTIONS TO [2]

After [2] was published, two mistakes were found in the proof of Theorem 1.1 by A. Kozlowski and M. Adamaszek.

It is stated in Section 2 of [2] that "two representations of a (p,q)-map by collections of (p,q)-polynomials need not coincide up to a constant but rather up to multiplication by a positive function." It is then concluded in Section 4 that, as the space of positive functions is convex, the map $\overline{\mathbf{Rat}}_f(p,q) \to \mathbf{Rat}_f(p,q)$ is a homotopy equivalence.

In fact, two representations of the same (p,q)-map coincide up to a *non-zero* function, which is not the same as a positive function since the functions in question are complex-valued. As a consequence, there is no easy description of the relationship between the spaces $\overline{\mathbf{Rat}}_f(p,q)$ and $\mathbf{Rat}_f(p,q)$.

To remedy this, note that we only need the space $\mathbf{Rat}_f(p,q)$ insofar as it is related to the space of $\Omega^{2m}\mathbf{CP}^n$ by Proposition 3. This proposition, however, is valid, with the same proof, for $\overline{\mathbf{Rat}}_f(p,q)$ rather than $\mathbf{Rat}_f(p,q)$, with the words "natural inclusion" replaced by "natural map". Then we just need to omit any mention of $\mathbf{Rat}_f(p,q)$ in Section 4.

The second substantial mistake is in Section 4 where it is claimed that "the condition that a (p,q)-polynomial in $W^i_{p,q}$ vanishes at r distinct points x_j produces exactly r independent conditions on its coefficients if and only if the convex hull of the points $v_{p,q}(x_j)$ in V is an $(r-1)$-dimensional simplex." This would be true if the r independent conditions were over the real numbers; however, in order to justify the dimension count in (2) we need r independent conditions over $\mathbf{C}$.

The solution to this problem is to replace the argument of [2] to that of the previous section. In fact, when $m < n$ one can still use degenerate simplicial resolutions but use the filtration by *complex* skeleta of the fibres. (By the complex k-skeleton of a simplex in $\mathbf{C}^N$ we mean the union of all its faces that are contained in complex affine subspaces of dimension at most k. The complex k-skeleton always contains the usual k-skeleton, and is a subset of the usual $2k$-skeleton.) This allows to recover the results of [2] when $m < n$.

In brief, if Z_r is the filtration of the *degenerate* simplicial resolution (denoted by Z^Δ in Section 4 of [2], by its complex $r-1$-skeleta, one gets the following estimate on the dimensions of the successive differences:

$$\dim Z_r \backslash Z_{r-1} \leq 2N_{p,q} - 2r(n-m+1) + r - 1$$

for $r \leq p+1$, and

$$\dim Z_r \backslash Z_{r-1} \leq 2N_{p,q} - 2r(n-m) - 2$$

for all r. Moreover, for all $r \leq [\frac{p+1}{2}]$ the space $Z_r\backslash Z_{r-1}$ is homeomorphic to a real vector bundle of rank

$$2N_{p,q} - 2(n+1)r + r - 1$$

over the configuration space $C_r(\mathbf{C}^m)$ of r distinct unordered points in $\mathbf{C}^m$. All this leads to the estimate of [2] when $m < n$. We should point out that this estimate is better than that of the present paper for $d \leq 3$. In particular, the methods discussed here are insufficient to establish the estimate of [2] for $d \leq 3$ and $m = n$.

We note that none of the problems discussed above arise in the real case, which is described in [1].

There is also a need for two further small corrections. In the proof of Lemma 4 the algebra A_0 which is supposed to separate the points of $\mathbf{CP}^m$ fails to do so. Instead, A_0 should be defined as generated by the functions of the form

$$\frac{v\bar{v}}{z_0\bar{z}_0 + \ldots + z_m\bar{z}_m}$$

where v is linear in the z_i. Finally, the formula on the last line of page 289 should involve twisted, rather than constant, coefficients on the left-hand side.

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Departamento de Matemáticas, CINVESTAV, Apartado Postal 14-740 C.P. 07000, México, D.F., México

E-mail address: jacob@math.cinvestav.mx