A TOY THEORY OF VASSILIEV INVARIANTS

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To Victor Vassiliev on the occasion of his 50th birthday with our best wishes

ABSTRACT. We set up a theory of finite type invariants for smooth hypersurfaces in $\mathbb{R}^n$. For $n = 1, 2, 3$ these invariants admit a complete description: they form a polynomial algebra on one generator.

1. Introduction

Methods developed by V. Vassiliev for the study of complements of discriminants [5] proved to be useful in diverse contexts. However, luck seems to be an important ingredient of Vassiliev's method: while some discriminants produce rich and mysterious theories of finite type invariants (think of knots), in other cases finite type invariants reveal next to nothing. In this note we shall illustrate the latter situation by describing the finite type invariants of *co-knots*.

Co-knots are smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ which are equal to infinity at infinity; in what follows we shall assume that $f(x) \sim |x|^2$ as $|x| \to \infty$. A co-knot is *singular* if 0 is its critical value; singular co-knots form the *discriminant* Σ in the space $\mathcal{F}$ of all co-knots. We consider two non-singular co-knots as equivalent if they belong to the same connected component of the complement $\mathcal{F} - \Sigma$. A co-knot is determined up to equivalence by its zero level set, so equivalence classes of co-knots are in 1-to-1 correspondence with smooth hypersurfaces in $\mathbb{R}^n$ considered up to isotopy.

An invariant of co-knots is a locally constant function on the space $\mathcal{F}-\Sigma$. Finite type invariants of co-knots are defined in the same way as in the more familiar case of knots: a function φ is a finite type invariant of degree $< n$ if the alternating sum

$$\varphi(f) = \sum_{\varepsilon_1,\ldots,\varepsilon_n} \varphi(f_{\varepsilon_1,\ldots,\varepsilon_n})$$

is zero for any function $f \in \Sigma$ having n non-degenerate critical points with critical value 0. Here $\varepsilon_1, \ldots, \varepsilon_n$ is a sequence of signs $\pm$ and $f_{\varepsilon_1,\ldots,\varepsilon_n} \in \mathcal{F} - \Sigma$ denotes the function obtained from f by a small perturbation near all zero level critical points according to the signs ε_i.

Knot theory in $\mathbb{R}^3$ is a subset of the theory of co-knots with $n = 3$: knotted tori can be thought of as both knots and co-knots. Nevertheless, finite type invariants of co-knots with $n = 3$ fail to distinguish any knottedness: it turns out that all they see is the Euler characteristic! (The picture is very similar for $n = 1, 2$.)

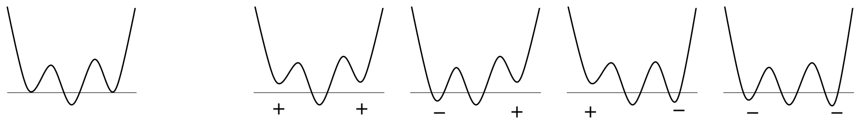
This can be proved by rather elementary means so the whole theory is no more than a toy. We prepared this note after agreeing that a toy makes a good birthday present, even for a gentleman of 50!

2. Statement of results

We begin by discussing the simplest case $n = 1$. A 1-dimensional co-knot is a function $f : \mathbb{R} \to \mathbb{R}$ that tends to $+\infty$ as $|x| \to \pm\infty$ and does not have 0 as its critical value. Equivalence classes of such functions are defined by the number of zeroes which is an even integer 0, 2, 4,... Let φ_k be the value of the invariant φ on the co-knot with $2k$ zeroes. The condition that φ is finite type of degree $< n$ transcribes as the identity

$$\varphi_k - n\varphi_{k+1} + \frac{n(n+1)}{2}\varphi_{k+2} - \cdots + (-1)^n \varphi_{k+n} = 0$$

for all $k = 0, 1, 2, \ldots$ (Here is an illustration in the particular case $n = 2$, $k = 1$:



— it shows a point of the discriminant and its 4 resolutions giving the identity $\varphi_1 - 2\varphi_2 + \varphi_3 = 0$ for an invariant φ of order 1.) Introducing the difference operator Δ in the space of real sequences by $\Delta(\varphi)_k := \varphi_k - \varphi_{k+1}$, we can rewrite this identity in a compact way as $(\Delta - 1)^n(\varphi) = 0$. The general solution to this equation is an arbitrary sequence $\{\varphi_k\}$ which depends on k as a polynomial of degree $< n$.

The algebra of real-valued finite type invariants in this case is thus the polynomial algebra $\mathbb{R}[k]$ where k is the basic invariant taking value k on nonsingular functions with $2k$ zeroes.

