Bol. Soc. Mat. Mexicana (3) Vol. 6, 2000

ALGEBRAIC CYCLES AND ANTIHOLOMORPHIC INVOLUTIONS ON PROJECTIVE SPACES

Jacob Mostovoy

1. Introduction

The first step in the development of what is known now as "Lawson homology" was H. B. Lawson's proof of the algebraic version of the Thom isomorphism for a certain class of bundles [9] and, as a consequence, a complete description of the groups of algebraic cycles on $\mathbf{CP}^n$ from the topological point of view.

In this paper we describe two real counterparts of the results of [9]. Usually, the term "real variety" means "complex variety with an antiholomorphic involution on it". In concrete examples, however, reality comes in very different flavours. On the Riemann sphere, for example, there are two essentially different antiholomorphic involutions: complex conjugation and the antipodal map. The analogue of the antipodal map exists on any odd-dimensional complex projective space; the "reality" associated with it has a close relationship to quaternions.

Taking this into account, by "real algebraic cycles" in $\mathbf{CP}^n$ we will mean those invariant under complex conjugation. Cycles that are invariant under the generalised antipodal map will be called "quaternionic cycles".

Our main results are the calculations of the homotopy groups of the groups of real and quaternionic cycles on complex projective spaces. The real case has already been treated by T. K. Lam [8], who calculated the homotopy type of what he called "groups of mod 2 cycles". We complement his calculation by computing the homotopy type of the groups of cycles with integral coefficients. (In a recent work [10] by H. B. Lawson, P. Lima-Filho and M.-L. Michelsohn the authors also calculate the homotopy type of these groups. The results of [10] are more complete than ours as they include a description of the multiplicative structure on the homotopy of the groups of real cycles.)

As for quaternionic cycle groups, we determine their homotopy type only rationally. Stronger statements about groups of quaternionic cycles of even dimension were obtained by Lawson, Lima-Filho and Michelsohn in [11], but the rational description of odd-dimensional cycle groups seems to be the best available at the moment.

The paper is organised as follows. The next section contains some background material on symmetric products and complex cycle spaces. We need it mostly to establish the notation.

2000 *Mathematics Subject Classification*: 14C25, 14P99, 55P20.

In section 3 we determine the homotopy type of real cycle spaces and state some of the results of [8]. Section 4 contains the description of quaternionic cycle spaces. First we state some basic facts about the quaternionic involution; in §4.3 the calculation of the rational homotopy groups is given; §4.4 is a discussion of the groups of quaternionic curves. Finally, in section 5 we state the real and quaternionic versions of a theorem of Lawson and Michelsohn from [12].

There is an appendix which treats the homology of the quotient space of $\mathbf{CP}^n$ by complex conjugation. The results given there ought to be known but seem to be missing in the literature.

It is not surprising that proofs of many facts in this paper are similar to those of analogous statements for complex cycle spaces. In such a situation we often only sketch the argument or omit it altogether. This should present no difficulty for a reader familiar with [9] and [12].

Some of the results presented here were announced in [15].

2. Preliminaries: symmetric products and complex cycle spaces

(2.1) Symmetric products. Here we give a very brief overview of symmetric products and related groups, the main references are [3], [7], [14] and [16].

Let $SP^k(X)$ be the k-fold symmetric product of a topological space X. We assume X to be a countable connected CW-complex (this condition is introduced so that all the generalisations of the Dold-Thom Theorem that we list below, hold). We think of points of $SP^k(X)$ as of formal linear combinations of points of X with the sum of coefficients (which are non-negative integers) equal to k. So, for example, we use the convention that $SP^0(X)$ which is an "empty linear combination" is a one-point space $\{\emptyset\}$. The union $SP(X) = \coprod_{k\geq 0} SP^k(X)$ is a topological abelian monoid, the addition being defined by the formal addition of linear combinations of points.

There are several ways of defining a completion of $SP(X)$. If X comes with a basepoint, one can introduce infinite symmetric products $SP^\infty(X)$. We will use a different (but equivalent, at least in good cases) completion, which is just the Grothendieck group of $SP(X)$ with the topology induced by that of $SP(X)$. Our notation for this group will be $\mathbf{Z}[X]$. By $\mathbf{F}_2[X]$ we denote the topological vector space over $\mathbf{F}_2$ generated by points of X. There is an exact sequence, which is also a principal fibration

$$0 \to \mathbf{Z}[X] \xrightarrow{\times 2} \mathbf{Z}[X] \to \mathbf{F}_2[X] \to 0\,.$$

The homotopical structure of these groups is given by the Dold-Thom Theorem [3]:

$$\mathbf{Z}[X] \simeq \prod_{k\geq 0} K(H_k(X), k) \quad \text{and} \quad \mathbf{F}_2[X] \simeq \prod_{k\geq 0} K(H_k(X, \mathbf{F}_2), k)\,,$$

where $K(G, k)$ are Eilenberg-Mac Lane spaces.

We will also consider topological groups $\mathbf{Q}[X]$ which we define as direct limits of sequences

$$\mathbf{Z}[X] \xrightarrow{\times 2} \mathbf{Z}[X] \xrightarrow{\times 3} \ldots \xrightarrow{\times n} \mathbf{Z}[X] \xrightarrow{\times n+1} \ldots$$

Points of $\mathbf{Q}[X]$ can be thought of as linear combinations of points of X with rational coefficients, this justifies the notation. Clearly, $\pi_*\mathbf{Q}[X] = \pi_*(\mathbf{Z}[X]) \otimes \mathbf{Q}$; so the Dold-Thom Theorem says that

$$\mathbf{Q}[X] \simeq \prod_{k \geq 0} K(H_k(X, \mathbf{Q}), k)\,.$$

Finally, if $Y \subset X$ is a subcomplex, the relative groups $\mathbf{Z}[X; Y]$ are defined as quotients

$$0 \to \mathbf{Z}[Y] \to \mathbf{Z}[X] \to \mathbf{Z}[X; Y] \to 0\,.$$

The group $\mathbf{Z}[X; Y]$ is isomorphic as a topological group to the connected component of $\mathbf{Z}[X/Y]$ which contains 0; this we can also write as $\mathbf{Z}[X/Y; *]$ where $*$ is a point (which should be thought of as the image of Y in X/Y). Similarly one can define relative groups with $\mathbf{F}_2$ and rational coefficients. The homotopy groups of these spaces are naturally isomorphic to the homology of the pair (X, Y).

More generally, if M is a free compactly graded abelian monoid and $\widetilde{M}$ is its group completion we will use the notation $\widetilde{M} \otimes \mathbf{F}_2$ to denote the quotient in the exact sequence

$$0 \to \widetilde{M} \xrightarrow{\times 2} \widetilde{M} \to \widetilde{M} \otimes \mathbf{F}_2 \to 0\,.$$

By a theorem of Lima-Filho [14] this sequence will always be a principal fibration. For example, $\mathbf{Z}[X] \otimes \mathbf{F}_2$ is the same thing as $\mathbf{F}_2[X]$. Sometimes we will refer to this fibration as "coefficient fibration".

By $M \otimes \mathbf{Q}$ we will mean the direct limit of the sequence $M \xrightarrow{\times 2} M \xrightarrow{\times 3} \ldots$. Thus $\mathbf{Z}[X] \otimes \mathbf{Q}$ coincides with $\mathbf{Q}[X]$. All our groups will be groups of cycles so this notation is natural. We will use as obvious the fact that $\pi_* M \otimes \mathbf{Q} = \pi_*(M \otimes \mathbf{Q})$.

Finally, let us recall the description of the homology transfer map in terms of symmetric products.

Consider a "nice" action of a finite group G on a connected topological space X. (For example, we may assume that X is a geometric realisation of a countable simplicial set and that G acts simplicially.) Define the transfer map

$$\mathrm{Tr} : \mathbf{Z}[X/G] \to \mathbf{Z}[X]$$

to be the homomorphism which sends a point $x \in X/G \subset \mathbf{Z}[X/G]$ to $\sum_{g \in G} gx'$, where x' is some inverse image of x under the projection $X \to X/G$. There is an induced map

$$\mathrm{Tr}_* : H_*(X/G) = \pi_*\mathbf{Z}[X/G] \to \pi_*\mathbf{Z}[X] = H_*(X)\,.$$

This map coincides with the usual homology transfer.

The action of G on X induces an action of G on $\mathbf{Z}[X]$. Denote by $\mathbf{Z}[X]^G$ the fixed set of this action. The map $\mathrm{Tr} : \mathbf{Z}[X/G] \to \mathbf{Z}[X]^G$ becomes a homeomorphism after tensoring with $\mathbf{Q}$, so $\pi_*(\mathbf{Q}[X]^G)$ can be identified with $H_*(X/G, \mathbf{Q})$.

