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ORIGINAL PAPER

Eigenvalue asymptotic expansion of large tetradiagonal Toeplitz matrices: cusp case

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Abstract

In a paper from 2021, Albrecht Böttcher, Juanita Gasca, Sergei M. Grudsky, and Anatoli V. Kozak gave a precise and complete description of all types of the limit Schmidt–Spitzer sets for tetradiagonal Toeplitz matrices. In this paper, we consider one of these possible cases, when the limit set consists of two analytic arcs that join at one point forming a cusp. For this family of Toeplitz matrices, we provide asymptotic formulas for every eigenvalue as the order of the matrix tends to infinity. Our analysis provides a theoretical understanding of the structural behavior of the eigenvalues, while the obtained formulas enable high-precision calculation of the eigenvalues.

Keywords Tetradiagonal Toeplitz matrix · Eigenvalue · Asymptotic formulas · Simple-loop method

Mathematics Subject Classification 15A18 · 15B05 · 41A60 · 65F15

1 Introduction

The behavior of the spectral characteristics of Toeplitz matrices as their dimension tends to infinity has been intensively studied since the beginning of the last century.

Dedicated to the memory of our dear friend Nikolai Vasilevski

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The starting point is Szegő's article [30] (see also monographs [7, 14, 15, 20] and the literature cited therein). Thereafter, there were numerous versions of Szegő's theorem on the asymptotic distribution of eigenvalues and the Avram–Parter-type theorems on the asymptotic distribution of singular values [2, 11, 25, 27, 32]. An extensive literature is devoted to the asymptotics of Toeplitz matrix determinants (see, for example, monographs [7, 15], papers [16, 18, 22, 36] and the literature cited there). Much attention is given to the asymptotics of the largest and smallest eigenvalues [13, 24, 28, 34, 35].

Such a great interest in the asymptotic behavior of spectral characteristics of large Toeplitz matrices is motivated, to a very significant extent, by numerous important applications, including the numerical solution of differential and integral equations [21], stochastic processes and time series analysis [20], signal and image processing [23, 26], quantum mechanics [21], etc.

However, despite the profound interest of many researchers in this area, the undeniably important problem of individual asymptotic formulas of all eigenvalues and eigenvectors was not studied until 2009.

A substantial body of work focuses on numerical methods for finding the spectrum of Toeplitz matrices for large values of their dimensions (see [19, 31, 37] and the literature cited there). In the work [3] (see also [5, 6, 17] and review [6]), asymptotic expansions were constructed for all eigenvalues of Toeplitz matrices with real-valued symbols (the self-adjoint case) satisfying the so-called SL (simple loop) condition. The latter means that the real-valued symbol under consideration has exactly one minimum and one maximum on the unit circle. In the indicated articles, classes of polynomial, infinitely differentiable, and finitely differentiable symbols were successively considered. The case of complex-valued symbols (non-self-adjoint Toeplitz matrices) is more delicate.

The classical work [29] of Schmidt and Spitzer gives a description of the set for which the eigenvalues of a sequence of Toeplitz matrices *converges* as the dimension of the matrices tends to infinity. There was also proved that this set consists of a finite number of analytic arcs (which, as was established somewhat later in [33], form a connected set in the complex plane) and their analytic description was given. We refer to the aforementioned set as *the limit set*.

Papers [4, 12] are particular cases of the application of [29], in both situations the limit set of eigenvalues coincides with the image of the symbols when the variable runs over the unit circle. Taking advantage of this situation, uniform asymptotic representations for all eigenvalues were constructed for matrices with symmetric symbols and symbols having power singularities on the unit circle. In the case of complex polynomial symbols, the limit set is a structure of a more complicated nature.

In the article [10], a significant refinement of the results from [29] was obtained for the case of a tetradiagonal Toeplitz matrix. There, it was proven that the set of all tetradiagonal Toeplitz matrices is divided into three classes, each of which has the limit set consisting of one, two, or three analytic arcs, respectively. Based on these results, asymptotic expansions of all eigenvalues were obtained in [8, 9] in the case of a limit set consisting of one arc.

In this paper, we consider the problem of deriving asymptotic expansions for all eigenvalues of a tetradiagonal Toeplitz matrix whose limit set consists of two analytic arcs. Note that in this case these arcs form a cusp. Outside the neighborhood of the cusp, we use a technique similar to the SL case, yielding asymptotic formulas of a similar structure. Near the cusp, however, the SL method fails, necessitating a different approach. Nevertheless, we still have a small set of eigenvalues not represented by any of these two kinds of asymptotic expansions. However, numerical experiments show that the asymptotic approximations work for all the eigenvalues. In a future paper, we will use another approach that reveals a unique asymptotic formula encompassing all of them.

