Uniform individual asymptotics for the eigenvalues and eigenvectors of large Toeplitz matrices

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Abstract

The asymptotic behavior of the spectrum of large Toeplitz matrices has been studied for almost one century now. Among this huge work, we can find the Szegö theorems on the eigenvalue distribution and the asymptotics for the determinants, as well as other theorems about the individual asymptotics for the smallest and largest eigenvalues.

Results about uniform individual asymptotics for all the eigenvalues and eigenvectors appeared only five years ago. The goal of the present lecture is to review this area, to talk about the obtained results, and to discuss some open problems.

This review is based on joint works with Manuel Bogoya, Albrecht Böttcher, and Egor Maximenko.
Main object.

Spectral properties of larger finite Toeplitz matrices

\[ A_n = (a_{j-k})_{j,k=0}^{n-1} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \cdots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix} . \]

\[ a(t) = \sum_{j=-\infty}^{\infty} a_j t^j, \quad t \in \mathbb{T} \text{-symbol of } \{A_n\}_{n=1}^{\infty} \]

Eigenvalues, eigenvectors singular values, condition numbers, invertibility and norms of inverses, e.t.c.

\( n \sim 1000 \) is a business of numerical linear algebra.

Statistical physics - \( n = 10^7 - 10^{12} \) - is a business of asymptotic theory.
I. Two parameters:
   \( n \)- dimensions of matrices;
   \( j \)- number of eigenvalue

   \[ 1 \leq j \leq n \]

Asymptotics by \( n \) uniformly in \( j \).

II. Distance between \( \lambda_j \) and \( \lambda_{j+1} \) is small:

\[ |\lambda_j - \lambda_{j+1}| = O \left( \frac{1}{n} \right) \quad \text{normal case} \]

\[ |\lambda_j - \lambda_{j+1}| = O \left( \frac{1}{n^\gamma} \right) \quad \text{special case} \]

\[ \lambda_j = \lambda_{j+1} \quad \text{exceptional case} \]
Publications about asymptotics of individual eigenvalues and eigenvectors.


Main results-simple loop case

For $\alpha \geq 0$, we denote by $W^\alpha$ the weighted Wiener algebra of all functions $a : \mathbb{T} \to \mathbb{C}$ whose Fourier coefficients satisfy

$$||a||_\alpha := \sum_{j=-\infty}^{\infty} |a_j|(|j| + 1)^\alpha < \infty.$$ 

Let $m$ be the entire part of $\alpha$. It is readily seen that if $a \in W^\alpha$ then the function $g$ defined by $g(\sigma) := a(e^{i\sigma})$ is a $2\pi$-periodic $C^m$ function on $\mathbb{R}$. In what follows we consider real-valued simple-loop functions in $W^\alpha$. To be more precise, for $\alpha \geq 2$, we let $SL^\alpha$ denote the set of all $a \in W^\alpha$ such that $g$ has the following properties: the range of $g$ is a segment $[0, M]$ with $M > 0$, $g(0) = g(2\pi) = 0$, $g'''(0) = g'''(2\pi) > 0$, and there is a $\varphi_0 \in (0, 2\pi)$ such that $g(\varphi_0) = M$, $g'(\sigma) > 0$ for $\sigma \in (0, \varphi_0)$, $g'(\sigma) < 0$ for $\sigma \in (\varphi_0, 2\pi)$, and $g'''(\varphi_0) < 0$. 
Let $a \in SL^\alpha$. Then for each $\lambda \in [0, M]$, there are exactly one
$\varphi_1(\lambda) \in [0, \varphi_0]$ such that $g(\varphi_1(\lambda)) = \lambda$ and exactly one $\varphi_2(\lambda) \in [\varphi_0, 2\pi]$ satisfying $g(\varphi_2(\lambda)) = \lambda$. For each $\lambda \in [0, M]$, the function $g$ takes values
less than or equal to $\lambda$ on the segments $[0, \varphi_1(\lambda)]$ and $[\varphi_2(\lambda), 2\pi]$. Denote
by $\varphi(\lambda)$ the arithmetic mean of the lengths of these two segments,

$$
\varphi(\lambda) := \frac{1}{2}(\varphi_1(\lambda) - \varphi_2(\lambda)) + \pi = \frac{1}{2} \mu \{\sigma \in [0, 2\pi] : g(\sigma) \leq \lambda\},
$$

where $\mu$ is the Lebesgue measure on $[0, 2\pi]$. The function
$\varphi : [0, M] \to [0, \pi]$ is continuous and bijective. We let $\psi : [0, \pi] \to [0, M]$ stand for the inverse function.