However, it is a standard result [4] that the transfer map of homology with rational coefficients is an isomorphism of $H_*(X/G, \mathbf{Q})$ onto the G-fixed subgroup of $H_*(X, \mathbf{Q})$.

Thus we have the following

PROPOSITION (2.2). *The inclusion map $\mathbf{Z}[X]^G \hookrightarrow \mathbf{Z}[X]$ induces an isomorphism between $\pi_*(\mathbf{Z}[X]^G) \otimes \mathbf{Q} = \pi_*(\mathbf{Q}[X]^G)$ and the G-fixed part of $\pi_*\mathbf{Q}[X] = H_*(X, \mathbf{Q})$.*

We will need to consider group quotients, topological quotients and orbit spaces. With the exclusion of "$\mathbf{Z}/4$", the notation "/" we reserve for the latter two. The notation for the cyclic groups of 2 and 4 elements is $\mathbf{F}_2$ and $\mathbf{Z}/4$ respectively; this apparent inconsistency is due to typographical reasons.

(2.3) Complex cycle spaces. All definitions and results of this section can be found in [9].

For X a projective subvariety of $\mathbf{CP}^n$ denote by $C_{p,d}(X)$ the Chow variety of p-dimensional effective algebraic cycles in X which have degree d in $\mathbf{CP}^n$. By convention, $C_{p,0}(X)$ is a point $\{\emptyset\}$. The union $\coprod_{k\geq 0} C_{p,k}(X)$ is an abelian monoid with respect to formal addition of cycles, so we can consider its Grothendieck group $\widetilde{\mathcal{C}}_p(X)$. The topology on $\widetilde{\mathcal{C}}_p(X)$ is induced by the natural topology of the Chow varieties considered as complex projective algebraic varieties.

If $X \subset \mathbf{CP}^n$ and $Y \subset \mathbf{CP}^m$ choose some linear inclusions of $\mathbf{CP}^n$ and $\mathbf{CP}^m$ into $\mathbf{CP}^{n+m+1}$ whose images are disjoint and consider the union of all lines which connect X with Y inside $\mathbf{CP}^{n+m+1}$. The subset thus obtained will be referred to as the *ruled join* of X and Y. The ruled join of X with a point is denoted by $\Sigma\!\!\!/ X$. Obviously, $\Sigma\!\!\!/ \mathbf{CP}^n = \mathbf{CP}^{n+1}$.

The ruled join with a point induces a continuous homomorphism $\Sigma\!\!\!/$ from $\widetilde{\mathcal{C}}_p(X)$ to $\widetilde{\mathcal{C}}_{p+1}(\Sigma\!\!\!/ X)$. The main result of [9] is the following theorem:

THEOREM (2.4). *If X is a closed subvariety of $\mathbf{CP}^n$, the maps*

$$\Sigma\!\!\!/ : \widetilde{\mathcal{C}}_p(X) \to \widetilde{\mathcal{C}}_{p+1}(\Sigma\!\!\!/ X)$$

are homotopy equivalences for all $p \geq 0$.

The group of 0-cycles on $\mathbf{CP}^n$ is just the free abelian topological group $\mathbf{Z}[\mathbf{CP}^n]$. Applying the Dold-Thom theorem one obtains

COROLLARY (2.5).

$$\widetilde{\mathcal{C}}_p(\mathbf{CP}^n) \simeq \prod_{i=0}^{n-p} K(\mathbf{Z}, 2i)\,.$$

Remarks. As in the case of symmetric products, there is more than one way to choose the group completion for the monoid of effective cycles. However, according to [6] and [14] all these methods give the same result. We will always use the "naïve" group completion, i.e. the Grothendieck group.

Since the paper [9] there have been various developments in the field. In particular, Lawson's results were generalised by E.Friedlander in [F] to the l-adic case.

3. Real cycles

(3.1) Definitions and $\mod 2$ **case.** Complex conjugation on $\mathbf{C}^{n+1}$ induces an antiholomorphic involution on $\mathbf{CP}^n$ which we denote by t. By the same letter t we will also denote the corresponding involution on the space of homogeneous polynomials in $n+1$ variables, which sends $P(z_i)$ to $\overline{P(\bar{z}_i)}$.

A subvariety of $\mathbf{CP}^n$ is called *real* if the ideal which defines it, is fixed by t; a *real* cycle is a formal sum of real irreducible subvarieties of the same dimension. The ruled join of a real cycle with a real point is again a real cycle; this makes it possible to adapt the methods of [9] to the study of real cycle spaces. The definitions and results below can be found in [8].

For a real variety X denote by $\widetilde{\mathcal{R}C}_p(X)$ the subgroup of real cycles in $\widetilde{\mathcal{C}}_p(X)$. The subgroup of $\widetilde{\mathcal{R}C}_p(X)$ which consists of the cycles of the form $c+t(c), c \in \widetilde{\mathcal{C}}_p(X)$ is denoted by $\widetilde{\mathcal{D}C}_p(X)$. The quotient group of $\widetilde{\mathcal{R}C}_p(X)$ by $\widetilde{\mathcal{D}C}_p(X)$ is denoted by $\widetilde{\mathcal{E}}_p(X)$. Following Lam, we refer to this group as to the "space of real cycles modulo 2"; this is justified for groups of 0-cycles, but in higher dimensions may be somewhat misleading.

THEOREM (3.2). [8] *If X is a real subvariety of $\mathbf{CP}^n$, the maps*

$$\not\Sigma : \widetilde{\mathcal{R}C}_p(X) \to \widetilde{\mathcal{R}C}_{p+1}(\not\Sigma X)\,,$$

$$\not\Sigma : \widetilde{\mathcal{D}C}_p(X) \to \widetilde{\mathcal{D}C}_{p+1}(\not\Sigma X)\,,$$

$$\not\Sigma : \widetilde{\mathcal{E}}_p(X) \to \widetilde{\mathcal{E}}_{p+1}(\not\Sigma X)$$

are homotopy equivalences for all $p \geq 0$.

The spaces of cycles of dimension 0 have the following description ([8]):

$$\widetilde{\mathcal{D}C}_0(X) = \mathbf{Z}[X/t],$$

$$\widetilde{\mathcal{E}}_0(X) = \mathbf{F}_2[X_{\mathbf{R}}]\,.$$

where $X_{\mathbf{R}}$ is the real locus of X. The space $\widetilde{\mathcal{R}C}_0(X)$ does not have such a simple description in terms of abelian groups generated by topological spaces. We will consider it in detail in the next section.

Combining Theorem (3.2) with the Dold-Thom Theorem, we can describe the spaces of real cycles modulo 2 on $\mathbf{CP}^n$:

COROLLARY (3.3). [8]

$$\widetilde{\mathcal{E}}_p(\mathbf{CP}^n) \simeq \prod_{i=0}^{n-p} K(\mathbf{F}_2, i)\,.$$

Remark. The topology of the groups of 0-cycles modulo 2 on a variety X is determined completely by the real locus $X_{\mathbf{R}}$. In general, there is no reason to expect such behaviour for cycles of higher dimension unless $X \subset \mathbf{CP}^n$ is irreducible and $\dim_{\mathbf{R}} X_{\mathbf{R}} = \dim_{\mathbf{C}} X$, in which case X itself is uniquely determined by $X_{\mathbf{R}}$.

(3.4) Real cycles with integral coefficients. Here we calculate the homotopy groups of $\widetilde{\mathcal{R}C}_p(\mathbf{CP}^n)$; by virtue of Theorem (3.2) it is enough to consider 0-cycles.

First we describe the homotopy of $\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n)$ rationally. As $\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n)$ is the t-invariant subgroup of $\mathbf{Z}[\mathbf{CP}^n]$ its rational homotopy groups coincide with the rational homology of $\mathbf{CP}^n/t$ (see Proposition (2.2)). The rational and mod 2 homology of $\mathbf{CP}^n/t$ are calculated in Appendix A. The next result follows directly from Proposition (A.6).