To conclude, we note that the asymptotic method for calculating eigenvalues is of fundamental importance in the case of complex-valued polynomial symbols. Standard numerical algorithms encounter the problem of large instability, as a consequence, for usual 16-precision digits, the results of calculations cease to be adequate already for matrix dimensions in the range of 500–1000. To overcome this effect, it is necessary to increase accuracy (64-precision digits and more) which sharply reduces the speed of the algorithms used. The presented formulas serve as the basis for algorithms, with significantly higher calculation speed. Namely, we have complexity of order $O(n)$, where n is the dimension of the matrix, and provide good accuracy already for n of the order of 100, and this accuracy, naturally, improves with the growth of n.

This paper is structured as follows: in Sect. 2, we present our main results, concerning the asymptotics of the eigenvalues for tetradiagonal Toeplitz matrices in the case, where the limit set consists of two analytic arcs. In Sect. 3, we recall some known facts about the limit set and provide new ones. In Sect. 4, we provide some properties of the main actors of our analysis. To get the asymptotic expansions of the eigenvalues, two different approaches are needed, one when the eigenvalues are *far* from the bifurcation point, and another when they are *close* to it; we call them *inner* and *cusp* eigenvalues, respectively. The analysis performed on the characteristic equation and the proofs for the construction of the eigenvalue asymptotic expansions, are presented in Sects. 5 and 6, for inner and cusp eigenvalues, respectively. Finally, in Sect. 7, we present the numerical results from a test conducted using our formulas.

2 Main results

Given a Laurent polynomial $b(z) := b_{-1}z^{-1} + b_0 + b_1 z + b_2 z^2$ with complex coefficients, let $T_n(b)$ be the $n \times n$ tetradiagonal Toeplitz matrix generated by b. For example, for $n = 5$

$$T_5(b) = \begin{bmatrix} [r]b_0 & b_{-1} & 0 & 0 & 0 \\ b_1 & b_0 & b_{-1} & 0 & 0 \\ b_2 & b_1 & b_0 & b_{-1} & 0 \\ 0 & b_2 & b_1 & b_0 & b_{-1} \\ 0 & 0 & b_2 & b_1 & b_0 \end{bmatrix}.$$

In [10], by the use of the Schmidt and Spitzer trick [29, Lemma 3.1], the authors showed that the analysis of the spectrum of $T_n(b)$ can be divided into two main cases.

For instance, if $b_1 = 0$, choosing ρ, so that $\rho^2 b_2 = \rho^{-1} b_{-1}$, then

$$\sigma(T_n(b)) = b_0 + \rho^2 b_2 \sigma(T_n(z^2 + z^{-1})).$$

If $b_1 \neq 0$, choosing ρ, so that $\rho b_1 = \rho^{-1} b_{-1}$ leads to

$$\sigma(T_n(b)) = b_0 + \rho^2 b_2 \sigma(T_n(z^2 + cz + cz^{-1})) \tag{2.1}$$

with $c = \rho b_1/(\rho^2 b_2)$.

Taking into consideration (2.1), for $c \in \mathbb{C} \setminus \{0\}$, we define

$$a(z) := z^2 + cz + cz^{-1} \qquad (z \in \mathbb{C} \setminus \{0\}). \tag{2.2}$$

For every λ in $\mathbb{C}$, we consider the equation

$$a(z) - \lambda = 0. \tag{2.3}$$

Let $z_1(\lambda)$, $z_2(\lambda)$ and $z_3(\lambda)$ be the solutions of (2.3) labeled in such a way that $|z_1(\lambda)| \leq |z_2(\lambda)| \leq |z_3(\lambda)|$. Following Schmidt–Spitzer [29], the limit set associated to (2.2) is defined by

$$\Lambda(a) := \{\lambda \in \mathbb{C} : \ |z_1(\lambda)| = |z_2(\lambda)| \leq |z_3(\lambda)|\}. \tag{2.4}$$

According to [29], the sequence of sets of eigenvalues of $T_n(a)$ converges to $\Lambda(a)$ in the Hausdorff metric as $n \to \infty$. Therefore, for every n sufficiently large, it is natural to *search* the eigenvalues of $T_n(a)$ in a neighborhood of $\Lambda(a)$.

