The Wiener-Hopf integral equation on a finite interval: asymptotic solution for large intervals with an application to acoustics

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(S.-Petersburg, DD03, June 2003)
\( (W_L(a)f)(x) = \int_0^L K(x-t)f(t)dt, \quad x \in (0, L) \)
\( a(\mu) = (FK)(\mu) \quad \text{symbol} \)

I
\[ a(\mu) = \frac{\sqrt{1 - \mu^2}}{\nu + \sqrt{1 - \mu^2}}, \quad \mu_1 = 1, \quad \mu_2 = -1 \]

A problem of sound propagation in an air half-space over the earth surface with motorway

II
\[ a(\mu) = 1 + c\sqrt{1 - \mu^2}, \quad c \in \mathbb{R}, \quad |a(\mu)| \sim c|\mu| \]
A problem of sound propagation in the water with the surface partially covered by ice

III
\[ a(\mu) = -ic\mu + c_1[2 - (1 + i\mu)^\nu - (1 - i\mu)^\nu], \quad \nu \in [0, 2) \]
The theory of barrier options

\[ \text{Re}(Af, f) \geq 0, \quad \forall f \in H \quad \iff \quad A \text{ is semisectorial} \]
\[ \varepsilon = \inf_{\|f\|=1} \text{Re}(Af, f) \]
\[ \varepsilon > 0 \quad \iff \quad A \text{ is sectorial} \]
\[ a(\mu) \in L_{\infty}(\mathbb{R}), \quad (\chi_{(0, L)} f)(x) = \begin{cases} f(x), & x \in (0, L); \\ 0, & x \in \mathbb{R} \setminus (0, L) \end{cases} \]

\[ W_L(a) = \chi_{(0, L)} \mathcal{F}^{-1} a \mathcal{F} \big|_{L^2(0, L)} : L^2(0, L) \to L^2(0, L) \quad (1) \]

\( \mathcal{F} \) is Fourier transform

\[ \hat{W}_L(a) = \mathcal{F} W_L(a) \mathcal{F}^{-1} : E_{2, L} \to E_{2, L} \quad (2) \]

\[ \Phi^+_L(\mu) \in E_{2, L} \iff \Phi^+_L(\mu) = \int_0^L f(x) e^{i\mu x} \, dx, \quad f(x) \in L^2(0, L) \]

\[ E_{2, L} \subset H_2 \]

\[ \Phi^+(\mu) \in H_2 \iff \Phi^+_L = \int_0^\infty f(x) e^{i\mu x} \, dx, \quad f(x) \in L^2(0, \infty) \]

\[ \hat{W}_L(a) = P_L a P_L \quad (3) \]

\[ P_L = \mathcal{F} \chi_{(0, L)} \mathcal{F}^{-1} : L^2(\mathbb{R}) \to E_{2, L} \]

\[ P_L^2 = P_L \quad \text{is projector} \]

\[ (\hat{W}_L(a) X^+_L)(\mu) = (P_L a X^+_L)(\mu) = \Phi^+_L(\mu) \quad (4) \]

\[ X^+_L(\mu), \Phi^+_L(\mu) \subset E_{2, L} \]
\[
H = L_2(0, L) \iff (f, g)_H = \int_0^L f(x)g(x)dx
\]
\[
(W_L(a)f, f)_H = (\chi_{(0,L)}\mathcal{F}^{-1}a\mathcal{F}f, f)_H = (\mathcal{F}^{-1}a\mathcal{F}f, f)_{L^2(\mathbb{R})}
\]
\[
= (a\mathcal{F}f, \mathcal{F}f)_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} a(\mu)\left|\mathcal{F}(\mu)\right|^2d\mu
\]
\[
\mathcal{F} = \mathcal{F}^*
\]

Re \(a \geq 0\) \implies \(W_L(a)\) is semisectorial

Re \(a \geq \varepsilon > 0\) \implies \(W_L(a)\) is sectorial

Re \((W_L(a)f, f) \geq \varepsilon(\mathcal{F}f, \mathcal{F}f) = \varepsilon(f, f)\)

**Theorem 1 (Brown and Halmos).** Let \(a \in L_\infty(\mathbb{R})\) and suppose that Re \(a \geq \varepsilon > 0\). Then \(W_L(a)\) is invertible and for all \(L > 0\) we have

\[
\|W_L^{-1}(a)\| \leq \frac{1}{\varepsilon} \left(1 + \sqrt{1 - \frac{\varepsilon^2}{\|a\|_\infty^2}}\right) \leq \frac{2}{\varepsilon}
\]

where

\[
\|a\|_\infty = \text{ess sup}_{\mu \in \mathbb{R}}|a(\mu)|.
\]