THEOREM (3.5).

$$\pi_i \widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \otimes \mathbf{Q} = \mathbf{Q} \quad \textit{for } i = 4k \le 2n \textit{ and 0 otherwise}.$$

For the corresponding spaces with coefficients in $\mathbf{F}_2$ we can define a map

$$I : \widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \otimes \mathbf{F}_2 \to \mathbf{F}_2[\mathbf{CP}^n/t; \mathbf{RP}^n] \times \mathbf{F}_2[\mathbf{RP}^n]$$

as follows: write a cycle $a \in \widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \otimes \mathbf{F}_2$ in the form $a = r + c + t(c)$, where r lies in $\mathbf{RP}^n$ and c is disjoint from $\mathbf{RP}^n$. Then r can be considered as a point in $\mathbf{F}_2[\mathbf{RP}^n]$ and c - as a point in $\mathbf{F}_2[\mathbf{CP}^n/t; \mathbf{RP}^n]$. The map I is a homeomorphism as it is continuous and has a continuous inverse. Hence, for $i \ge 0$

$$\pi_i(\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \otimes \mathbf{F}_2) = H_i(\mathbf{CP}^n/t, \mathbf{RP}^n, \mathbf{F}_2) \oplus H_i(\mathbf{RP}^n, \mathbf{F}_2).$$

From Proposition (A.3) it follows that

$$\mathrm{rk}(\pi_k(\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \otimes \mathbf{F}_2)) = \begin{cases} [\frac{1}{2}k] + 1 & \text{for } k \le n, \\ [\frac{1}{2}k] + 1 - (k - n) & \text{for } n < k \le 2n, \end{cases}$$

where $[x]$ denotes the integer part of x.

Finally, we have a "coefficient fibration"

$$\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \overset{\times 2}{\to} \widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \to \widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \otimes \mathbf{F}_2.$$

If we know $\pi_* \widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \otimes \mathbf{Q}$ and $\pi_*(\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \otimes \mathbf{F}_2)$, then from the exact sequence of the coefficient fibration we can determine the homotopy groups of $\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n)$, given that these groups contain no torsion apart from $\mathbf{F}_2$ - torsion, i.e. apart from direct sums of copies of $\mathbf{F}_2$. This is established by the following lemma:

LEMMA (3.6). *Let X be a real projective variety. Then if $H_*(X)$ is torsion-free, $\pi_*\widetilde{\mathcal{R}C}_0(X)$ does not contain any torsion apart from $\mathbf{F}_2$ -torsion.*

Proof. Let $S^-(X)$ be the subgroup of $\mathbf{Z}[X]$, consisting of all c such that $t(c) = -c$. There is a fibration

$$\widetilde{\mathcal{R}C}_0(X) \times S^-(X) \to \mathbf{Z}[X] \to \mathbf{F}_2[(X/t)/X_{\mathbf{R}}]\,.$$

As $H_*(X)$ is torsion-free and $H_*((X/t)/X_{\mathbf{R}}, \mathbf{F}_2)$ contains only $\mathbf{F}_2$-torsion, the statement of the lemma follows from the exact sequence of this fibration. □

Calculations which use the homotopy exact sequence of the coefficient fibration now give us

THEOREM (3.7). *The free part of $\pi_k\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n)$ is isomorphic to $\mathbf{Z}$ when k is divisible by 4 and $k \leq 2n$; and is equal to 0 otherwise. The torsion is a direct sum of $a(k)$ copies of $\mathbf{F}_2$, where*

$$a(k) = \begin{cases} [\frac{k+3}{4}] & \text{for } 0 \leq k \leq n, \\ [\frac{2n-k+2}{4}] & \text{for } n < k \leq 2n,\ n \text{ even}, \\ [\frac{2n-k}{4}] + \frac{1}{2}(1-(-1)^k) & \text{for } n < k \leq 2n,\ n \text{ odd}, \\ 0 & \text{otherwise}; \end{cases}$$

The general formulae for the ranks of the torsion in $\pi_k\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n)$ are not too pleasant. Notice, however, that the rank of torsion in $\pi_k\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n)$ does not depend on n when $k \leq n$. This suggests that there is a simpler "stable" version of Theorem (3.7). One can indeed speak of stability here:

LEMMA (3.8). *The natural inclusion*

$$\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \hookrightarrow \widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1})$$

induces isomorphisms in homotopy groups of dimensions $\leq n$.

Proof. Let $\widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1}, \mathbf{CP}^n)$ be the group quotient of $\widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1})$ by the subgroup $\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n)$; by Lima-Filho's Theorem [14] there is a fibration

$$\widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \hookrightarrow \widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1}) \to \widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1}, \mathbf{CP}^n)\,.$$

The space $\widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1}, \mathbf{CP}^n)$ can be considered as a space of "relative" real 0-cycles of degree zero on $\mathbf{CP}^{n+1}/\mathbf{CP}^n = S^{2n+2}$. In particular, we have a fibration

$$\mathbf{Z}[S^{2n+2}/t; *] \to \widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1}, \mathbf{CP}^n) \to \mathbf{F}_2[S^{n+1}; *]\,,$$

where the first map is a transfer map and S^{n+1} should be thought of as $\mathbf{RP}^{n+1}/\mathbf{RP}^n$.

By Proposition (A.4) $\widetilde{H}_i(S^{2n+2}/t) = 0$ for $i \leq n$ so $\pi_i \widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1}, \mathbf{CP}^n) = 0$ for $i \leq n$ as well. This implies that

$$\pi_i \widetilde{\mathcal{R}C}_0(\mathbf{CP}^n) \to \pi_i \widetilde{\mathcal{R}C}_0(\mathbf{CP}^{n+1})$$

is an isomorphism for $i < n$ and an epimorphism for $i = n$. The latter turns out to be an isomorphism by Theorem (3.7). $\square$

Remark. Groups of cycles on projective spaces carry a multiplicative structure on the homotopy groups [7], and it is convenient to describe the cycle spaces in terms of this ring structure. Such description of complex cycles can be found in [7], of real mod 2 cycles - in [8]. The multiplicative structure on the homotopy groups of the spaces of real cycles with integer coefficients is computed in [10].

4. Spaces of quaternionic cycles

(4.1) Action of the quaternion j on $\mathbf{CP}^{2n+1}$. The antipodal map on $\mathbf{CP}^1$ is given in homogeneous coordinates by

$$(z_0, z_1) \to (-\bar{z}_1, \bar{z}_0)\,.$$

It has a natural generalisation to $\mathbf{CP}^{2n+1}$:

$$j : (z_0, z_1, z_2, z_3, \dots) \to (-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2, \dots)\,.$$

In fact, if $\mathbf{CP}^{2n+1}$ is considered as the complex projectivisation of an $(n+1)$-dimensional quaternion vector space $\mathbf{H}^{n+1}$ the involution j is induced by the left multiplication by the quaternion j.

The involution j is fixed point-free. It reverses the orientation of $\mathbf{CP}^{2n+1}$, so the quotient space $\mathbf{CP}^{2n+1}/j$ is a non-orientable manifold. Notice that j induces involutions on the fibres of the map

$$\mathbf{CP}^{2n+1} \to \mathbf{HP}^n\,.$$

This means that there is a fibration over the quaternionic projective space

$$\mathbf{RP}^2 \to \mathbf{CP}^{2n+1}/j \to \mathbf{HP}^n\,.$$

The spectral sequence of this fibration collapses at the term E^2, and thus the homology groups of $\mathbf{CP}^{2n+1}/j$ are as follows:

$$H_i(\mathbf{CP}^{2n+1}/j) = \begin{cases} \mathbf{Z} & \text{when } i \equiv 0 \ (mod\ 4) \quad i \leq 4n+2, \\ \mathbf{F}_2 & \text{when } i \equiv 1 \ (mod\ 4) \quad i \leq 4n+2, \\ 0 & \text{otherwise;} \end{cases}$$

The rational homology groups are equal to $\mathbf{Q}$ in dimensions $4i$ for $i \leq n$ and 0 otherwise. Notice also that the natural inclusions $\mathbf{CP}^{2n+1}/j \hookrightarrow \mathbf{CP}^{2n+3}/j$ induce monomorphisms on homology.

Remark. As $\mathbf{CP}^{\infty}/j$ has $\mathbf{CP}^{\infty}$ as double cover, only two of its homotopy groups are non-trivial: $\pi_1 = \mathbf{F}_2$ and $\pi_2 = \mathbf{Z}$. The homology groups tell us that $\mathbf{CP}^{\infty}/j$ does not split as a product of Eilenberg-Mac Lane spaces.

A homogeneous polynomial P of $2n+2$ variables is called j-invariant if $P(z_i) = \overline{P(j(z_i))}$. We denote the graded ring of j-invariant polynomials by $\mathbf{C}[z_0, \ldots, z_{2n+1}]^j$. Here the letter j is used to denote the generator of the action of $\mathbf{Z}/4$ on the space of polynomials which sends $P(z_i)$ to $\overline{P(j(z_i))}$.

