GEOMETRY OF TRUNCATED SYMMETRIC PRODUCTS AND REAL ROOTS OF REAL POLYNOMIALS

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ABSTRACT

We study the conditions for truncated symmetric products of manifolds to be manifolds. In particular, we show that suitably defined spaces of systems of real roots of real polynomials are homeomorphic to real projective spaces.

1. *Definitions and statement of results.*

Unless otherwise stated, in this paper 'manifold' will mean 'manifold without boundary'.

DEFINITION. The n-th symmetric product of a topological space X (denoted by $SP^n(X)$) is the quotient of the n-fold Cartesian product $(X)^n$ by the action of the symmetric group S_n, which permutes the factors in $(X)^n$. The topology on $SP^n(X)$ is the quotient topology with respect to the natural map

$$(X)^n \xrightarrow{\rho_1} SP^n(X).$$

$SP^0(X)$ is defined to be a point.

We can write points of $SP^n(X)$ as formal sums $\sum k_i x_i$, where $x_i \in X$, $i \in \mathbf{Z}$, k_i are non-negative integers and $\sum k_i = n$. We require the points $x_i \in X$ to be distinct, but, in order to simplify the definition below, we allow zero coefficients in these expressions, so the representation of a point of $SP^n(X)$ by a formal sum is not unique.

DEFINITION. The n-th truncated symmetric product of a topological space X (denoted by $TP^n(X)$) is the quotient of $SP^n(X)$ by the following equivalence relation:

$$\sum k_i x_i \sim \sum k_i' x_i \text{ if } k_i \equiv k_i' \ (mod\ 2) \text{ for all } i.$$

The topology on $TP^n(X)$ is the quotient topology with respect to the natural map

$$SP^n(X) \xrightarrow{\rho_2} TP^n(X).$$

Other notations for truncated products, such as $SP^n(X, \mathbf{Z}/2)$ and $XP^n(X)$ can be found in the literature. We use the notation of [**1**].

Choose an arbitrary point $x \in X$ and for any $0 \leq k \leq n$ let $\Delta_k \subset SP^n(X)$ be the subset of points which can be written as $mx + \sum k_i x_i$ for some $m \geq k$. Then $\rho_2(\Delta_k)$ can

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be identified with a copy of $TP^{n-k}(X)$ inside $TP^n(X)$. When k is even, this inclusion does not depend on the choice of the point x, so there is a natural filtration on truncated symmetric products:

$$\ldots \subset TP^{n-2}(X) \subset TP^n(X).$$

We can write points of $TP^n(X)$ as formal sums $\sum k_i x_i$, where $x_i \in X$, $k_i \in \mathbf{Z}/2$ and the number of non-zero terms does not exceed n and has the same parity as n.

Truncated symmetric products were introduced in [**4**] by Dold and Thom (they studied only *infinite* products); a topological description of finite truncated products (i.e. the ones in question here) can be found in [**7**] and [**1**], where they were used as a tool for studying configuration spaces. Truncated products are also relevant as spaces of real algebraic 0-cycles modulo 2 in the context of Lawson's theory of algebraic cycles, see [**6**].

There are two well-known facts about symmetric products, we state them as one theorem:

THEOREM 1. *A) Let M be a manifold. Then $SP^n(M)$ is a manifold for $n > 1$ if and only if $\dim M = 2$.*
B) $SP^n(\mathbf{CP}^1)$ is homeomorphic to $\mathbf{CP}^n$.

Theorem 1B is easily proved as follows: $\mathbf{CP}^n$ can be interpreted as the space of homogeneous polynomials in two variables of degree n modulo multiplication by a non-zero complex constant. Each polynomial is determined up to a constant by its n roots on the Riemann sphere $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$; on the other hand, $SP^n(\mathbf{CP}^1)$ is precisely the space of n-tuples of unordered points in $\mathbf{CP}^1$. Clearly this correspondence between polynomials and root systems is continuous; hence the result.

Here we prove an analogous theorem for truncated symmetric products.

THEOREM 2. *A) Let M be a compact connected manifold. Then $TP^n(M)$ is a manifold for $n > 1$ if and only if $\dim M = 1$.*
B) $TP^n(\mathbf{RP}^1)$ is homeomorphic to $\mathbf{RP}^n$.

