# LATTICES IN $\mathbb{C}$ AND FINITE SUBSETS OF A CIRCLE

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In this note we give a new proof of the following surprising fact: the space of all non-empty subsets of a circle of cardinality at most 3 is homeomorphic to a 3-sphere, while the subspace corresponding to one-element subsets, is a trefoil knot.

## 1. *Spaces of finite subsets*

For $X$ a topological space let $\exp_k X$ be the set of all non-empty finite subsets of $X$ of cardinality at most $k$. There is a map from the Cartesian product of $k$ copies of $X$ with itself to $\exp_k X$ which sends $(x_1, \ldots, x_k)$ to $\{x_1\} \cup \ldots \cup \{x_k\}$. The quotient topology gives $\exp_k X$ the structure of a topological space. Notice that for any $m \leqslant k$ the space $\exp_m X$ is canonically embedded into $\exp_k X$. Clearly, $\exp_1 X = X$.

The simplest non-trivial example is provided by the space $\exp_2 S^1$ which is homeomorphic to the Möbius band. One way to see it is as follows ([3]). Let us identify $S^1$ with the boundary of an open disk $D$ in the projective plane. Notice that $M = \mathbf{RP}^2 \backslash D$ is a Möbius band. For each point $x \in M$ there exist at most two lines that pass through $x$ and have a tangency with $S^1$. Let $T(x) \in \exp_2 S^1$ be the corresponding set of tangency points. Then

$$T : M \to \exp_2 S^1$$

is the desired homeomorphism. Notice that the boundary of the band corresponds to one-point subsets.

The space $\exp_3 S^1$ is described by a theorem of R. Bott:

**Theorem 1.** (Bott, [2]) *The space* $\exp_3 S^1$ *is homeomorphic to a 3-sphere* $S^3$.

The fact that $\exp_3 S^1$ is a 3-sphere is not obvious at all. In particular, K. Borsuk claimed in [1] that $\exp_3 S^1$ is homeomorphic to $S^1 \times S^2$. In fact, Bott's paper [2] is a part of a letter to Borsuk where Bott points out the mistake in [1] and corrects the argument.

An interesting illustration of the non-triviality of Bott's theorem is the following result, due to E. Shchepin. Recall that $\exp_1 S^1$ is a circle, and thus the inclusion map $\Delta : \exp_1 S^1 \hookrightarrow \exp_3 S^1$ is a knot in $S^3$.

**Theorem 2.** (Shchepin, [unpublished]) *The embedding* $\Delta : S^1 \to S^3$ *is a trefoil knot*.

*Remark*. The trefoil $\Delta$ comes equipped with a canonical "non-orientable Seifert surface" which is the Möbius band $\exp_2 S^1 \subset \exp_3 S^1$ .

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In his proof of Theorem 1 Bott used a rather complicated "cut-and-paste" argument. Shchepin's proof of Theorem 2 was based on a direct calculation of the fundamental group of $S^3\backslash\Delta(S^1)$. The purpose of this note is to show that the above theorems are equivalent to a well-known fact about lattices in a plane.

## 2. *Spaces of lattices*.

A closed discrete additive non-trivial subgroup of $\mathbf{C}$ is either isomorphic to $\mathbf{Z}\oplus\mathbf{Z}$ or to an infinite cyclic group. Such subgroups will be referred to as *(non-degenerate) lattices* and *degenerate lattices* respectively and the set of all these subgroups will be denoted by $\Lambda$.

The set $\Lambda$ is, in fact, a metric space. For $L\in\Lambda$ let $\overline{L}$ be the closure of the set of the elements of $L$ in the Riemann sphere $\mathbf{C}\cup\{\infty\}$. Assume that the Riemann sphere is given its usual "round" metric. Then the distance between two subgroups $L, L'\in\Lambda$ is defined as the minimal such $\varepsilon$ that $\overline{L'}$ is contained in the closed $\varepsilon$-neighbourhood of the set $\overline{L}$, and $\overline{L}$ is contained in the closed $\varepsilon$-neighbourhood of $\overline{L'}$. (This distance function is known as the *Hausdorff metric*, §28 of [4] is the classical reference for it.)

The multiplicative group $\mathbf{R}^+$ of positive real numbers acts on $\Lambda$ freely by re-scaling the lattices. The quotient space by this action can be represented as the union

$$\Lambda/\mathbf{R}^+ = L_0\cup L_1$$

where $L_0$, $L_1$ are the spaces of degenerate, respectively non-degenerate lattices considered up to re-scaling.

**Theorem 3.** *The space of all lattices up to re-scaling $\Lambda/\mathbf{R}^+$ is homeomorphic to $S^3$. The subspace $L_0$ of degenerate lattices is a circle and the inclusion $L_0\hookrightarrow\Lambda/\mathbf{R}^+$ is a trefoil knot.*

*Remark*. The space $L_1$ can be thought of as the space of *unimodular* lattices, that is, lattices whose fundamental domain has unit area. Thus $L_1$ can be identified with the homogeneous space $SL(2,\mathbf{R})/SL(2,\mathbf{Z})$. Indeed, applying elements of $SL(2,\mathbf{R})$ to the standard basis in $\mathbf{R}^2$ one obtains all unimodular lattices, and every lattice has $SL(2,\mathbf{Z})$ as its automorphism group.

The proof of the above theorem can be found in [7], see also page 84 of [6]. We sketch the argument very briefly below.