Proposition (4.2). *The real vector space $\mathbf{C}[z_0, \ldots, z_{2n+1}]_d^j$ of j-invariant polynomials of degree d has dimension 0 if d is odd and $\binom{2n+1+d}{d}$ if d is even.*

The map $j : P(z_i) \to \overline{P(j(z_i))}$ is real linear, so to describe it completely it is enough to consider the action of j on monomials. A direct calculation then verifies the statement.

(4.3) Quaternionic subvarieties and cycles. We say that X is a quaternionic (or j-invariant) subvariety of $\mathbf{CP}^{2n+1}$ if its ideal is fixed by the action of j on the space of polynomials. If $X \subset \mathbf{CP}^{2n+1}$ is quaternionic, then j induces involutions on the cycle spaces:

$$j : \mathcal{C}_{p,d}(X) \to \mathcal{C}_{p,d}(X)\,.$$

The fixed subspaces are called the spaces of quaternionic cycles; we denote them by $\mathcal{Q}C_{p,d}(X)$.

In a certain sense, quaternionic varieties and cycles are a particular case of real varieties and cycles. Indeed, define a map

$$v : \mathbf{CP}^{2n+1} \to \mathbf{CP}^{(n+1)(2n+3)-1}$$

as follows: choose a basis $P_i,\ 1 \leq i \leq (n+1)(2n+3)$ for the vector space $\mathbf{C}[z_0, \ldots, z_{2n+1}]_2^j$ of j-invariant polynomials of degree 2; then v sends a point $z \in \mathbf{CP}^{2n+1}$ to the point $(P_1(z), \ldots, P_{(n+1)(2n+3)}(z))$. It is clearly a regular embedding (in fact, it is essentially the Veronese embedding). The following diagram is commutative:

$$\begin{array}{ccc} \mathbf{CP}^{2n+1} & \hookrightarrow & \mathbf{CP}^{2n^2+5n+2} \\ \downarrow j & & \downarrow t \\ \mathbf{CP}^{2n+1} & \hookrightarrow & \mathbf{CP}^{2n^2+5n+2} \end{array}$$

This means that any j-invariant subset of $\mathbf{CP}^{2n+1}$ can be embedded into $\mathbf{CP}^{2n^2+5n+2}$ as a real subset.

The simplest examples of quaternionic subvarieties are j-invariant linear subspaces. If a j-invariant linear subspace satisfies $F(z) = 0$, where F is linear, then it also satisfies $F(j(z)) = 0$. These two equations can be written as a single linear equation in quaternions; hence we get

PROPOSITION (4.4). *All j-invariant linear subspaces of $\mathbf{CP}^{2n+1}$ are inverse images of linear subspaces in $\mathbf{HP}^n$ under the map $\mathbf{CP}^{2n+1} \to \mathbf{HP}^n$. In particular, j-invariant complex lines are just the fibres of this map.*

A corollary of this is that all j-invariant linear subspaces are odd-dimensional. More generally, the following is true:

PROPOSITION (4.5). *The degree of an even-dimensional quaternionic cycle is even.*

The geometric reason of this is rather transparent. As j acts freely on $\mathbf{CP}^{2n+1}$, every 0-dimensional quaternionic cycle can be written as a sum $c + j(c)$; hence, its degree is even. Now, for any even-dimensional quaternionic subvariety there exists a j-invariant linear subspace of complementary dimension, which intersects it at a finite number of points. (This is obvious for a quaternionic hypersurface; the case of an arbitrary even-dimensional

subvariety can be reduced to that of a hypersurface by projecting away from j-invariant lines.) The intersection with such a subspace is a j-invariant 0-dimensional cycle whose degree is even, and this proves the proposition.

Alternatively, one can use projections away from j-invariant subspaces to reduce the statement of Proposition (4.5) to the case of divisors; we already know that j-invariant divisors are all of even degree.

There are two basic examples of quaternionic cycle spaces: 0-cycles and divisors on $\mathbf{CP}^{2n+1}$. For 0-cycles we have

$$\mathcal{Q}C_{0,d}(X) = SP^d(X/j)\,.$$

Spaces of divisors are described by Proposition (4.2):

$$\mathcal{Q}C_{2n,d}(\mathbf{CP}^{2n+1}) = \mathbf{RP}^D, \quad \text{where } D = \binom{2n+1+d}{d} - 1$$

for d even and

$$\mathcal{Q}C_{2n,d}(\mathbf{CP}^{2n+1}) = \emptyset$$

for d odd. When $n = 0$, divisors and 0-cycles are the same thing, so we recover the "topological Maxwell theorem" (see [Ar2]):

$$SP^n(\mathbf{RP}^2) = \mathbf{RP}^{2n}\,.$$

For a j-invariant subvariety X the topological groups of quaternionic cycles $\widetilde{\mathcal{Q}C}_p(X)$ are defined as the Grothendieck groups of the abelian monoids

$\{\emptyset\}\bigcup_d \mathcal{Q}C_{p,d}(X)$, where the addition is just the formal addition of cycles. Clearly, $\widetilde{\mathcal{Q}C}_p(X)$ is just the j-fixed subgroup of $\widetilde{\mathcal{C}}_p(X)$, so it inherits its topology from $\widetilde{\mathcal{C}}_p(X)$.

(4.6) Quaternionic cycles on $\mathbf{CP}^{2n+1}$. The ruled join with a fixed j-invariant complex line (we denote it by $\#\!\!\!/$) carries quaternionic cycles on $X \subset \mathbf{CP}^{2n+1}$ to quaternionic cycles on $\#\!\!\!/\,\mathbf{CP}^{2n+1} = \mathbf{CP}^{2n+3}$. This gives maps

$$\#\!\!\!/ : \widetilde{\mathcal{Q}C}_p(X) \to \widetilde{\mathcal{Q}C}_{p+2}(\#\!\!\!/ X)\,.$$

To show that these maps are homotopy equivalences one could try to adapt Lawson's proof of Theorem (2.4); this approach was taken in [11]. However, we will work with spaces of cycles with rational coefficients and for our purposes it will be sufficient to use a weaker result of [13], which we state below.

If a finite group G acts on $\mathbf{C}^{n+1}$ linearly or antilinearly, there is an induced action of G on $\mathbf{CP}^n$ by linear and antilinear automorphisms and a corresponding action of G on the cycle spaces. Extend the action of G on $\mathbf{C}^{n+2}$ by complex conjugation on the last coordinate in case of an antilinear automorphism or trivially in case of a linear automorphism; there are corresponding extensions of the induced actions on $\mathbf{CP}^{n+1}$ and the cycle spaces. Let X be a G-invariant subvariety of $\mathbf{CP}^n$; denote by $\widetilde{\mathcal{C}}_p(X)^G \otimes \mathbf{Q}$ the space of G-invariant cycles with rational coefficients on X.

It is clear that if a cycle c on $\mathbf{CP}^n$ is G-invariant, then $\Sigma\!\!\!\!/\, c \subset \mathbf{CP}^{n+1}$ is also G-invariant.

Theorem (4.7). *For each p the maps*

$$\Sigma\!\!\!\!/ : \widetilde{\mathcal{C}}_p(X)^G \otimes \mathbf{Q} \to \widetilde{\mathcal{C}}_{p+1}(\Sigma\!\!\!\!/ X)^G \otimes \mathbf{Q}$$

are homotopy equivalences.

Remark. In fact, the statement of the above theorem in [13] is
slightly different. In particular, Lawson and Michelsohn considered in [13] only *linear* actions of G. However, the antilinear case comes essentially for free; the proof is unchanged.

Here we prove the following statement:

Theorem (4.8). *If X is a quaternionic subvariety of $\mathbf{CP}^{2n+1}$, the maps*

$$\#\!\!\!/ : \widetilde{\mathcal{Q}C}_p(X) \otimes \mathbf{Q} \to \widetilde{\mathcal{Q}C}_{p+2}(\#\!\!\!/ X) \otimes \mathbf{Q}$$

are homotopy equivalences for all p.

Proof. Extend the action of $\mathbf{F}_2$ by j on $\mathbf{CP}^{2n+1}$ to an action of $\mathbf{Z}/4$ on $\mathbf{CP}^{2n+3}$:

$$j_2 : (z, z_{2n+2}, z_{2n+3}) \to (jz, \bar{z}_{2n+2}, \bar{z}_{2n+3})\,.$$

Applying Theorem (4.7) twice we get a homotopy equivalence

$$\widetilde{\mathcal{Q}C}_p(X) \otimes \mathbf{Q} \to \widetilde{\mathcal{C}}_{p+2}(\not\# X)^{\mathbf{Z}/4} \otimes \mathbf{Q}\,.$$

The cycle space on the right is invariant under j_2, which changes the last two coordinates only by conjugation. Now our purpose is to show that $\widetilde{\mathcal{C}}_{p+2}(\not\# X)^{\mathbf{Z}/4} \otimes \mathbf{Q}$ is homotopy equivalent to $\widetilde{\mathcal{Q}C}_{p+2}(\not\# X) \otimes \mathbf{Q}$.