Theorem 2B has the following refinement. Consider $\mathbf{RP}^n$ as the space of real homogeneous polynomials in two variables of degree n modulo multiplication by a real constant. Then we have a map

$$P : \mathbf{RP}^n \to TP^n(\mathbf{RP}^1),$$

which sends a polynomial to the system of its real roots with multiplicities reduced modulo 2. This map is clearly onto, continuous and closed, but not one-to-one. We shall prove the following statement:

THEOREM 3. *The map P is homotopic to a homeomorphism.*

2. *Proof of Theorem 2A.*

Denote by Δ the 'big diagonal' in $(M)^n$, i.e. the set of points with at least two coinciding coordinates. Consider the sequence of natural maps

$$(M)^n \xrightarrow{\rho_1} SP^n(M) \xrightarrow{\rho_2} TP^n(M).$$

The image of Δ under $\rho_2\rho_1$ is $TP^{n-2}(M) \subset TP^n(M)$. Also notice that the composite map $\rho_2\rho_1$, restricted to $(M)^n \backslash \Delta$ is an $n!$-sheeted covering

$$\rho_2\rho_1 : (M)^n \backslash \Delta \to TP^n(M) \backslash TP^{n-2}(M).$$

Now suppose $TP^n(M)$ is a manifold, $\dim M \geq 2$. Clearly, $\dim TP^n(M) = n \dim M$.

Choose any point $y \in M$. Then the point $\rho_2(ny)$ (or, in other words, $\rho_2\rho_1(y, y, \dots, y)$) belongs to $TP^{n-2}(M) \subset TP^n(M)$. Take a small ball B around $\rho_2(ny)$. Then

$$\dim B \cap TP^{n-2}(M) = (n-2) \dim M$$

and, hence, $H^{n \dim M - 2}(B \cap TP^{n-2}(M)) = 0$. By Alexander duality this implies that the first homology group of

$$W = B \backslash (B \cap TP^{n-2}(M))$$

is zero. Also, as $\dim M \geq 2$, it is clear that $\widetilde{W} = \rho_1^{-1}\rho_2^{-1}(W)$ is connected. But as

$$\rho_2\rho_1 : \widetilde{W} \to W$$

is an $n!$-sheeted covering with S_n being the monodromy group, $H_1(W)$ cannot be trivial; so we get a contradiction.

(Indeed, we have a surjection

$$\pi_1(W) \to S_n.$$

For any n there is a non-trivial map $S_n \to \mathbf{Z}/2$. As $H_1(W) = \pi_1(W)/[\pi_1(W), \pi_1(W)]$, the composite map $\pi_1(W) \to \mathbf{Z}/2$ must factor through a non-trivial map $H_1(W) \to \mathbf{Z}/2$, so $H_1(W) \neq 0$.)

The only 1-dimensional compact connected manifold is the circle $\mathbf{RP}^1$. In the next section we will show that $TP^n(\mathbf{RP}^1)$ is a manifold for any n; that will establish Theorem 2A.

3. *Proof of Theorem 2B.*

Here we will verify Theorem 3 and, hence, prove Theorem 2B. First we state an auxiliary lemma about the symmetric products of a 2-disk:

LEMMA 1. *$SP^k(D^2)$ is homeomorphic to a $2k$-dimensional ball D^{2k}.*

Proof. Let $D^2 = \{z \mid z \in \mathbf{C}, z\bar{z} \leq 1\}$. Define a map $\psi_1 : SP^k(D^2) \to \mathbf{C}^k$ by sending a point $(a_1, \dots, a_k) \in SP^k(D^2)$ to the coefficients of the polynomial $(z - a_1) \cdot \dots \cdot (z - a_k)$; and a map $\psi_2 : \mathbf{C}^k \to \mathbf{C}^k = \mathbf{R}^{2k}$ by

$$(r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, \dots, r_k e^{i\phi_k}) \to (r_1 e^{i\phi_1}, \sqrt{r_2} e^{i\phi_2}, \dots, \sqrt[k]{r_k} e^{i\phi_k}).$$

Then the composite map $\psi_2\psi_1 : SP^k(D^2) \to \mathbf{R}^{2k}$ is one-to-one and carries $SP^k(D^2)$ onto some neighbourhood of zero in $\mathbf{R}^{2k}$. Any ray starting at $0 \in \mathbf{R}^{2k}$ meets the boundary of $\psi_2\psi_1 SP^k(D^2)$ precisely in one point. This implies that $\psi_2\psi_1 SP^k(D^2)$ and, hence, $SP^k(D^2)$ is homeomorphic to a ball.

Let us study the structure of the map $P : \mathbf{RP}^n \to TP^n(\mathbf{RP}^1)$. The space $TP^n(\mathbf{RP}^1)$ is filtered by the subspaces $TP^m(\mathbf{RP}^1)$, where $m \leq n$ and $m \equiv n \ (mod\ 2)$. The space

$$C_m = TP^m(\mathbf{RP}^1) \backslash TP^{m-2}(\mathbf{RP}^1)$$

can be identified with the configuration space of m unordered distinct points on the circle. Clearly C_m is an (open) manifold.