Put

$$
\sigma_1(s) = \varphi_1(\psi(s)) \text{ and } \sigma_2(s) = \varphi_2(\psi(s)).
$$

Then

$$
g(\sigma_1(s)) = g(\sigma_2(s)) = \psi(s).
$$
Let further

\[
\beta(\sigma, s) := \frac{(g(\sigma) - \psi(s))e^{is}}{(e^{i\sigma} - e^{i\sigma_1(s)})(e^{-i\sigma} - e^{-i\sigma_2(s)})} = \frac{\psi(s) - g(\sigma)}{4 \sin \frac{\sigma - \sigma_1(s)}{2} \sin \frac{\sigma - \sigma_2(s)}{2}}.
\]

We will show that \(\beta\) is a continuous and positive function on \([0, 2\pi] \times [0, \pi]\). We define the function \(\eta : [0, \pi] \to \mathbb{R}\) by

\[
\eta(s) := \theta(\psi(s)) = \frac{1}{4\pi} \int_{0}^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - \sigma_2(s)}{2}} d\sigma - \frac{1}{4\pi} \int_{0}^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - \sigma_1(s)}{2}} d\sigma,
\]

the integrals taken in the principal-value sense.
Theorem

Let $a \in SL^\alpha$ with $\alpha \geq 2$ and let $\lambda_1^{(n)} \leq \ldots \leq \lambda_n^{(n)}$ be the eigenvalues of $T_n(a)$. If $n$ is sufficiently large, then

(i) the eigenvalues of $T_n(a)$ are all distinct, i.e., $\lambda_1^{(n)} < \lambda_2^{(n)} < \ldots < \lambda_n^{(n)}$,

(ii) the numbers $s_j^{(n)} := \psi(\lambda_j^{(n)})$ $(j = 1, \ldots, n)$ satisfy

$$(n + 1)s_j^{(n)} + \eta(s_j^{(n)}) = \pi j + \Delta_1^{(n)}(j)$$

with $\Delta_1^{(n)}(j) = o(1/n^{\alpha-2})$ as $n \to \infty$, uniformly with respect to $j$,

(iii) this equation has exactly one solution $s_j^{(n)} \in [0, \pi]$ for each $j = 1, \ldots, n$. 
To write down the individual asymptotics of the eigenvalues, we introduce the parameter

\[ d := \frac{\pi j}{n + 1}. \]

Note that the dependence of \( d \) on \( j \) and \( n \) is suppressed.

**Theorem**

Let \( a \in SL^\alpha \) with \( \alpha \geq 2 \) and let \( s_j^{(n)} \) be as in previous Theorem. Then

\[ s_j^{(n)} = d + \sum_{k=1}^{[\alpha]-1} \frac{p_k(d)}{(n + 1)^k} + \Delta_2^{(n)}(j) \]

where \( \Delta_2^{(n)}(j) = o\left(\frac{1}{n^{\alpha-1}}\right) \) as \( n \to \infty \) uniformly in \( j \),

\[ p_1(d) = -\eta(d), \quad p_2 = \eta(d)\eta'(d). \]

\([\alpha]\) is integer part of \( \alpha \).
That is for $2 \leq \alpha < 3$ we have

$$s_j^{(n)} = d - \frac{\eta(a)}{n + 1} + o\left(\frac{1}{n^{\alpha-1}}\right), \quad d = \frac{\pi j}{n + 1}. $$

For $3 \leq \alpha < 4$ we have

$$s_j^{(n)} = d - \frac{\eta(a)}{n + 1} + \frac{\eta(d)\eta'(d)}{(n + 1)^2} + o\left(\frac{1}{n^{\alpha-1}}\right).$$

e.t.c.
Theorem

Let $\alpha \geq 2$ and $a \in SL^\alpha$. Then

$$\lambda_j^{(n)} = \psi(d) + \sum_{k=1}^{[\alpha]-1} \frac{c_k(d)}{(n+1)^k} + \Delta_3^{(n)}(j)$$

where $\Delta_3^{(n)}(j) = o\left(d(\pi - d)/n^{\alpha-1}\right)$ as $n \to \infty$, uniformly in $j = 1, 2, \ldots, n$, and

$$c_1(d) = -\psi'(d)\eta(d),$$

$$c_2(d) = \psi''(d)\eta^2(d)/2 + \psi'(d)\eta(d)\eta'(d).$$
Here is the result for the extreme eigenvalues.