The answer in the cases $n = 2$ and $n = 3$ is quite similar. Recall that a planar co-knot $f : \mathbb{R}^2 \to \mathbb{R}$ is determined by the nonsingular curve $f^{-1}(0)$ which is a configuration of ovals, while a spatial co-knot $f : \mathbb{R}^3 \to \mathbb{R}$ is described by a nonsingular configuration of closed surfaces in $\mathbb{R}^3$.

Theorem 1. *Given a configuration of ovals in the plane, paint the plane in the checkerboard manner, starting with white at infinity and changing colour at each crossing of a line. Let* $e = b - w$ *where* b *is the number of compact black regions and* w *is the number of compact white regions. Then (1)* e *is a finite type invariant of degree 1, (2) the whole algebra of finite type invariants of 2-dimensional co-knots coincides with* $\mathbb{R}[e]$.

Theorem 2. *Let* χ *be the Euler charactristic of a nonsingular compact hypersurface in* $\mathbb{R}^3$. *Then (1)* χ *is a finite type invariant of degree 1, (2) the entire algebra of finite type invariants of 3-dimensional co-knots coincides with* $\mathbb{R}[\chi]$.

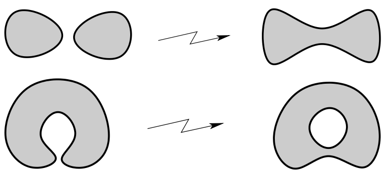
3. Proof in dimension 2

An invariant of co-knots in dimension 2 is a function of a plane configuration of disjoint ovals considered up to topological equivalence. The discriminant consists of curves with a finite number of singularities of two types: isolated points and transversal self-intersections. Equivalence types of non-singular co-knots are clearly in 1-to-1 correspondence with rooted trees, but we are not going to use this language since it is easier to state all the necessary facts directly in terms of ovals.

As usual, the space of invariants of degree 0 is 1-dimensional and consists of constants, since such invariants preserve their value under arbitrary passes through the discriminant.

Lemma 1. *The function e described in Theorem 1 is an invariant of degree 1.*

Proof. We need to prove the identity $e(K_{++})-e(K_{+-}) = e(K_{-+})-e(K_{--})$, where the letter K with subscripts denotes co-knots obtained by all resolutions from a singular co-knot with 2 singular points. In plain English this means that the jump of the function e corresponding to the $+ \to -$ change of the second subscript does not depend on the value of the first subscript. Recall that $e = b - w$ where b (resp. w) is the number of white (resp. black) connected components of the complement to the curve. The fact that the jump of e is determined by the local behaviour of the co-knot near the point in question, is evident for isolated points where birth/death of a circle takes place. For singular points of the second type corresponding to saddle surgery, one must consider two cases, depending on whether or not there are two regions that merge when passing through the discriminant. The following picture shows that in either case the jump is the same:



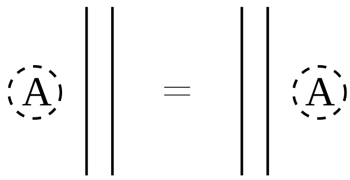
: $b \mapsto b-1, \quad w \mapsto w.$

: $b \mapsto b, \quad w \mapsto w+1.$

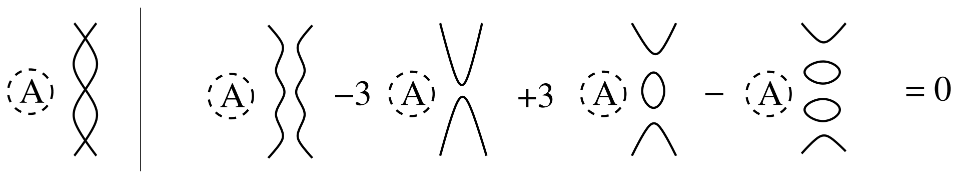
□

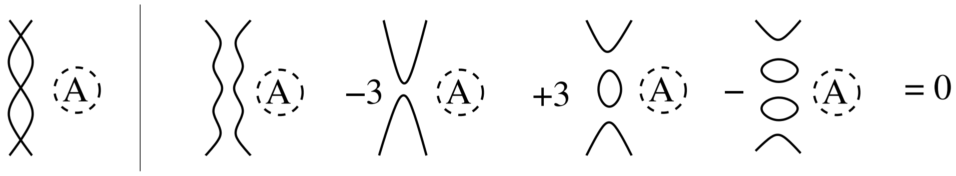
In order to prove that *any* finite type invariant of plane co-knots is a polynomial in e, we first describe two local transformations of co-knots which do not affect the finite-type invariants. Let us call two co-knots *FT-equivalent* if no finite type invariant can distinguish them.