$\mathbf{RP}^n$ can be decomposed as the union of subspaces $\bigcup R_i$, where R_i is the space of polynomials, which have *precisely* i real roots of odd multiplicity. Explicitly,

$$\mathbf{RP}^n = \begin{cases} R_0 \cup R_2 \cup \dots R_n & \text{if } n \text{ is even} \\ R_1 \cup R_3 \cup \dots R_n & \text{if } n \text{ is odd} \end{cases}$$

Notice that $\bigcup \partial R_i$ is the hypersurface of polynomials with multiple real zeros; it is a subset of the zero set of the discriminant. The map P respects the decompositions we have introduced on $\mathbf{RP}^n$ and $TP^n(\mathbf{RP}^1)$. Denote by p_i the restriction of P to R_i:

$$p_i : R_i \to C_i.$$

Every polynomial with i roots of odd multiplicity can be uniquely factorised as a product of a non-negative (i. e. positive semi-definite) polynomial of degree $n - i$ (normalised in such a way that the sum of squares of its coefficients is equal to 1) and a polynomial of the form

$$(a_1 x - b_1 y) \dots (a_i x - b_i y),$$

where (a_k, b_k) are distinct points of $\mathbf{RP}^1$.

The space of normalised non-negative polynomials of degree $n-i$ is homeomorphic to $SP^{\frac{1}{2}(n-i)}(D^2)$, where D^2 is considered as the closure of the upper half-plane $\mathrm{Im}(z) > 0$ in $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$. This means that

$$R_i = C_i \times SP^{\frac{1}{2}(n-i)}(D^2) = C_i \times D^{n-i}$$

and p_i is the projection on the first factor. The boundary of R_i can be described as a union of two components. The first one is $F_i \overset{def}{=} C_i \times \partial D^{n-i} = C_i \times S^{n-i-1}$, obviously $F_i \subset R_i$. The second one is $\partial R_i \backslash F_i = \partial R_i \bigcap \overline{R_{i-2}}$. The following lemmas can be easily verified directly from the definition of R_i.

LEMMA 2. *Any $x \in F_i$ has a neighbourhood N_x such that $N_x \bigcap R_k$ is empty for $k < i$.*

LEMMA 3. *For each i, $\overline{F}_i$ divides $\mathbf{RP}^n$ into two components; the closure of one of them is $\bigcup_{k \le i} R_k$; the closure of the other is $\bigcup_{k>i} \overline{R_k}$.*

The map P can be described as a composition of collapsing maps

$$\mathbf{RP}^n = Q_0 \overset{q_0}{\to} Q_2 \overset{q_2}{\to} \dots \overset{q_{n-2}}{\to} Q_n = TP^n(\mathbf{RP}^1),\ n \text{ even},$$

or

$$\mathbf{RP}^n = Q_1 \overset{q_1}{\to} Q_3 \overset{q_3}{\to} \dots \overset{q_{n-2}}{\to} Q_n = TP^n(\mathbf{RP}^1),\ n \text{ odd},$$

where q_i are the maps, which collapse the fibres of p_i to single points, and Q_i are the corresponding quotient spaces. We will see that these maps are actually homotopic to homeomorphisms.

Let us first look at an example: $n = 2$. Here the map P is 1-1 everywhere, apart from the disk, bounded by the conic

$$x_1^2 - 4x_0 x_2 = 0,$$

i.e. the conic, defined by the condition that the discriminant of the polynomial is zero; here

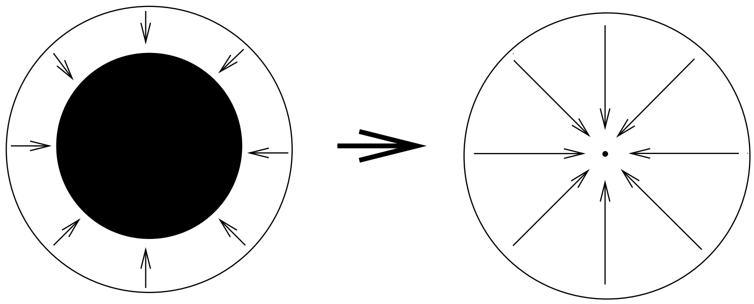

FIG. 1. The deformation retraction ϕ_t

x_i are the homogeneous coordinates on $\mathbf{RP}^2$. This disc, which describes polynomials with double roots or no real roots, maps into a single point.

Take a 'one-sided tubular neighbourhood' of this disk and define the deformation retraction

$$\phi_t : \mathbf{RP}^2 \times [0,1] \to \mathbf{RP}^2$$

so that ϕ_t is the identity map outside the neighbourhood of the disk for all t and is the contraction of the disk to a single point when $t = 1$, see Fig. 1. Then $P(\phi_t^{-1})$ is a homotopy of P to a homeomorphism.

Our strategy is to use the same method for the maps q_i 'fibrewise'. In general, however, the hypersurface of polynomials with multiple real roots is singular, so we need some results concerning one-sided tubular neighbourhoods in such a situation.