**Corollary**

Let $a \in SL^\alpha$ with some $\alpha \geq 3$.

(i) If $j/(n + 1) \to 0$ then

$$
\lambda_j^{(n)} = \frac{c_5 j^2}{(n + 1)^2} + \frac{c_6 j^2}{(n + 1)^3} + \Delta_5^{(n)}(j),
$$

where $c_5 = \pi^2 g''(0)/2$, $c_6 = -\pi^2 g''(0)\eta'(0)$, and $\Delta_5^{(n)}(j) = o(j/n^3)$ as $n \to \infty$.

(ii) If $j/(n + 1) \to 1$ then

$$
\lambda_j^{(n)} = M + \frac{c_7 (n + 1 - j)^2}{(n + 1)^2} + \frac{c_8 (n + 1 - j)^2}{(n + 1)^3} + \Delta_6^{(n)}(j),
$$

where $c_7 = \pi^2 g''(\varphi_0)/2$, $c_8 = -\pi^2 g''(\varphi_0)\eta'(\pi)$, and $\Delta_6^{(n)}(j) = o(n + 1 - j/n^3)$ as $n \to \infty$.  

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Simple-loop

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This theorem is close to a result by Widom 1958, who considered the case where $g$ is an even function and $j$ is fixed.
Local nature of the asymptotics

Symmetric symbol: \( \psi(\varphi) = g(\varphi) = (a(e^{i\varphi})) \)

1. Normal case: \( g'(\varphi) \neq 0 \) \( (\varepsilon < \frac{\pi j}{n+1} < \pi - \varepsilon) \). Inner eigenvalues

\[
\lambda_j^{(n)} = g \left( \frac{\pi j}{n+1} \right) + \frac{c_1 \left( \frac{\pi j}{n+1} \right)}{n+1} + O \left( n^{-2} \right).
\]

Distance between next eigenvalues is

\[
O \left( \frac{1}{n} \right)
\]
2. Exceptional case: \( \left( \frac{\pi j}{n+1} \right) \leq \varepsilon \). Extreme eigenvalue

\[
\lambda_j^{(n)} = g \left( \frac{\pi j}{n+1} \right) + O \left( \frac{1}{n^3} \right), \quad \frac{\pi j}{n+1} \leq \varepsilon.
\]

Distance between next eigenvalues is

\[
O \left( \frac{1}{n^2} \right)
\]

Asymptotics: is defined by behavior of function \( g(\varphi) \) in neighborhood of point \( \varphi_0 \), where eigenvalues are located.
More general restriction

1. Normal case:

\[ g(\varphi) - g(\varphi_0) = (\varphi - \varphi_0) \tilde{g}(\varphi), \quad \tilde{g}(\varphi) \in W(= W^0). \]

That is \( g(\varphi) \in W^1 \)

2. Exceptional case:

\[ g(\varphi) - g(\varphi_0) = (\varphi - \varphi_0)^2 \tilde{g}(\varphi), \quad \tilde{g}(\varphi) \in W(= W^0). \]
Main ideas of the Proof

Lemma

Let $a \in SL^\alpha$, $\alpha \geq 2$ and $n \geq 1$. A number $\lambda = \psi(s)$ is an eigenvalue of $T_n(a)$ if and only if

$$e^{i(n+1)\sigma_2(s)}\Theta_{n+2}(e^{i\sigma_1(s)}, s)\hat{\Theta}_{n+2}(e^{i\sigma_2(s)}, s)$$

$$-e^{i(n+1)\sigma_1(s)}\Theta_{n+2}(e^{i\sigma_2(s)}, s)\hat{\Theta}_{n+2}(e^{i\sigma_1(s)}, s) = 0,$$

where, for every $k \geq 1$, the functions $\Theta_k$ and $\hat{\Theta}_k$ are defined by