**Theorem 2 (A. Böttcher, 1994).** Let

\[
W_\infty(a) = \chi_{(0,\infty)}\mathcal{F}^{-1}a\mathcal{F}\big|_{L^2(0,\infty)} : L^2(0, \infty) \to L^2(0, \infty)
\]

be a convolution operator on semi-axis and symbol \(a(\mu)\) is piecewise continuous on \(\mathbb{R}\) function such that \(W_\infty(a)\) is invertible. Then for \(L\) large enough \(W_L(a)\) is invertible and

\[
\lim_{L \to \infty} \|W_L^{-1}(a)\| = \|W_\infty^{-1}\|.
\]
Semisectoriality

\[ \tilde{a}(\mu) = a(\mu) + e^{iL\mu}h^+(\mu) + e^{-iL\mu}h^-(\mu) \]

\[ h^+(\mu) \in H_2, \quad h^-(\mu) \in \overline{H}_2 \]

\[ e^{iL\mu}h^+(\mu) = \int_{-L}^{\infty} f_+(x)e^{ix} \, dx \]

\[ e^{-iL\mu}h^-(\mu) = \int_{-\infty}^{-L} f_-(x)e^{ix} \, dx \]

\[ \downarrow \]

\[ W_L(a) = W_L(\tilde{a}) \]

If \( \tilde{a} \) is sectorial \( \Rightarrow \) \( W_L(a) \) is invertible

\[ u(\mu) := \text{Re} \, a(\mu), \quad v := \text{Im} \, a(\mu) \]

**Def.:** \( \mu_j \in \mathbb{R} \) is said to be a (finite) zero of \( u \) if

\[ \text{ess inf} \{ u(\mu) : |\mu - \mu_j| < \delta \} = 0 \text{ for each } \delta > 0. \]

\[
\frac{1}{w_j(L)} := \text{ess inf} \{ u(\mu) : 1/L < |\mu - \mu_j| < \delta \}
\]

\[
\lim_{L \to \infty} w_j(L) = \infty
\]

**Upper estimates**

**Theorem 3.** Let \( a \in L_\infty(\mathbb{R}) \). Suppose \( u \geq 0 \) a.e. and \( u \) has exactly \( n \geq 1 \) finite zeros \( \mu_1, \ldots, \mu_n \) on \( \mathbb{R} \). Put \( w(L) = \max(w_1(L), \ldots, w_n(L)) \). Then for all \( L \) large enough the operator \( W_L(a) \) is invertible and

\[ \|W_L^{-1}(a)\| \leq 8(\|v\|_\infty + 1)w(9L/(2\pi)). \]
Sound propagation in an air half-space with motorway

\[ \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = -\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad (5) \]

\[ \frac{\partial p}{\partial z}(x, y, 0) = 0, \quad (x, y) \in (0, a) \times \mathbb{R} \quad \text{— motorway} \quad (6) \]

\[ \frac{\partial p}{\partial z}(x, y, 0) + \nu p(x, y, 0) = 0, \quad (x, y) \in (\mathbb{R} \setminus (0, a)) \times \mathbb{R} \quad \text{— ground} \quad (7) \]

Re \( \nu > 0 \)

\[ k^2 = k_0^2 + i\varepsilon \quad (k_0 > 0, \varepsilon > 0) \quad \text{— wave number} \quad (8) \]

LAP (Limit Absorption Principle)

\[ \varepsilon > 0 \Rightarrow p_\varepsilon(x, y, z) \]

\[ \lim_{\varepsilon \to 0} p_\varepsilon(x, y, z) = p_0(x, y, z) \]
\[ p(x, y, z) = p_0(x, y, z) + \frac{k_0^2 \nu}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{X_L^+(\mu, \beta)}{\nu + \gamma(\mu, \beta)} \exp(-ik_0(\mu x + \beta y - \gamma(\mu, \beta)z)) d\mu d\beta \]  

\[ \gamma(\mu, \beta) = \sqrt{n^2 - \mu^2 - \beta^2}, \quad L = k_0 a, \quad n^2 = \frac{k_0^2}{k_0^2} = 1 + \frac{i\varepsilon}{k_0^2} \]  

\[ (\mathcal{W}G_\varepsilon X_{\varepsilon,L}^+)(\mu, \beta) = \Phi_L(\mu, \beta) \]  

\[ \Phi_L(\mu, \beta) = P_L \left\{ \frac{1}{2\pi i k_0} \cdot \frac{\exp(ik_0(\mu x_0 + \gamma(\mu, \beta)z_0))}{\nu + \gamma(\mu, \beta)} \right\} \]  