Define a map

$$T : \mathbf{CP}^{2n+3} \to \mathbf{CP}^{2n+3}$$

by

$$T : (z_0, \dots , z_{2n+1}, z_{2n+2}, z_{2n+3}) \to (z_0, \dots , z_{2n+1}, -z_{2n+3}, z_{2n+2})\,.$$

The automorphism T has order 4. It can be checked that

$$j_2 T = T j_2 = j \ \text{ and } jT^3 = T^3 j = j_2\,.$$

If c is a j-invariant cycle, then it is easy to see from the properties of T, that $c + Tc + T^2c + T^3c$ is a j_2-invariant cycle and if c is j_2-invariant then $c + Tc + T^2c + T^3c$ is j-invariant. This means that we have maps

$$P_1 : \widetilde{\mathcal{Q}C}_{p+2}(\not\# X) \otimes \mathbf{Q} \xrightarrow{1+T+T^2+T^3} \widetilde{\mathcal{C}}_{p+2}(\not\# X)^{j_2} \otimes \mathbf{Q}$$

and

$$P_2 : \widetilde{\mathcal{C}}_{p+2}(\not\# X)^{j_2} \otimes \mathbf{Q} \xrightarrow{1+T+T^2+T^3} \widetilde{\mathcal{Q}C}_{p+2}(\not\# X) \otimes \mathbf{Q}\,.$$

These maps establish the desired homotopy equivalence. To see this, notice that

$$P_1 P_2 = (1 + T + T^2 + T^3)^2$$

induces multiplication by 16 in $\pi_* \widetilde{\mathcal{Q}C}_{p+2}(\not\# X) \otimes \mathbf{Q}$, as T is homotopic to the identity through linear maps

$$T_t : (\dots , z_{2n+2}, z_{2n+3}) \to (\dots , z_{2n+2} \cos t - z_{2n+3} \sin t, z_{2n+2} \sin t + z_{2n+3} \cos t)\,,$$

which commute with j. (Here $T_0 = \mathrm{Id}$, $T_{\pi/2} = T$.) But in a rational vector space multiplication by 16 is an isomorphism, hence P_1P_2 is a homotopy equivalence. The same is true for P_2P_1, so P_1 and P_2 are homotopy equivalences as well.

The map

$$\widetilde{\mathcal{Q}C}_p(X) \otimes \mathbf{Q} \to \widetilde{\mathcal{C}}_{p+2}(\not\# X)^{j_2} \otimes \mathbf{Q} \xrightarrow{P_2} \widetilde{\mathcal{Q}C}_{p+2}(\not\# X) \otimes \mathbf{Q}$$

is a composition of homotopy equivalences; this means that

$$\not\# : \widetilde{\mathcal{Q}C}_p(X) \otimes \mathbf{Q} \to \widetilde{\mathcal{Q}C}_{p+2}(\not\# X) \otimes \mathbf{Q}$$

is also a homotopy equivalence. □

Theorem (4.8) allows us to describe the spaces of quaternionic cycles on $\mathbf{CP}^{2n+1}$ rationally:

THEOREM (4.9).

$$\widetilde{\mathcal{Q}C}_{2p}(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q} \simeq \prod_{i=0}^{n-p} K(\mathbf{Q}, 4i)$$

and

$$\widetilde{\mathcal{Q}C}_{2p+1}(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q} \simeq \prod_{i=0}^{n-p} K(\mathbf{Q}, 4i)\,.$$

Proof. The statement for $\widetilde{\mathcal{Q}C}_{2p}(\mathbf{CP}^{2n+1})$ is a mere corollary of the Dold-Thom theorem. It is less trivial for $\widetilde{\mathcal{Q}C}_{2p+1}(\mathbf{CP}^{2n+1})$, as all we can get from Theorem (4.8) is

$$\widetilde{\mathcal{Q}C}_{2p+1}(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q} \simeq \widetilde{\mathcal{Q}C}_{1}(\mathbf{CP}^{2(n-p)+1}) \otimes \mathbf{Q}\,.$$

To describe the space $\widetilde{\mathcal{Q}C}_1(\mathbf{CP}^{2k+1}) \otimes \mathbf{Q}$ we can use the same averaging trick as in the proof of Theorem (4.8). Namely, the space $\widetilde{\mathcal{Q}C}_1(\mathbf{CP}^{2k+1}) \otimes \mathbf{Q}$ is homotopy equivalent to the space $\widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k})^{\mathbf{Z}/4} \otimes \mathbf{Q}$, where the generator of $\mathbf{Z}/4$ acts on $\mathbf{CP}^{2k}$ by sending (z, z_{2k}) to $(jz, \bar{z}_{2k})$.

To calculate the homotopy groups of $\widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k})^{\mathbf{Z}/4} \otimes \mathbf{Q}$ notice that there is an inclusion map

$$\widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k})^{\mathbf{Z}/4} \otimes \mathbf{Q} \hookrightarrow \widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k}) \otimes \mathbf{Q}\,,$$

which, by Proposition (2.2) induces an isomorphism from the homotopy groups of $\widetilde{\mathcal{C}}_0(\mathbf{CP}^{2k})^{\mathbf{Z}/4} \otimes \mathbf{Q}$ onto the $\mathbf{Z}/4$-fixed part of the rational homology of $\mathbf{CP}^{2k}$.

Choosing appropriate representatives for the homology classes, one can easily see that $4k$-dimensional classes in $H_*(\mathbf{CP}^{2k})$ are fixed by the action of $\mathbf{Z}/4$ and $4k+2$-dimensional classes change sign under the action of a generator. So

$$H_i(\mathbf{CP}^{2k}/(\mathbf{Z}/4), \mathbf{Q}) = \mathbf{Q} \quad \text{ if } \; i = 4m, i \leq 4k$$

and 0 otherwise. □

Remark. According to the Theorem (4.9), $\pi_0 \widetilde{\mathcal{Q}C}_p(\mathbf{CP}^{2n+1}) \otimes \mathbf{Q} = \mathbf{Q}$. One can check that the spaces $\mathcal{Q}C_{p,d}(\mathbf{CP}^{2n+1})$ are connected, the proof of this is similar to the proof of the analogous statement in [9] for complex Chow varieties. This implies that $\pi_0 \widetilde{\mathcal{Q}C}_p(\mathbf{CP}^{2n+1}) = \mathbf{Z}$; the components are indexed by the degree of cycles for odd p and by half the degree for even p.

(4.10) More on quaternionic 1-cycles. As we have seen, there are two essentially different types of quaternionic cycles: cycles of even and odd dimension. The "basic" case of a space of even-dimensional cycles is the case of 0-cycles. The space of 0-cycles on a quaternionic variety X is just the abelian group generated by the points of X/j; such groups are well-understood. The "basic" case of odd-dimensional cycles seems to be more obscure: even though we have already calculated the rational homotopy type of the spaces of quaternionic 1-cycles on $\mathbf{CP}^{2n+1}$, it is still desirable to get a better understanding of their structure. The following construction provides natural geometric representatives for the rational homotopy of $\widetilde{\mathcal{Q}C}_1(\mathbf{CP}^{2n+1})$.

There is an inclusion map

$$\mathbf{HP}^n = \mathcal{Q}C_{1,1}(\mathbf{CP}^{2n+1}) \hookrightarrow \widetilde{\mathcal{Q}C}_1(\mathbf{CP}^{2n+1})\,,$$

as every point of $\mathbf{HP}^n$ corresponds to a j-invariant line in $\mathbf{CP}^{2n+1}$. This map can be extended linearly to give a continuous homomorphism

$$h : \mathbf{Z}[\mathbf{HP}^n] \hookrightarrow \widetilde{\mathcal{Q}C}_1(\mathbf{CP}^{2n+1})\,.$$

Theorem (4.11). *The map h induces isomorphisms between $\pi_k\mathbf{Z}[\mathbf{HP}^n]$ and the quotient of $\pi_k\widetilde{\mathcal{Q}C}_1(\mathbf{CP}^{2n+1})$ by torsion for all $k > 0$.*

Proof. Consider the composite map

$$\mathbf{Z}[\mathbf{HP}^n] \xrightarrow{h} \widetilde{\mathcal{Q}C}_1(\mathbf{CP}^{2n+1}) \xrightarrow{i'} \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}),$$

where i' is the inclusion map of the space of quaternionic cycles into the complex cycle space.