$$\Theta_k(t, s) := [T_k^{-1}(b(\cdot, s))\chi_0](t), \quad \hat{\Theta}_k(t, s) := [T_k^{-1}(\tilde{b}(\cdot, s))\chi_0](t^{-1}),$$

and $\tilde{b}(t, s) := b(1/t, s)$, $\chi_\ell(t) = t^\ell$, $\chi_0(t) = 1$. 
Proof. We are searching for all values of $\lambda$ belonging to $[0, M]$ such that the equation $T_n(a)X = \lambda X$ has non-zero solutions $X$ in $L_2^{(n)}$. Using the change of variable $\lambda = \psi(s)$ we can rewrite the latter equation as

$$T_n(a - \psi(s))X = 0. \quad (2)$$

Equation (2) is equivalent to

$$P_n b(\cdot, s)p(\cdot, s)X = 0, \quad (3)$$

where $p(t, s) := e^{-is}(t - e^{i\sigma_1(s)})(t^{-1} - e^{-i\sigma_2(s)})$. Multiply (3) by the function $\chi_1$ to get

$$(P_{n+1} - P_1)b(\cdot, s)\chi_1 p(\cdot, s)X = 0. \quad (4)$$
Here $P_{n+1} - P_1$ is just one way to write the orthogonal projection of the space $L_2(\mathbb{T})$ onto the span of $\chi_1, \ldots, \chi_n$. Note that $\chi_1 p(\cdot, s) X \in L^{(n+2)}_2$ and put

$$Y := T_{n+2}(a - \psi(s)) \chi_1 X = P_{n+2} b(\cdot, s) \chi_1 p(\cdot, s) X = T_{n+2}(b(\cdot, s)) \chi_1 p(\cdot, s) X.$$  

Then (4) can be written as $(P_{n+1} - P_1) Y = 0$. This means that $Y$ has the form

$$Y = y_0 \chi_0 + y_{n+1} \chi_{n+1}.$$  

Since $T_{n+2}(b(\cdot, s))$ is invertible, it follows that $T_{n+2}^{-1}(b(\cdot, s)) Y = \chi_1 p(\cdot, s) X$, that is,

$$y_0 [T_{n+2}^{-1}(b(\cdot, s)) \chi_0](t) + y_{n+1} [T_{n+2}^{-1}(b(\cdot, s)) \chi_{n+1}](t) = tp(t, s) X(t). \quad (5)$$
Now recall notation (5). Taking into account the identity
\[ W_{n+2} T_{n+2}(b) W_{n+2} = T_{n+2}(\tilde{b}), \]
it is easy to verify that
\[ [T_{n+2}^{-1}(b(\cdot, s))\chi_{n+1}](t) = t^{n+1}\Theta_{n+2}(t, s). \]
Therefore (5) can be written as
\[ y_0\Theta_{n+2}(t, s) + y_{n+1}t^{n+1}\Theta_{n+2}(t, s) = tp(t, s)X(t). \]
Thanks to the factor $p(t, s)$, the right-hand side vanishes at both $t = e^{i\sigma_1(s)}$ and $t = e^{i\sigma_2(s)}$. Consequently, $y_0$ and $y_{n+1}$ must satisfy the homogeneous system of linear algebraic equations given by

$$
\Theta_{n+2}(e^{i\sigma_1(s)}, s)y_0 + e^{i(n+1)\sigma_1(s)}\hat{\Theta}_{n+2}(e^{i\sigma_1(s)}, s)y_{n+1} = 0,
$$
$$
\Theta_{n+2}(e^{i\sigma_2(s)}, s)y_0 + e^{i(n+1)\sigma_2(s)}\hat{\Theta}_{n+2}(e^{i\sigma_2(s)}, s)y_{n+1} = 0.
$$