Lemma 2. *A local collection of ovals can freely move through a double wall: if K and K' are two configurations of ovals that differ as shown in the picture (A is an arbitrary cluster of ovals inside a small disk), then K and K' are FT-equivalent.*



Proof. Let φ be an invariant of degree n. Replace the parallel lines by two lines with $n+1$ intersections and use the definition of finite type invariants. We get two linear combinations which are both equal to zero and coincide save for one term corresponding to the co-knots K and K'. This is illustrated in the following picture where we take $n = 2$. On the left a singular co-knot is displayed, on the right the corresponding identity for the the values of φ (to simplify notation, we write x instead of $\varphi(x)$):

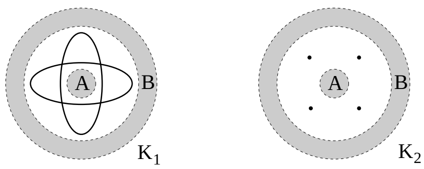




□

Lemma 3. *Concentric circles annihilate: if K is a configuration that contains two circles, one inside another and such that the annulus between them is empty, and K' is obtained from K by deleting these two circles, then K and K' are FT-equivalent. In particular, the configuration consisting of two concentric ovals is FT-equivalent to the empty plane.*

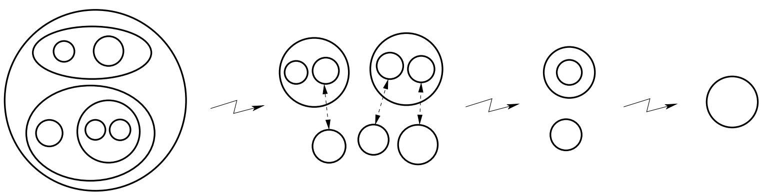
Proof. The proof is similar to that of the previous lemma. Let φ is an invariant of degree n. Introduce two auxiliary singular co-knots K_1, K_2 replacing the concentric circles by (1) a circular curve with $n+1$ self-intersections, (2) $n+1$ isolated points, then write down the linear combinations of values of φ on the resolutions of K_1 and K_2. As before, only two terms in these expressions are different, and they are exactly $\varphi(K)$ and $\varphi(K')$. The picture shows singular co-knots K_1 and K_2 adapted for invariants of degree ≤ 3:



In either case, shaded regions A and B designate the same collections of ovals. □

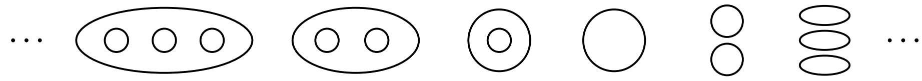
Lemma 4. *If $n = e(K) \geq 0$, then K is FT-equivalent to a collection of n circles, one outside another (if $n = 0$, this is just an empty set). If $n = e(K) < 0$, then K is FT-equivalent to a circle containing $1 - n$ smaller circles, one outside another.*

Proof. Given a set of ovals, use repeatedly the move from Lemma 2 to drag the innermost circles outside, passing through two lines at a time. In this way we get a configuration nested to depth at most 1, i.e. a collection of some empty circles not contained in other circles (call them "outside circles") and some circles containing other ("inner") circles. If there is at least one outside circle and one inner circle, then we can move one inside the other and delete the pair according to the rule of Lemma 3. This process is illustrated in the picture (annihilating circles are connected by arrows):



□

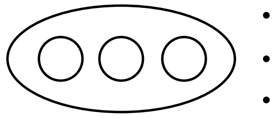
The function e thus establishes a 1-to-1 correspondence between the FT-equivalence classes and all integers:



Therefore an invariant φ is the same thing as a numeric sequence $\{\varphi_k \mid k \in \mathbb{Z}\}$. The following lemma completes the proof of Theorem 1.