By the Dold-Thom Theorem

$$\pi_m\mathbf{Z}[\mathbf{HP}^n] = H_m(\mathbf{HP}^n) = \mathbf{Z} \quad \text{for } 0 \le m = 4k \le 4n \ \text{ and } \ 0 \ \text{ otherwise.}$$

We also know from Theorem (4.9) that

$$\mathrm{rk}\,\pi_m\widetilde{\mathcal{Q}C}_1(\mathbf{CP}^{2n+1}) = 1 \ \text{ for } 0 \le m = 4k \le 4n \ \text{ and } \ 0 \text{ otherwise.}$$

Finally, by Theorem (2.5)

$$\pi_m\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}) = \mathbf{Z} \ \text{ for } \ 0 \le m = 2k \le 4n \ \text{ and 0 otherwise.}$$

This implies that it is sufficient to prove the following

Lemma (4.12). *Under the composite map*

$$i'h : \mathbf{Z}[\mathbf{HP}^n] \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1})$$

a generator of $\pi_{4k}\mathbf{Z}[\mathbf{HP}^n]$ *maps to a generator of* $\pi_{4k}\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1})$.

We will use induction by n.

For $n = 0$ the statement is trivial. Suppose it is true for some $n = p - 1$, i.e. for all k there is an isomorphism

$$\pi_{4k}\mathbf{Z}[\mathbf{HP}^{p-1}] \to \pi_{4k}\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p-1})$$

and assume now that $n = p$.

First of all, notice that the statement of the lemma is obviously true for $k > p$ as all groups in question are trivial.

Further, there is a commutative diagram

$$(*)\qquad \begin{array}{ccccc} \mathbf{Z}[\mathbf{HP}^{p-1}] & \hookrightarrow & \mathbf{Z}[\mathbf{HP}^{p}] & \to & \mathbf{Z}[S^{4p};*] \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p-1}) & \hookrightarrow & \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p+1}) & \to & \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p+1}, \mathbf{CP}^{2p-1}) \end{array}$$

where all maps are natural inclusions and both rows are principal fibrations. The group $\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p+1}, \mathbf{CP}^{2p-1})$ is easily seen (as in [12]) to be homotopy equivalent to $K(\mathbf{Z}, 4p) \times K(\mathbf{Z}, 4p-2)$ and $\mathbf{Z}[S^{4p};*]$ is $4p$-connected; this together with the 5-lemma verifies the statement of Lemma (4.12) for $k < p$.

To deal with the remaining case $k = p$ we first introduce some notation: we fix an embedding $\mathbf{CP}^{2p-1} \hookrightarrow \mathbf{CP}^{2p+1}$ such that $\mathbf{CP}^{2p-1}$ is defined by the equations $z_{2p} = z_{2p+1} = 0$; by L and M we denote the hyperplanes $z_{2p} = 0$ and $z_{2p+1} = 0$ respectively. Notice that $\mathbf{CP}^{2p-1}$ is j-invariant and $\mathbf{CP}^{2p-1} = M \cap L$.

The space $\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p+1}, L)$ is homotopy equivalent to $K(\mathbf{Z}, 4p)$. We need to prove that the composition

$$p' : S^{4p} \hookrightarrow \mathbf{Z}[\mathbf{HP}^p/\mathbf{HP}^{p-1};*] \hookrightarrow \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p+1}, \mathbf{CP}^{2p-1}) \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p+1}, L)$$

represents a generator in homotopy. Here S^{4p} is identified with $\mathbf{HP}^p/\mathbf{HP}^{p-1}$, the middle arrow is from $(*)$ above and the last map is the obvious quotient map.

Let us recall that a generator of $\pi_{4p}\widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p+1}, L)$ is given by the composite map

$$p : S^{4p} \hookrightarrow \mathbf{Z}[M/\mathbf{CP}^{2p-1};*] = \widetilde{\mathcal{C}}_0(M, \mathbf{CP}^{2p-1}) \xrightarrow{\Sigma\!\!\!/} \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2p+1}, L)\,.$$

Here we identify S^{4p} with $M/\mathbf{CP}^{2p-1}$ and $\Sigma\!\!\!/$ denotes the ruled join with the point $(0, \dots, 0, 1)$.

To see that the two maps p and p' are in the same homotopy class, recall that we consider $\mathbf{HP}^n$ to be the space of j-invariant lines in $\mathbf{CP}^{2n+1}$. Define a map

$$\mu : S^{4n} = \mathbf{HP}^n/\mathbf{HP}^{n-1} \to M/\mathbf{CP}^{2n-1} = S^{4n}$$

by sending a j-invariant line into its point of intersection with M. All j-invariant lines which are not transversal to M (i.e. are contained in it) are also contained in $\mathbf{CP}^{2n-1}$; so, clearly, μ is continuous. Moreover, as any j-invariant line is defined uniquely by specifying a point on it, μ is a homeomorphism. It is also easy to see that if the orientations on S^{4n} are defined by the natural orientations of $\mathbf{HP}^n$ and M, then μ is orientation-preserving.

Now we will find the homotopy of the map p' to the map $p \circ \mu$ and this will establish the lemma. The construction below follows Lawson's argument in section 4 of [9].

Define a map

$$\phi_t : \mathbf{CP}^{2n+1} \times [1, \infty) \to \mathbf{CP}^{2n+1}$$

as the multiplication of the last homogeneous coordinate by a real number t:

$$(z_0, \dots , z_{2n}, z_{2n+1}) \to (z_0, \dots , z_{2n}, t \cdot z_{2n+1})\,.$$

All the points of M are fixed by ϕ_t and L is carried into itself. As L is invariant under ϕ_t and because ϕ_t is a linear transformation for any $t \in [1, \infty)$, there is an induced deformation of the cycle spaces

$$\Phi_t : \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}, L) \times [1, \infty) \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}, L)\,.$$

Consider a map

$$\Phi_t \circ p' : \mathbf{HP}^n/\mathbf{HP}^{n-1} \times [1, \infty) \to \widetilde{\mathcal{C}}_1(\mathbf{CP}^{2n+1}, L)\,.$$

One can extend this map by continuity to $t = \infty$. Indeed, all quaternionic lines that are not contained in $L \cap M = \mathbf{CP}^{2n-1}$ are transversal to M. If l is such a line then as t tends to infinity $\Phi_t(l)$ tends to the line passing through $(0, \dots , 0, 1)$ and the point $l \cap M$. The continuity at the basepoint $* \in \mathbf{HP}^n/\mathbf{HP}^{n-1}$ follows from the fact that one can choose an arbitrarily small neighbourhood U of L such that $\phi_t(U) \subset U$ for all $t \geq 1$. Clearly,

$$\Phi_\infty \circ p' = p \circ \mu$$

so Φ_t gives the desired homotopy. $\square$

5. Spaces of cycles of low degrees.

(5.1) Degree one: infinite Grassmannians. An immediate application of Theorem (2.4) and Corollary (2.5) is a construction of a classifying map for the Chern classes in $H^*(BU(q), \mathbf{Z})$, see [12]. Namely, spaces of effective p-cycles of degree 1 in $\mathbf{CP}^n$ are the Grassmannians $\mathrm{G}(n{+}1, p{+}1)$, so fixing the codimension of cycles and passing to the limit $n \to \infty$ one obtains an inclusion of the infinite Grassmannian $BU(q)$ into the space $\widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty)$ of all cycles of codimension q

on $\mathbf{CP}^\infty$. By Theorem (2.4) the space $\widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty)$ is a product of Eilenberg-Mac Lane spaces $K(\mathbf{Z}, 2k)$, where $0 \le k \le q$, so this inclusion map determines certain cohomology classes in $H^*(BU(q), \mathbf{Z})$.

There is a similar inclusion map for the mod 2 real cycles (see Section 3.1): it sends the real Grassmannian $BO(q)$ to the space $\widetilde{\mathcal{E}}^q(\mathbf{CP}^\infty)$ of mod 2 real cycles of codimension q on $\mathbf{CP}^\infty$. This map was considered in [8].

Theorem (5.2). [12, 8] *The maps*

$$BU(q) \hookrightarrow \prod_{i=1}^{q} K(\mathbf{Z}, 2i)$$

and

$$BO(q) \hookrightarrow \prod_{i=1}^{q} K(\mathbf{F}_2, i)$$

represent respectively the total Chern class of the universal bundle over $BU(q)$ and the total Stiefel-Whitney class of the universal bundle over $BO(q)$.