If $y_0 = y_{n+1} = 0$, then, by (6), the function $X$ is zero. Therefore the initial equation (2) has a non-trivial solution $X$ if and only if the determinant of system (7) is zero. □
Recall that $b_{\pm}(\cdot, s)$ are the Wiener-Hopf factors of $b(\cdot, s)$:

$$b(t, s) = b_+(t, s) b_-(t, s)$$

$$b_+(t, s) = \sum_{j=0}^{\infty} u_j(s) t^j \quad \text{and} \quad b_-(t, s) = \sum_{j=0}^{\infty} v_j(s) t^{-j}$$

$$T^{-1}(b(\cdot, s)) = b_+^{-1}(\cdot, s) Pb_-^{-1}(\cdot, s),$$

$$[T^{-1}(b(\cdot, s)) \chi_0](t) = [b_+^{-1}(\cdot, s) Pb_-^{-1}(\cdot, s) \chi_0](t) = b_+^{-1}(t, s).$$
\[ T_n(a) \bar{x}_n = \bar{f}_n \]

\[ L_n^* \left( \text{diag} \left( \lambda_j^{(n)} \right)_{j=1}^n \right) L_n \bar{X}_n = \bar{f}_n, \quad L^* = L_n^{-1} \]

\[
\begin{pmatrix}
\lambda_1^{(n)} & 0 & 0 & \ldots & 0 \\
0 & \lambda_2^{(n)} & 0 & \ldots & 0 \\
0 & 0 & \lambda_3^{(n)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_n^{(n)} 
\end{pmatrix}
\begin{pmatrix}
Y_1 \\
Y_2 \\
Y_3 \\
\vdots \\
Y_n 
\end{pmatrix}
= 
\begin{pmatrix}
F_1 \\
F_2 \\
F_3 \\
\vdots \\
F_n 
\end{pmatrix}
\]

\[ \bar{Y} = L_n \bar{X}, \quad \bar{F} = L_n \bar{f}. \]

Singular Value Decomposition.
Example 1.

Consider the non-rational symbol

\[ a(e^{i\sigma}) = g(\sigma) = g_2\sigma^2 + g_3\sigma^3 + g_4\sigma^4 + \beta + g_5\sigma^5 + g_6\sigma^6 + g_7\sigma^7, \quad \sigma \in [0, 2\pi], \]

where \( \beta \in [0, 1) \) and the coefficients \( g_2, \ldots, g_7 \) are chosen in such a manner that

\[ g(2\pi) = g'(2\pi) = 0 \quad \text{and} \quad g^{(k)}(2\pi) = g^{(k)}(0) \quad \text{for} \quad k = 2, 3, 4. \]

Elementary computations yield

\[
\begin{align*}
g_2 &= (24 - 38\beta + 13\beta^2 + 2\beta^3 - \beta^4)/(2\pi)^2, \\
g_3 &= (24 - 50\beta + 35\beta^2 - 10\beta^3 + \beta^4)/(2\pi)^3, \\
g_4 &= 240/(2\pi)^4 + \beta, \\
g_5 &= (360 + 42\beta - 201\beta^2 + 42\beta^3 + 3\beta^4)/(2\pi)^5, \\
g_6 &= (-216 + 66\beta + 209\beta^2 - 54\beta^3 - 5\beta^4)/(2\pi)^6, \\
g_7 &= (48 - 20\beta - 50\beta^2 + 20\beta^3 + 2\beta^4)/(2\pi)^7.
\end{align*}
\]
Figure: Graph of $g(\sigma) = a(e^{i\sigma})$ (left), and $\eta(s)$ (right) for $\beta = 1/5$. 

Figure: The functions $c_1(d)$ (left) and $c_2(d)$ (right).
<table>
<thead>
<tr>
<th>$n$</th>
<th>64</th>
<th>128</th>
<th>512</th>
<th>1024</th>
<th>2048</th>
<th>4096</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon^{(n,1)}_{(n+1)\varepsilon^{(n,1)}}$</td>
<td>$2.0 \cdot 10^{-4}$</td>
<td>$9.8 \cdot 10^{-5}$</td>
<td>$2.5 \cdot 10^{-5}$</td>
<td>$1.2 \cdot 10^{-5}$</td>
<td>$6.2 \cdot 10^{-6}$</td>
<td>$3.1 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$\varepsilon^{(n,2)}_{(n+1)2\varepsilon^{(n,2)}}$</td>
<td>$3.2 \cdot 10^{-8}$</td>
<td>$8.1 \cdot 10^{-9}$</td>
<td>$5.1 \cdot 10^{-10}$</td>
<td>$1.3 \cdot 10^{-10}$</td>
<td>$3.2 \cdot 10^{-11}$</td>
<td>$8.1 \cdot 10^{-12}$</td>
</tr>
<tr>
<td>$\varepsilon^{(n,3)}_{(n+1)3\varepsilon^{(n,3)}}$</td>
<td>$2.3 \cdot 10^{-10}$</td>
<td>$1.3 \cdot 10^{-11}$</td>
<td>$4.1 \cdot 10^{-14}$</td>
<td>$2.2 \cdot 10^{-15}$</td>
<td>$2.4 \cdot 10^{-16}$</td>
<td>$3.0 \cdot 10^{-17}$</td>
</tr>
<tr>
<td>$\hat{\varepsilon}^{(n)}_{(n+1)^{4.2}\hat{\varepsilon}^{(n)}}$</td>
<td>$2.3 \cdot 10^{-10}$</td>
<td>$1.3 \cdot 10^{-11}$</td>
<td>$4.1 \cdot 10^{-14}$</td>
<td>$2.2 \cdot 10^{-15}$</td>
<td>$1.2 \cdot 10^{-16}$</td>
<td>$6.7 \cdot 10^{-18}$</td>
</tr>
</tbody>
</table>