Lemma 5. *A function $\varphi(e)$ defines an invariant of finite type if and only if it is polynomial in e. The degree of the polynomial is the degree of the invariant.*

Proof. Assume $\varphi(e)$ is of degree $< n$. Denote by N_m the collection of circles described in Lemma 4 with $e(N_m) = m$. Consider the series of singular co-knots $S_{n,m}$ consisting of the curve N_m and n isolated points — the picture, by way of example, shows the co-knot $S_{3,-2}$:



Writing the finite type condition on φ for such point of the discriminant, we see at once that it is equivalent to the identity $\Delta^n(f) = 0$, where Δ is the difference operator on integer sequences defined in Section 2. This identity is, in turn, equivalent to the fact that φ is a polynomial function. □

4. Proof in dimension 3

In this Section we prove Theorem 2 about finite type invariants of spatial co-knots, i.e. smooth closed surfaces in $\mathbb{R}^3$. Any such surface in $\mathbb{R}^3$ is the union of a finite number of connected surfaces Q_g of genus $g \geq 0$. A reasonable classification of the embeddings of Q_g into $\mathbb{R}^3$ is known only for $g = 0, 1$ (see Problem 3.11 in Kirby's list of open problems [2]). A feeling of what such an embedding may look like is provided by the following example: take a sphere, drill two knotted and interlinked tunnels inside, add a torus that passes through both tunnels and attach it to the sphere by a handle. Although this picture looks rather frightening, we shall see that the finite type invariants are insensitive to such complications.

The argument we use here to classify finite type invariants is similar to what we did in Section 3: we introduce certain moves of co-knots within the same class of FT-equivalence and, using these moves, reduce any co-knot to a normal form uniquely labelled by the Euler characteristic.

Lemma 6. *The Euler characteristic $\chi(S)$ of the surface S is a degree 1 invariant.*

Proof. As in Lemma 1, we must check that the jump of the Euler characteristic on a simple passage through the discriminant does not depend on the behaviour of the surface far away from the point in question. For an isolated point (where birth/death of a small sphere takes place) this is clear. For singularities of the second kind (conical points, where a small two-sheeted hyperboloid becomes one-sheeted) there are two cases to consider:

(1) The two pieces that are merged belong to different connected components. This is the transformation $Q_g \cup Q_h \mapsto Q_{g+h-1}$.
(2) The two pieces belong to the same component Q_g which becomes Q_{g+1}.

In either case the total Euler characteristic increases by 1, i.e. its change is determined locally. □

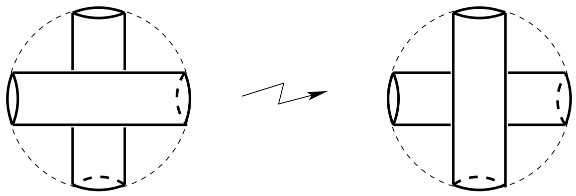
Now we show that *any* finite type invariant of 3-dimensional co-knots is actually a function of χ.

Let S be a surface embedded in $\mathbb{R}^3$. A *tube* of S is a full cylinder $T \subset \mathbb{R}^3$ such that $S \cap T$ is exactly the side of T. Assume that S and S' are two surfaces and T and T' are tubes of S and S' respectively, lying both on the inside or both on the outside of the corresponding surface. If the cylinders T and T' have the same bases and

$$S - S \cap T = S' - S' \cap T'$$

we say that a surface S' is obtained from S by *re-routing a tube* T.

The simplest example of tube re-routing is a "crossing change":



Lemma 7. *By re-routing tubes, any surface in $\mathbb{R}^3$ can be transformed into a union of unlinked (but possibly nested) surfaces, each bounding a ball or a handlebody in $\mathbb{R}^3$.*

Proof. This statement is implicit in Fox's paper [1] though it is not stated there in this form. Let S be a (not necessarily connected) surface that can be "undone" by re-routing tubes and let A be the bounded region in R^3 such that $S = \partial A$. If A' is obtained from A by either "drilling a tunnel" inside A or by adding a handle to A, then $S' = \partial A'$ also can be "undone". Fox proves in [1] that any region bounded by a surface in R^3 can be obtained from a region bounded by a union of spheres by drilling tunnels and adding handles, so Lemma 7 follows. □

The next lemma implies that tube re-routing does not change the values of finite type invariants.

Lemma 8. *Two co-knots that differ by a re-routed tube are FT-equivalent.*

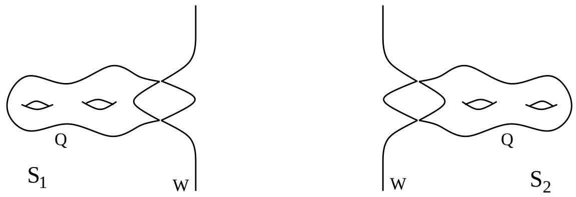
Proof. Assume that S_- is obtained from S_+ by tube re-routing. To prove that $\varphi(S_+) = \varphi(S_-)$ for an invariant φ of degree $< n$, consider singular co-knots $S^*_\pm$ obtained from $S_\pm$ by squeezing one of the cylinders at n places to form n conical points. The finite type condition on φ says that the corresponding sums of 2^n terms are both zero. But it is easy to see that these sums are identical save for one term: $\varphi(S_+)$ in one case and $\varphi(S_-)$ in another. □

According to Lemmas 7 and 8, any surface within its class of FT-equivalence can be transformed to a collection of surfaces Q_g, embedded in $\mathbb{R}^3$ in a standard way (up to isotopy) and unlinked between themselves, but possibly nested.