Remark. In general, there are certain choices involved in representing cohomology classes by maps into Eilenberg-Mac Lane spaces: one has to choose the fundamental class in the cohomology of a $K(\pi, n)$. An implicit claim in the theorem above is that there is a natural choice of the fundamental classes in the homotopy of the cycle spaces. Namely, there is an inclusion map

$$\mathbf{CP}^q \hookrightarrow \widetilde{\mathcal{C}}^q(\mathbf{CP}^\infty),$$

and the natural orientation of $\mathbf{CP}^n$ determines the fundamental classes in the homotopy of the space of cycles.

Quite analogously one can consider spaces of cycles of fixed codimension $\widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty)$ and $\widetilde{\mathcal{QC}}^q(\mathbf{CP}^{2n+1})$. There are inclusion maps of $BO(n)$ into $\widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty)$ and of $BSp(n)$ into $\widetilde{\mathcal{QC}}^{2q}(\mathbf{CP}^{2n+1})$.

Theorem (5.3). *The composite maps*

$$BSp(q) \hookrightarrow \widetilde{\mathcal{QC}}^{2q}(\mathbf{CP}^\infty) \hookrightarrow \widetilde{\mathcal{QC}}^{2q}(\mathbf{CP}^\infty) \otimes \mathbf{Q} = \prod_{i=0}^{q} K(\mathbf{Q}, 4i)$$

and

$$BO(q) \hookrightarrow \widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) \longrightarrow \widetilde{\mathcal{RC}}^q(\mathbf{CP}^\infty) \otimes \mathbf{Q} = \prod_{i=0}^{[\frac{1}{2}q]} K(\mathbf{Q}, 4i)$$

represent, respectively, the total rational symplectic Pontrjagin class of the universal bundle over $BSp(q)$ and, up to signs, the total rational Pontrjagin class of the universal bundle over $BO(q)$.

By the "total rational symplectic Pontrjagin class" we understand the universal characteristic class for quaternionic vector bundles in the cohomology with rational coefficients. We assume that the fundamental classes in the cohomology of the spaces of quaternionic cycles are induced by the natural orientation of $\mathbf{HP}^n$; the fundamental classes for real cycle spaces are chosen to agree with those of complex cycle spaces under the inclusion map.

The proofs (details of which we omit) utilise Theorem (5.2), the existence of commutative squares

$$\begin{array}{ccc} BSp(q) & \hookrightarrow & \widetilde{\mathcal{Q}C}^{2q}(\mathbf{CP}^\infty) \\ \downarrow & & \downarrow \\ BU(2q) & \hookrightarrow & \widetilde{\mathcal{C}}^{2q}(\mathbf{CP}^\infty) \end{array} \qquad \begin{array}{ccc} BO(q) & \hookrightarrow & \widetilde{\mathcal{R}C}^{q}(\mathbf{CP}^\infty) \\ \downarrow & & \downarrow \\ BU(q) & \hookrightarrow & \widetilde{\mathcal{C}}^{q}(\mathbf{CP}^\infty) \end{array}$$

and standard facts about the cohomology of infinite Grassmannians.

(5.4) Quaternionic cycles of degree two and odd codimension. The spaces of quaternionic cycles in Theorem (5.3) are all of even codimension. Indeed, in odd codimension, (i.e. in even dimension) there are no cycles of degree one. However, a statement, similar to Theorem (5.3) can be made about spaces of *irreducible* cycles of degree 2 and odd codimension.

Denote by $A^q(\mathbf{CP}^{2n+1})$ the closure of the subset of irreducible varieties in $\mathcal{Q}C_{2(n-q),2}(\mathbf{CP}^{2n+1})$. There are maps

$$\Sigma\!\!\!/\!\!/ : A^q(\mathbf{CP}^{2n+1}) \hookrightarrow A^q(\mathbf{CP}^{2n+3})$$

and the direct limit of these maps as the dimension goes to infinity is the stabilised space A^q.

Every irreducible j-invariant quadric of codimension $2q+1$ is contained in a unique quaternionic linear subspace of complex codimension $2q$. The same is true for the reducible quadrics which are contained in $A^q(\mathbf{CP}^{2n+1})$. This means that there are fibrations

$$\mathbf{RP}^D \to A^q(\mathbf{CP}^{2n+1}) \to \mathbf{H}\mathrm{G}(n+1, n+1-q)$$

over quaternionic Grassmannians. Here $D = \binom{2(n-q)+3}{2} - 1$ is the dimension of the space of quadratic j-invariant divisors in $\mathbf{CP}^{2(n-q)+1}$. In the limit these fibrations give a map

$$A^q \to BSp(q)$$

which induces an isomorphism on rational cohomology. So one can speak of symplectic Pontrjagin classes in $H^*(A^q, \mathbf{Q})$.

THEOREM (5.5). *The composite map*

$$A^q \hookrightarrow \widetilde{\mathcal{Q}C}^{2q+1}(\mathbf{CP}^\infty) \hookrightarrow \widetilde{\mathcal{Q}C}^{2q+1}(\mathbf{CP}^\infty) \otimes \mathbf{Q} = \prod_{i=0}^{q} K(\mathbf{Q}, 4i)$$

represents the total symplectic Pontrjagin class in $H^*(A^q, \mathbf{Q})$

The proof is similar to that of Theorem (5.2) and we refer for details to [12].

Appendix A. On the homology of the quotient of $\mathbf{CP}^n$ by complex conjugation

The quotient of a complex projective space by complex conjugation is a rather well-studied object; nevertheless, we failed to find any references for $H_*(\mathbf{CP}^n/t)$ in the literature. Here we calculate the homology of $\mathbf{CP}^n/t$ with coefficients in $\mathbf{Q}$ and $\mathbf{F}_2$. All proofs that are sketched in section A.1 are due to Elmer Rees.

(A.1) Mod 2 homology of $\mathbf{CP}^n/t$.

Proposition (A.2).

$$\operatorname{rk}(\widetilde{H}_k(\mathbf{CP}^n/t, \mathbf{F}_2) = \begin{cases} [\frac{1}{2}k] - 1 & \textit{for } 1 < k \leq n+1, \\ [\frac{1}{2}k] + 1 - (k-n) & \textit{for } n+1 < k \leq 2n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition (A.3).

$$\operatorname{rk}(\widetilde{H}_k((\mathbf{CP}^n/t)/\mathbf{RP}^n, \mathbf{F}_2)) = \begin{matrix} [\frac{1}{2}k] & \textit{for } 1 < k \leq n+1, \\ [\frac{1}{2}k] + 1 - (k-n) & \textit{for } n+1 \leq k \leq 2n, \\ 0 & \textit{otherwise} \end{matrix}$$

Sketch of the proof. Let $S^{2n+1} \subset \mathbf{C}^{n+1}$ be the sphere

$$|z_0|^2 + |z_1|^2 + \ldots + |z_n|^2 = 2$$

and Λ be the subset of S^{2n+1} defined by

$$z_0^2 + z_1^2 + \ldots + z_n^2 = 0\,.$$

Each point of Λ can be thought of as representing a pair of orthogonal unit vectors in $\mathbf{R}^{n+1}$. Indeed, let $(x_i), (y_i) \in \mathbf{R}^{n+1}$ be the real and imaginary parts, respectively, of a vector $(z_i) \in \mathbf{C}^{n+1}$. Then it follows from the equations of Λ that $\sum x_i y_i = 0$ and that $\sum x_i^2 = \sum y_i^2 = 1$. Associating to a pair of orthogonal unit vectors the plane that contains them we obtain a map $\Lambda \to \mathrm{G}(n+1, 2)$ to the Grassmannian of real unoriented 2-planes in $\mathbf{R}^{n+1}$.

The complex conjugation, as well as the action of S^1 on S^{2n+1}, which sends $(z_0, \ldots, z_n)$ to $(z_0 e^{i\theta}, \ldots, z_n e^{i\theta})$, preserves not only Λ but also the fibres of the map $\Lambda \to \mathrm{G}(n+1, 2)$. One may easily see that the image of Λ in $\mathbf{CP}^n/t$ under the composite map

$$S^{2n+1} \to \mathbf{CP}^n \to \mathbf{CP}^n/t$$

is exactly $\mathrm{G}(n+1, 2)$. Note that its intersection with $\mathbf{RP}^n \subset \mathbf{CP}^n/t$ is empty, so the image of Λ can be regarded as a subset of $(\mathbf{CP}^n/t)/\mathbf{RP}^n$.