**Table:** Maximum errors and normalized maximum errors for the eigenvalues of $T_n(a)$ obtained with our formula (1), $\varepsilon^{(n,p)}$ with $p = 1, 2, 3$, and by fixed-point iterations, $\hat{\varepsilon}^{(n)}$, for different values of $n$. The data were obtained by comparison with the solutions given by Wolfram Mathematica.

Note that Table 1 shows that $\hat{\varepsilon}^{(n)} = O(1/(n+1)^{4.2})$ as $n \to \infty$. 
For \( \gamma \in \mathbb{R} \), we define \( z_k^\gamma \) as \( e^{i\sigma_k(s_j^{(n)})\gamma} \). Given a function \( f: \mathbb{T} \to \mathbb{C} \), let \( f_p \) be its \( p \)th Fourier coefficient, and for a vector \( X \), let \( X_p \) stand for its \( p \)th component. Let \( \theta = (\theta_p)^{n+1}_{p=0} \) be the vector in the first column of the matrix \( T_{n+2}^{-1}(b(\cdot, s_j^{(n)})) \). For \( t \in \mathbb{C} \), we put

\[
\theta(t) = \theta_0 + \theta_1 t + \cdots + \theta_{n+1} t^{n+1}.
\]

The following theorem describes the components of the eigenvectors of \( T_n(a) \).
Theorem

Let \( a \in SL^\alpha \). The vector

\[ X^{(n,j)} = M^{(n,j)} + L^{(n,j)} + R^{(n,j)} \]  \hspace{1cm} (8)

whose \( p \)-th component, \( p = 0, 1, \ldots, n - 1 \), is given by

\[ M_p^{(n,j)} := z_1^{\frac{n-1}{2} - p} |\theta(z_1)| + (-1)^{n-j} z_2^{\frac{n-1}{2} - p} |\theta(z_2)|, \]

\[ L_p^{(n,j)} := -\frac{z_1^{\frac{n+1}{2}} \theta(z_1)}{2\pi i |\theta(z_1)|} \int_{\mathbb{T}} \left( \frac{\theta(t) - \theta(z_1)}{t - z_1} - \frac{\theta(t) - \theta(z_2)}{t - z_2} \right) \frac{dt}{t^{p+1}}, \]

\[ R_p^{(n,j)} := L_{n-p-1}^{(n,j)}, \]

is an eigenvector of \( T_n(a) \) corresponding to the eigenvalue \( \lambda_j^{(n)} \). Moreover, \( M^{(n,j)} \) is conjugate symmetric, i.e., \( M_p^{(n,j)} = M_{n-p-1}^{(n,j)} \).
Theorem

Let \( a \in SL^\alpha \). For \( k = 1, 2 \), let \( \hat{z}_k := e^{i\sigma_k(s_j^{(n)})} \), and, for \( p = 0, 1, \ldots, n - 1 \), put

\[
\hat{M}_p^{(n,j)} := \frac{\hat{z}_1^{n-1} - p}{|b_+(\hat{z}_1)|} + (-1)^{n-j} \frac{\hat{z}_2^{n-1} - p}{|b_+(\hat{z}_2)|},
\]

\[
\hat{L}_p^{(n,j)} := -\frac{\hat{z}_1^{n+1}}{2\pi i|b_+(\hat{z}_1)|} \int_\mathbb{T} \left( \frac{b_+^{-1}(t) - b_+^{-1}(\hat{z}_1)}{t - \hat{z}_1} - \frac{b_+^{-1}(t) - b_+^{-1}(\hat{z}_2)}{t - \hat{z}_2} \right) \frac{dt}{t^{p+1}},
\]

\[
\hat{R}_p^{(n,j)} := \hat{L}_{n-1-p}^{(n,j)}.
\]