DEFINITION. ([**2**]) Let X be a topological space and B a subset of X. Then B is *collared* in X if there is a homeomorphism h carrying $B \times [0,1)$ onto an open neighbourhood of B such that $h(b,0) = b$ for all $b \in B$. If B can be covered by a collection of open subsets (relative to B) each of which is collared in X, then B is *locally collared* in X.

B is said to be *bi-collared* in X if there is a homeomorphism h carrying $B \times (-1,1)$ onto an open neighbourhood of B such that $h(b,0) = b$ for all $b \in B$. The notion of B being *locally bi-collared* is defined similarly.

An example of a locally collared (in fact, collared) subset is the boundary of a manifold with boundary. An example of a locally bi-collared subset is a smooth submanifold of codimension 1 in a smooth manifold.

THEOREM 4. ([**2**]) *A locally collared subset of a metric space is collared.*

THEOREM 5. ([**8**], see also [**2**]) *If $B \to Int(X)$ is a piecewise linear embedding of an $(n-1)$-dimensional PL-manifold into a PL n-manifold X, then B is locally bi-collared in X.*

Now we are in the position to prove the following

PROPOSITION 1. *For each i, the map $q_i : Q_i \to Q_{i+2}$ is homotopic to a homeomorphism.*

Proof. First of all notice that $\mathbf{RP}^n$ together with the hypersurface of polynomials with multiple real roots is a compact stratified set, hence by the main theorem of [**5**] it has a triangulation, compatible with the stratification. The subsets F_i are not closed, but they are locally triangulable.

As we mentioned above, F_i is homeomorphic to $C_i \times S^{n-i-1}$, which is a manifold. Thus, by Lemma 2, for any point $x \in F_i$ we can find a triangulation of $\mathbf{RP}^n$, such that there exists a triangulated neighbourhood N of x, homeomorphic to a ball D^n, $N \cap F_i$ is homeomorphic to a ball D^{n-1} and $N \cap \partial R_k$ is empty for $k < i$. Applying Theorem 5 we get that F_i is bi-collared in some neighbourhood of x inside $\mathbf{RP}^n$ and, hence, collared in some neighbourhood of x inside each of the components into which $\overline{F}_i$ divides $\mathbf{RP}^n$. In particular, it is collared in $\mathbf{RP}^n \backslash Int(\bigcup_{k \le i} R_k)$. As x is an arbitrary point of F_i, it follows that F_i is locally collared and, by Theorem 4, is collared in $\mathbf{RP}^n \backslash Int(\bigcup_{k \le i} R_k)$.

Now we construct a 'spindle neighbourhood' of F_i as follows. Let λ be a continuous function on $\overline{F_i}$, such that $\lambda = 0$ on $\overline{F_i} - F_i$ and $0 < \lambda < 1$ on F_i. Then our spindle neighbourhood S_i is the subset of the collar $F_i \times [0,1)$ defined as

$$S_i = \{(x,t) \mid x \in F_i; 0 \le t \le \lambda\}.$$

There is a deformation retraction

$$\Phi_t : (S_i \cup R_i) \times [0,1] \to S_i \cup R_i,$$

which commutes with the projection p_i and in fibres looks like the retraction ϕ_t of Fig.1. The union of R_i and the spindle neighbourhood of F_i maps homeomorphically into Q_i under the composite collapsing map $\mathbf{RP}^n \to Q_i$. We extend Φ_t to Q_i by the identity map; it is immediately clear that this extension of Φ_t is continuous everywhere, possibly apart from the image of $\bigcup_{k<i} R_k$. And continuity there follows from the following fact:

LEMMA 4. *The decomposition of $\mathbf{RP}^n$ into fibres of P is upper semicontinuous, i.e. for any neighbourhood U of a disk $P^{-1}(y)$, $y \in TP^n(\mathbf{RP}^1)$ there exists a neighbourhood V, such that any fibre of P which intersects V lies in U.*

(This lemma directly follows from Prop.1, Ch.1 of [**3**], which says that if a map between topological spaces is closed, the decomposition of the source space into the fibres of the map is upper semicontinuous. It can be easily verified directly.)

Now $q_i \Phi_t^{-1}$ is a homotopy, carrying q_i into a homeomorphism; this proves Proposition 1.

If two maps are homotopic to homeomorphisms, their composite is also homotopic to a homeomorphism. This proves Theorem 3.

REMARKS. There are powerful tools, such as Edwards' cell-like approximation theorem [**3**], to treat situations like ours. However, Edwards' theorem works in dimensions ≥ 5 and requires some conditions, which are not totally trivial to verify in our case; so we chose the direct approach.

Theorem 2B in a weaker form, namely, that $TP^n(\mathbf{RP}^1)$ is *homotopy equivalent* to $\mathbf{RP}^n$, has been proved by B. Mann and R. J. Milgram. Their proof (unpublished) was based on completely different arguments.

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