*Sketch of the proof.* For $L\in\Lambda$ define the complex numbers $G_4(L)$ and $G_6(L)$ by

$$G_k(L) = \sum_{\omega}\omega^{-k}$$

where the sum is taken over all non-zero points of $L$. The following statements are well-known in the theory of elliptic functions:

(a) the map $\Lambda\to\mathbf{C}^2\backslash 0$ given by sending the subgroup $L$ to the pair $(G_4(L), G_6(L))$ is a homeomorphism;
(b) a pair $(u,v)\in\mathbf{C}^2\backslash 0$ is the image of a non-degenerate lattice if and only if $20u^3-49v^2\neq 0$.

(For a non-degenerate lattice $L$ the numbers $G_4(L)$ and $G_6(L)$ are the parameters in the differential equation for the Weierstraß function $\wp_L$:

$$(\wp_L')^2 = 4\wp_L^3 - 60G_4(L)\wp_L - 140G_6(L).$$

The statements (a) and (b) are equivalent to saying that a differential equation of the above form determines a non-degenerate lattice provided that the cubic polynomial on the right-hand side has distinct roots.)

Now, consider a unit sphere $S^3$ centred at the origin in $\mathbf{C}^2$. For any $L \in \Lambda$ and any $t > 0$

$$G_4(tL) = t^{-4}G_4(L),$$
$$G_6(tL) = t^{-6}G_6(L).$$

This means that for any $L$ there exists the unique $L' \in \Lambda$ such that $L$ and $L'$ are homothetic, and such that the point $(G_4(L'), G_6(L'))$ lies on $S^3 \in \mathbf{C}^2 \backslash 0$. It follows from (a) that the map which sends $L$ to $(G_4(L'), G_6(L'))$ gives a homeomorphism between $\Lambda/\mathbf{R}^+$ and $S^3$. Finally, the intersection of $S^3$ with the curve $20u^3 - 49v^2 = 0$ is easily seen to be a trefoil knot; according to (b) this set corresponds precisely to the degenerate lattices. □

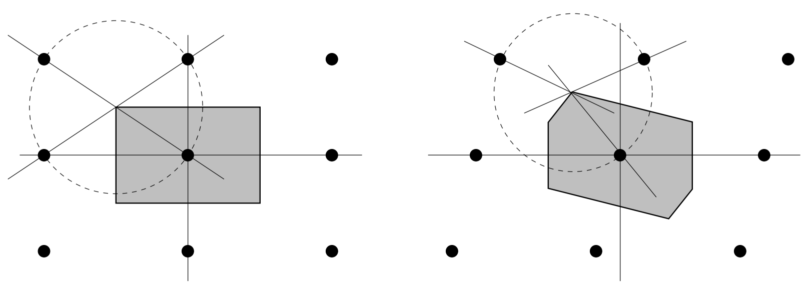
## 3. *Main theorem*

Our main result relates Theorem 3 to Theorems 1 and 2.

**Theorem 4.** *There is a homeomorphism* $\Phi : \Lambda/\mathbf{R}^+ \to \exp_3 S^1$ *which identifies the circle* $L_0$ *with* $\Delta(S^1)$.

*Remark*. The circle $S^1$ acts as a subgroup of $PSL(2,\mathbf{R})$ on $\Lambda$ rotating the lattices. It will be clear from the construction below that $\Phi$ sends the action of $S^1$ on $\Lambda/\mathbf{R}^+$ to the action of $SO(2)$ by rotations on $\exp_3 S^1$.

*Proof*. The *Voronoi cell*[1] $V(L)$ of a non-degenerate lattice $L$ is defined as the set of such $z \in \mathbf{C}$ that $|z| \leqslant |z - \omega|$ for all non-zero lattice points $\omega \in L$. The Voronoi cell of a rectangular lattice is a rectangle, for any other lattice it is a hexagon ([5]).



Let $v$ be a vertex of the polygon $V(L)$. The minimal distance between $v$ and points of $L$ is attained generically on three (in case of a rectangular lattice four)

[1] also known as the *Dirichlet region*

lattice points $\omega_i$. The set of lines connecting $v$ to these points consists of either 3 or 2 lines, as shown on the figure.

Consider the translation of $\mathbf{C}$ which sends the vertex $v$ to the origin. It sends the lines connecting $v$ to $\omega_i$ to lines passing through the origin. It is easy to see that the resulting set of lines only depends on $L$ and not on the choice of the vertex $v$.

Recall now that lines passing through the origin in $\mathbf{R}^2$ can be thought as points of a circle $S^1 = \mathbf{RP}^1$. Thus the above construction associates to every non-degenerate lattice $L$ a point $\phi(L)$ in $\exp_3 S^1$. For homothetic lattices $L, L'$ it is clear that $\phi(L) = \phi(L')$ so, in fact, $\phi$ descends to a map

$$\Phi : L_1 \to \exp_3 S^1.$$

It remains to define the map $\Phi$ on degenerate lattices. Each additive cyclic subgroup of $\mathbf{C}$ is contained in the unique line. The map $\Phi$ is defined on $L_0$ by sending a degenerate lattice $L$ to the one-point subset of $\mathbf{RP}^1$ that corresponds to the line *orthogonal* to the line containing $L$.

The continuity of $\Phi$ needs to be checked on $L_0$ and on the subspace of rectangular lattices. In both cases it is straightforward.

□

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