In the case of co-knots in $\mathbb{R}^2$ Lemma 2 ensures that a subobject can go through a pair of adjacent walls. In 3-space, even individual walls are transparent:

Lemma 9. *Let S and S' be two spatial co-knots that differ only in the position of one (local) connected component: in S it is on one side of a certain surface, while in S' it is on the other side. Then S and S' are FT-equivalent.*

Proof. Let Q be the moving component and W be the "wall". Using the same argument as before, we consider two singular co-knots: one (S_1) formed by connecting the component Q with the wall W from one side, another (S_2) from the other side, by a series of n tubes squeezed in the middle to a conical point:



Finite type identities for S_1 (resp. S_2) consist of S (resp. S') and a number of surfaces where Q and W are joined into one connected component whose genus is the same in both cases and whose actual embedding into $\mathbb{R}^3$ is irrelevant due to the previous lemmas. Therefore, S and S' are FT-equivalent. □

Using the two lemmas, any configuration of surfaces (possibly knotted, linked and embedded one inside another) can be transformed into a collection of standard not nested handlebodies. A further reduction is possible due to the following "fusion" lemma.

Lemma 10. *Let Q_g denote a closed surface of genus g embedded in $\mathbb{R}^3$ in a standard way. If at least one of the numbers g, h is greater than 0, then the disjoint union $Q_g \cup Q_h$ is FT-equivalent to a single surface Q_{g+h-1}.*

Proof. Join the two surfaces Q_g and Q_h by a series of n tubes, each with a conical singular point. Then attach n similar handles to the surface Q_{g+h-1}. Comparing the finite type identities for these two singular co-knots we get the desired result. □

It follows that any co-knot in $\mathbb{R}^3$ is equivalent, modulo finite-type invariants, either to a collection of several disjoint spheres or to a single standard surface of genus $g > 1$ (the torus is equivalent to an empty space, i.e. a collection of 0 spheres — this follows from the comparison between a torus pinched at n points and a collection of n isolated points). These normal forms have different Euler characteristic χ (ranging over all even integers), so no two of them can be equivalent. This implies that *any* finite type invariant is a function of χ. The argument similar to the one used in Lemma 5 proves the following statement and thereby concludes the proof of Theorem 2.

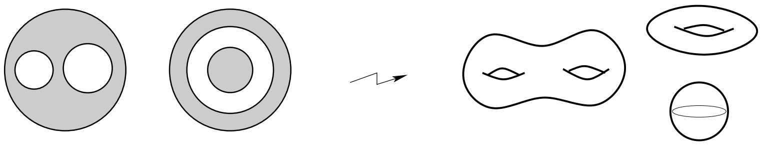
Lemma 11. *A function $\varphi(\chi)$ defines a finite type invariant if and only if it is polynomial in χ. The degree of the polynomial is the degree of the invariant.*

5. Comments and questions

5.1. **Completeness of the toy theories.** Finite type invariants do distinguish co-knots in $\mathbb{R}^n$ for $n = 1$ and do not distinguish them for $n = 2$ or 3.

5.2. **Relation between the dimensions 1, 2 and 3.** The toy theories in all three dimensions where we considered them are isomorphic. The isomorphism between the planar theory ($n = 2$) and the spatial theory ($n = 3$) has a nice geometric interpretation. Given a configuration of ovals, paint the plane in the checkerboard manner. Embed the plane into the 3-space and, on either side of the plane, top the black regions with smooth caps that orthogonally project in a

1-1 way onto the plane. We get a collection of closed surfaces whose total Euler characteristic is twice the invariant e (difference between the number of compact black and white regions) of the initial configuration of ovals.



5.3. **Other theories of finite type invariants.** There are other approaches to finite type invariants of hypersurfaces. A theory of invariants for immersions of a fixed surface in $\mathbb{R}^3$ is developed in [3, 4]. It is very different from our toy theory, but both have one feature in common: all invariants are functions of invariants of degree one.

5.4. **Further questions.** What is the analog of our toy theory if $\mathbb{R}^2$ or $\mathbb{R}^3$ is replaced by another 2– or 3–manifold? What is the algebra of the finite type invariants for co-knots $\mathbb{R}^n \to \mathbb{R}$, where $n > 3$?

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