Consider a function $f : S^{2n+1} \to \mathbf{R}$ given by

$$f(z_0, \ldots, z_n) = |z_0^2 + z_1^2 + \ldots + z_n^2|\,.$$

Clearly, f is invariant under the action of S^1 on S^{2n+1}; it is also invariant under the complex conjugation in $\mathbf{C}^{n+1}$. This implies that f descends to a function on $\mathbf{CP}^n/t$. The value of this function on $\mathbf{RP}^n \subset \mathbf{CP}^n/t$ is constant (and equal to 2) so, in fact, f descends to a function $\tilde{f}$ on $(\mathbf{CP}^n/t)/\mathbf{RP}^n$. The minimal value of $\tilde{f}$ is 0, it is attained on the subset of real codimension 2, which, in fact, is the image of Λ, i. e. the Grassmannian $\mathrm{G}(n+1,2)$. One can also check that $\tilde{f}$ has one maximum and no other critical points. This means that $(\mathbf{CP}^n/t)/\mathbf{RP}^n$ is homeomorphic to the Thom space of a real 2-plane bundle over $\mathrm{G}(n+1,2)$ and thus the homology of $(\mathbf{CP}^n/t)/\mathbf{RP}^n$ can be expressed via the homology of $\mathrm{G}(n+1,2)$ with the help of the Thom isomorphism.

Notice that the inclusion $\mathbf{RP}^n \hookrightarrow \mathbf{CP}^n$ induces zero maps on reduced homology. This implies that all maps $\widetilde{H}_i(\mathbf{RP}^n, \mathbf{F}_2) \to \widetilde{H}_i(\mathbf{CP}^n/t, \mathbf{F}_2)$ are also zero. Thus we obtain Proposition (A.2) from the homology exact sequence of the cofibration

$$\mathbf{RP}^n \hookrightarrow \mathbf{CP}^n/t \to (\mathbf{CP}^n/t)/\mathbf{RP}^n$$

and Proposition (A.3). □

The action of t respects the inclusions $\mathbf{CP}^{n-1} \hookrightarrow \mathbf{CP}^n$, so there is an involution on the quotient $S^{2n} = \mathbf{CP}^n/\mathbf{CP}^{n-1}$. If we represent $S^{2n} = \mathbf{CP}^n/\mathbf{CP}^{n-1}$ as a 1-point compactification of $\mathbf{R}^{2n}$, we can write t as follows:

$$t : (x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n, -y_1, \ldots, -y_n)$$

and $t(\infty) = \infty$. From this it is clear that S^{2n}/t is just the $(n+1)$-fold suspension of $\mathbf{RP}^{n-1}$, hence we have

Proposition (A.4).

$$\widetilde{H}_i(S^{2n}/t) = 0 \quad \textit{for } i < n\,.$$

(A.5) Rational homology of $\mathbf{CP}^n/t$. To calculate the rational homology of $\mathbf{CP}^n/t$, recall that the transfer homomorphism in homology

$$H_i(\mathbf{CP}^n/t, \mathbf{Q}) \to H_i(\mathbf{CP}^n, \mathbf{Q})$$

provides the isomorphism between $H_i(\mathbf{CP}^n/t, \mathbf{Q})$ and the t-fixed part of $H_i(\mathbf{CP}^n, \mathbf{Q})$. It is straightforward to see that the action of t on the homology of $\mathbf{CP}^n$ is trivial on $4k$-dimensional classes and is multiplication by -1 on the classes of dimension $4k+2$. Thus we have:

Proposition (A.6).

$$H_i(\mathbf{CP}^n/t, \mathbf{Q}) = \mathbf{Q} \textit{ for } i = 4k \leq 2n, \textit{ and } 0 \textit{ otherwise}.$$

(A.7) Case $n = 2$. The quotient $\mathbf{CP}^n/t$ can be explicitly identified when $n = 1, 2$. Case $n = 1$ is trivial: $\mathbf{CP}^1/t$ is a 2-disk D^2. Case $n = 2$ is a well-known statement that $\mathbf{CP}^2/t$ is homeomorphic[1] to a 4-sphere. This can be established by the following argument.

Recall that $\mathbf{CP}^2$ can be regarded as the symmetric square of a Riemann sphere $\mathbf{CP}^1$. In this set-up t acts by reflecting pairs of points in $\mathbf{CP}^1$ with respect to the real line. Denote by D_+ the subspace of $\mathbf{CP}^2$ formed by pairs of points (z_1, z_2) such that z_1 and z_2 lie in the same open hemisphere; similarly, let D_- be the subspace formed by pairs (z_1, z_2) where z_1 and z_2 lie in different hemispheres. Both D_+ and D_- together with their closures $\overline{D_+}$ and $\overline{D_-}$ are invariant with respect to t. Notice that

$$\mathbf{CP}^2/t = (\overline{D_+}/t) \cup (\overline{D_-}/t)\,.$$

Let us identify the upper closed hemisphere of the Riemann sphere with the 2-disk D^2. Then the orbit spaces $\overline{D_+}/t$ and $\overline{D_-}/t$ can be identified with the symmetric square of D^2. It can be easily seen that the symmetric square of a 2-disk is homeomorphic a 4-disk. The intersection of $\overline{D_+}/t$ and $\overline{D_-}/t$ is precisely the boundary of $\overline{D_+}/t$ and $\overline{D_-}/t$, so $\mathbf{CP}^2/t$ is homeomorphic to a union of two 4-disks glued together along the boundary, that is, to a 4-sphere.

Acknowledgments

I am deeply indebted to Elmer Rees for much help and encouragement; without his advice this work would not have been possible. I would also like to thank Blaine Lawson and Paulo Lima-Filho to whom I owe my understanding of some important points of the theory.

The work presented here was carried out with the support of the University of Edinburgh and an ORS award from the Committee of Vice-Chancellors and Principals of the Universities of the United Kingdom.

Instituto de Matemáticas (Unidad Cuernavaca)
Universidad Nacional Autónoma de México
Apartado Postal 273-3
62251 Cuernavaca, Morelos
México
jacob@matcuer.unam.mx

References

[1] V.I. Arnol'd, *Topological content of the Maxwell theorem on multipole representation of spherical functions*, Topol. Methods Nonlinear Anal. **7** (2) (1996), 205–217.

[1]actually, even *diffeomorphic*, see [2].

[2] V.I. Arnolp1d, *Razvetvle0nnoe nakrytie* $\mathbf{CP}^2 \to S^4$*, giperbolichnostp1 i proektivnaya topologiya*, Sib. Mat. Zh. **29** (5) (1988), 36–47; English translation: V.I. Arnol'd *The branched covering* $\mathbf{CP}^2 \to S^4$*, hyperbolicity and projective topology*, Siberian Math. J. **29** (5) (1978), 717–726.

[3] A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. **67** (2) (1958), 239–281.

[4] E.E. Floyd, *Periodic maps via Smith theory* (1960 No. 46), Borel's seminar on transformation groups, Princeton; Annals of Mathematical Studies. 35–48.

[5] E. Friedlander, *Algebraic cycles, Chow varieties, and Lawson homology*, Compositio Math. **77** (1991), 55–93.

[6] E. Friedlander and O. Gabber, *Cycle spaces and intersection theory*, Topological methods in modern mathematics, Stony Brook, NY, 1991; 325–370, Publish or Perish, Houston, TX, 1993.

[7] E. Friedlander and B. Mazur, *Filtrations on homology of algebraic varieties*, Memoirs of the AMS **110** (529) (1994).

[8] T.K. Lam, Spaces of real algebraic cycles and homotopy theory, Ph. D. Thesis, SUNY, 1990.

[9] H.B. Lawson, *Algebraic cycles and homotopy theory*, Ann. of Math. **129** (1989), 253–291.

[10] H. B. Lawson, P. Lima-Filho and M. -L. Michelsohn, *Algebraic cycles and the classical groups — Part I*; Real Cycles, `math.AT/9912154`.

[11] H.B. Lawson, P. Lima-Filho and M.-L. Michelsohn, *On equivariant algebraic suspension*, J. Algebraic Geom. **7** (4) (1998), 627–650.

[12] H. B. Lawson and M. -L. Michelsohn, *Algebraic cycles, Bott periodicity and the Chern characteristic map*, Proc. Symp. Pure Math **48** (1988), 241–263.

[13] H. B. Lawson and M. -L. Michelsohn, *Algebraic cycles and group actions*, Differential geometry, Longman, 1993.

[14] P. Lima-Filho, *Completions and fibrations for topological monoids*, Trans. Am. Math. Soc. **340** (1) (1993), 127–147.

[15] Mostovoi0, Ya., *O prostranstvah veshchestvennyh algebraicheskih ciklov na* $\mathbf{CP}^n$, Uspehi Mat. Nauk **53** No. 1 (319) (1998), 217–218.

[16] E. Spanier, *Infinite symmetric products, function spaces and duality*, Ann. of Math. **69** (1959), 142–198.