Then there is a vector \( \Omega_1^{(n,j)} \) such that \( [\Omega_1^{(n,j)}]_p = o(1/n^{\alpha-3}) \) as \( n \to \infty \), uniformly in \( j \) and \( p \), and such that

\[
X^{(n,j)} = \hat{M}^{(n,j)} + \hat{L}^{(n,j)} + \hat{R}^{(n,j)} + \Omega_1^{(n,j)} \tag{9}
\]

is an eigenvector of \( T_n(a) \) corresponding to the eigenvalue \( \lambda_j^{(n)} \).
Symbols with Fisher–Harturg singularity.

\[ a_{\alpha, \beta}(t) = (1 - t)^\alpha(-t)^\gamma, \quad 0 < \alpha < |\beta| < 1. \]

Conjecture of

\[ \lambda_j^{(n)} \sim a_{\alpha, \beta} \left( \omega_j \cdot \exp \left\{ (2\alpha + 1) \frac{\log n}{n} \right\} \right), \]

where \( \omega_j = \exp \left( -i \frac{2\pi j}{n} \right) \).
Complex value case

\[ a(t) = t^{-1}(1 - t)^\alpha f(t), \quad \alpha \in R_+ \setminus N \]

where

1. \( f(t) \in H^\infty \cap C^\infty \).
2. \( f \) can be analytically extended to a neighborhood of \( \mathbb{T}\setminus\{1\} \).
3. The range of the symbol \( a \mathcal{R}(a) \) is a closed Jordan curve without loops and winding number -1 around each interior point.
Figure: The map $a(t)$ over the unit circle.
Lemma

Let \( a(t) = t^{-1}h(t) \) be a symbol that satisfies the following conditions:

1. \( h \in H^\infty \).
2. \( \mathcal{R}(a) \) is a closed Jordan curve in \( \mathbb{C} \) without loops.
3. \( \text{wind}_\lambda(a) = -1 \), for each \( \lambda \) in the interior of \( \text{sp} \ T(a) \).

Then, for each \( \lambda \) in the interior of \( \text{sp} \ T(a) \), we have the equality

\[
D_n(a - \lambda) = (-1)^n h_o^{n+1} \left[ \frac{1}{h(t) - \lambda t} \right]^n,
\]

for every \( n \in \mathbb{N} \).
Theorem

We have the following asymptotic expression for \( \lambda_j \):

\[
\lambda_j = a(\omega_j) + (\alpha + 1) \omega_j a'(\omega_j) \frac{\log(n)}{n} + \frac{\omega_j a'(\omega_j)}{n} \log \left( \frac{a^2(\omega_j)}{c_0 a'(\omega_j) \omega_j^2} \right) \\
+ \mathcal{O} \left( \frac{\log(n)}{n} \right)^2, \quad n \to \infty,
\]

where \( \omega_j = \exp \left( -i \frac{2\pi j}{n} \right) \).
Figure: The solid blue line is the range of $a$. The black dots are $T_n(a)$ calculated by Matlab. The red crosses and the green stars are the approximations, for 1 and 2 terms respectively. Here we took $\alpha = 3/4$. 
New problems

1. Symbols of the kind:

\[ a(t) = \frac{a_{-1}}{t} + a_0 + a_1 t + a_2 t^2 \]

\[ a_0(t) = \frac{1}{t} + t^2 \]

2. Symbols of the kind:

\[ a(t) = \frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + a_0 + a_1 t + a_2 t^2 \]

\[ a_0(t) = \left( \frac{1}{t} - 2 + t \right)^2 \]
3. No simpleloop case.

4. Fisher-Harturg general case

\[ a(t) = (t - t_0)^\alpha t^\beta, \quad \alpha, \beta \in \mathbb{R} \ (\alpha, \beta \in \mathbb{C}). \]