\[ G_\varepsilon(\mu, \beta) = \frac{\gamma(\mu, \beta)}{\nu + \gamma(\mu, \beta)} \quad \text{symbol} \]  

\[ G_0(\mu, 0) = \frac{\sqrt{1 - \mu^2}}{\nu + \sqrt{1 - \mu^2}} \]  

\[ G_0(\mu, \beta), \quad \mu \in \mathbb{R}, \ |\beta| < 1 \quad G_\varepsilon(\mu, \beta), \quad \mu \in \mathbb{R}, \ |\beta| \geq 1 \]  

\[ G_0(\pm\sqrt{1 - \beta^2}, \beta) \quad G_\varepsilon(\pm\sqrt{1 - \beta^2}, \beta) \]  

semisectorial \quad \text{sectorial}
Theorem 6. Let \( L \in (0, \infty) \) and \( \varepsilon \in [0, \varepsilon_0] \) where \( \varepsilon_0 > 0 \) and small enough. Then equation (10) has a unique solution in the space \( E_{2,L} \) for every \( \beta \in \mathbb{R} \). Moreover for each \( L \in (0, \infty) \) there exists an independent of \( \varepsilon \in [0, \varepsilon_0] \) constant \( M_L \) such that

\[
\|X_{\varepsilon,L}(\mu, \beta)\|_{L^2(\mathbb{R}^2)} \leq M_L
\]

and

\[
\lim_{\varepsilon \to 0} \|X_{\varepsilon,L}(\mu, \beta) - X_{0,L}(\mu, \beta)\|_{L^2(\mathbb{R}^2)} = 0.
\]

**DF.** \( f(x, y, z) \in ML_2 \iff \)

1. \( f(x, y, z) \in C^2(\mathbb{R}^2 \times (0, \infty)) \);

2. \( \xi(z), \xi_1(z) \in C[0, \infty] \)

\[
\xi(z) := \|f(\cdot, \cdot, z)\|_{L^2(\mathbb{R}^2)}, \quad \xi_1(z) := \left\| \frac{\partial f}{\partial z}(\cdot, \cdot, z) \right\|_{L^2(\mathbb{R}^2)},
\]

and

\[
\lim_{z \to 0} \|f(\cdot, \cdot, z) - f(\cdot, \cdot, 0)\|_{L^2(\mathbb{R}^2)} = 0,
\]

\[
\lim_{z \to 0} \left\| \frac{\partial f}{\partial z}(\cdot, \cdot, z) - \frac{\partial f}{\partial z}(\cdot, \cdot, 0) \right\|_{L^2(\mathbb{R}^2)} = 0;
\]

3. \( \lim_{z \to \infty} \xi(z) = 0, \lim_{z \to \infty} \xi_1(z) = 0. \)
Theorem 7. The problem (5)–(8) for arbitrary $\varepsilon > 0$ has a unique solution of the form (9)

$$p_\varepsilon(x, y, z) = p_{\varepsilon, \delta}(x, y, z) + \varphi_\varepsilon(x, y, z)$$

where $\varphi_\varepsilon(x, y) \in ML_2$ and for every $(x, y, z) \in \mathbb{R}^2 \times (0, \infty)$ there exist

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon(x, y, z) = \varphi_0(x, y, z),$$

$$\lim_{\varepsilon \to 0} p_{\varepsilon, \delta}(x, y, z) = p_0(x, y, z).$$
1. Along the motorway

**Theorem 8.** Let the coordinates \((x_0, y_0, z_0)\) of a source be fixed and fix the coordinates \(x\) and \(z\) of the receiver \((x, y, z)\). Then

\[
\varphi_0(x, y, z) = c \frac{e^{i \omega (y - y_0)}}{(y - y_0)^2} + o \left( \frac{1}{(y - y_0)^2} \right) \quad \text{as} \quad |y| \to \infty
\]

where

\[
c = \frac{1 - ik_0 z \nu}{\nu} X_{0, L}^+ (0, 1) + 2 i \nu c_0 b_0.
\]

2. Outside the motorway

**Theorem 9.** Let \(\psi \in (-\pi, \pi) \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}\). Then for fixed \(z\) and \((x_0, y_0, z_0)\) we have

\[
\varphi_0(R \cos \psi, R \sin \psi) = \frac{1 - ik_0 z \nu}{\nu} X_{0, L}^+ (-\cos \psi, -\sin \psi) e^{i \omega R} + o \left( \frac{1}{R^2} \right)
\]

where \(R = \sqrt{x^2 + y^2} \to \infty\).
Asymptotics by $L \to \infty$

Two-dimensional problem

$$\frac{\partial^2 p(x, z)}{\partial x^2} + \frac{\partial^2 p(x, z)}{\partial z^2} + k^2 p(x, z) = -\delta(x - x_0)\delta(z - z_0),$$ \hspace{1cm} (12)

$$(x, z) \in \mathbb{R} \times (0, \infty)$$

$$\frac{\partial p(x, 0)}{\partial z}(x, 0) = 0, \quad x \in (0, a)$$ \hspace{1cm} (13)

$$\frac{\partial p(x, 0)}{\partial z} + i\nu p(x, 0) = 0, \quad x \in \mathbb{R} \setminus (0, a)$$ \hspace{1cm} (14)

$$\text{Re} \ \nu > 0$$

$LAP$ \hspace{1cm} (15)

$$p(x, z) = p_0(x, z) + \varphi_L(x, z)$$

$$\varphi_L(x, z) = \frac{k_0 \nu}{2\pi} \int_{\mathbb{R}} \frac{X_L^-(\mu)}{\nu + \gamma(\mu)} e^{-ik(\mu x - \gamma(\mu)z)} \, d\mu$$ \hspace{1cm} (17)

$$\gamma(\mu) = \sqrt{1 - \mu^2}$$

$$(W_L G)(\mu) X_L^+(\mu) = (P_L G)(\mu) X_L^+(\mu) = f_L(\mu)$$ \hspace{1cm} (18)

$$G(\mu) = \frac{\gamma(\mu)}{\nu + \gamma(\mu)}, \quad f_L(\mu) = P_L \left( e^{ik_0(x_0 + z_0 \gamma(\mu))} \right) \frac{1}{2\pi k(\nu + \gamma(\mu))}$$

$$X_L^+(\mu) \in \mathcal{E}_{2, \mu, \mathcal{E}_5}$$

$$\varphi_s(\mu) = (|\mu| + 1)^{-2s}|1 - \mu|^s|1 + \mu|^s, \quad s \in (-1, 1)$$
$P^+ = P_{(0,\infty)}$, \( T(G) = P^+ G P^+ \)

\[ J := J(f(x)) = f(-x) \quad W_L = e^{iLx} J P_L \]

\[ G(\mu) = G_+(\mu)G_-(\mu) \quad (19) \]

\[
G_+(\mu) = \frac{\sqrt{1+\mu}}{a_+(\mu)}, \quad G_-(\mu) = \frac{\sqrt{1-\mu}}{a_-(\mu)}
\]

\[ \nu + \sqrt{1 - \mu^2} = a_+(\mu)a_-(\mu) \]

\[ T^{-1}(G) = G^{-1}_+ P^+ G^{-1}_- : P^+ L_2(\mathbb{R}, \varrho_{s+1}) \to P^+ L_2(\mathbb{R}, \varrho_s) \]

\[ K(G) = T^{-1}(G) - T(G^{-1}) \]

\[ B_L(G) = P_L T^{-1}(G) P_L + W_L(K(G)) W_L \quad (20) \]

**Theorem 10.** Let \( \varepsilon = 0 \). Then for arbitrary \( L > 0 \) operator

\[ \hat{W}_L(G) = P_L G P_L : L_2(\mathbb{R}, \varrho_s) \to L_2(\mathbb{R}, \varrho_s), \ s \in (-1, 1), \]

is invertible and for \( L \geq 1 \) the following evaluation holds

\[ \| T_L^{-1}(G) \|_{L^2(\mathbb{R}, \varrho_s)} \leq c_1 L^{1/2} \]

where \( c_1 \) does not dependent on \( L \).
\[ \|X_L^+(\mu) - (B_L(G)f_L)(\mu)\|_{L^2(\mathbb{R},e_\theta)} \leq c_2 L^{-s/2}, \quad s \in (0,1) \]

\[ \varphi_\infty(x,z) = \frac{k_0 \nu}{2\pi} \int_{\mathbb{R}} \frac{B_L(G)f_L(\mu)}{\nu + \gamma(\mu)} e^{-ik(\mu x - \gamma(\mu)z)} d\mu \]

\[ |\varphi_L(x,z) - \varphi_\infty(x,z)| \leq c_3 L^{-s/2} \]

where \( \varepsilon_3 \) does not dependent on \( L \geq 1 \) and \( (x,z) \in \mathbb{R} \times (0,\infty) \)

\[ B_L(G) \sim P^+(T^{-1}(G) + T^{-1}(\bar{G}) - G^{-1})P^+ \]