In [10, Theorem 4.5], it was stated that $\Lambda(a)$ is the union of two analytic arcs with one intersection point, if and only if the parameter c belongs to the set

$$\Gamma := \left\{ \pm \frac{2(1+\ell+\ell^2)^{3/2}}{\ell} : \ \ell \in \gamma \right\}, \tag{2.5}$$

where

$$\gamma := \left\{ \rho e^{i\vartheta} : \ \cos\vartheta = \frac{4\rho^4 - \rho^2 - 1}{2\rho}, \ \rho \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2} \right] \right\}.$$

In [10, Theorem 4.5], it was also established that if c lies *inside* the region that Γ encloses, then $\Lambda(a)$ has three analytic arcs, while if c is *outside*, then $\Lambda(a)$ consists of only one analytic arc. The curve (2.5) is shown in Fig. 1.

The case where $\Lambda(a)$ consists of only one analytic arc was considered by Bogoya, Gasca, and Grudsky in [8, 9]; they provide a complete description of the eigenvalue asymptotic expansion of $T_n(a)$ as $n \to \infty$.

In this paper, we study the individual behavior of the eigenvalues of $n \times n$ Toeplitz matrices $T_n(a)$ as $n \to \infty$, where a is given by (2.2) with the parameter c in (2.5). We remark that, we use ideas and techniques similar to the ones employed in [8, 9].

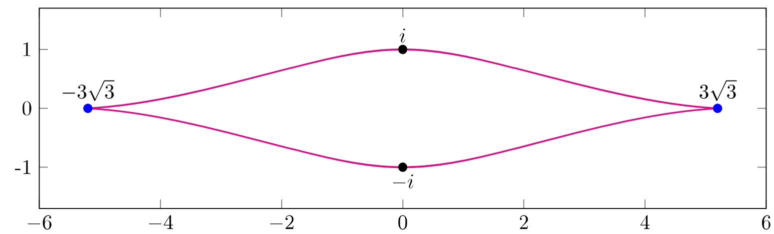

Fig. 1 Set Γ in pink

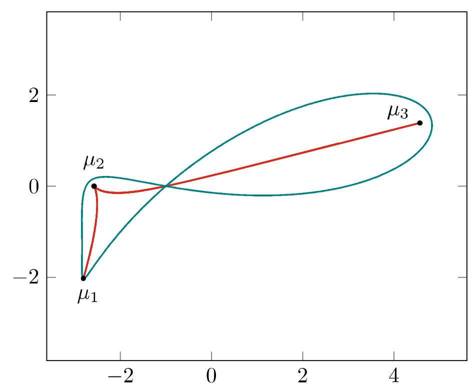

Fig. 2 In teal the set $a(\mathbb{T})$, in red $\Lambda(a)$ and in black the values μ_1, μ_2, μ_3, for some parameter c in Γ

Denote by t_1, t_2, t_3 the solutions of $a'(z) = 0$ labeled in such a way that $|t_2| \geq |t_1| \geq |t_3|$. By [10, Theorems 4.3, 4.5] (cf. Theorem 3.1 and Proposition 3.4), $\Lambda(a)$ is the union of two analytic arcs, one that initiates at $\mu_1 \equiv a(t_1)$ and terminates at $\mu_2 \equiv a(t_2)$, and the second going from $\mu_3 \equiv a(t_3)$ to μ_2; moreover, these arcs form a cusp at μ_2. Figure 2 shows the limit set for some parameter c in Γ, notice the *skeleton* shape with respect to the range of a restricted to the unit circle $\mathbb{T}$.

From [10, Theorems 4.3, 4.5], it is guaranteed the existence of a value φ_2 in $(0, 2\pi)$, such that the roots of $a(z)-\mu_2$ are the numbers $t_2, t_2, e^{i\varphi_2}t_2$. Let $\varphi_1 := 0$ and $\varphi_3 := 2\pi$. Now, we define the *natural* parametrization function of the limit set by

$$\psi(s) := a(u(s)), \tag{2.6}$$

where u is an analytic function defined on some complex open neighborhood of $[0, 2\pi]$ that satisfies $a(u(s)) - a(e^{is}u(s)) = 0$ for every $s \in [0, 2\pi]$. Proposition 3.13 proves that u restricted to $[0, 2\pi]$ is one to one, $u(\varphi_1) = t_1$, $u(\varphi_2) = t_2$, $u(\varphi_3) = t_3$, and $a(u[0, 2\pi]) = \Lambda(a)$. In Proposition 3.14, we show that ψ inherits these properties.

In Remark 3.16, we show how to numerically compute u on the grid $2\pi k/N$ ($j = 1, \ldots, N$) of $[0, 2\pi]$, making it possible to construct a polynomial approximation of u. Figure 3 shows the range of this approximation restricted to $[0, 2\pi]$, for some parameter c; exclusively for visual reference, we also plot the unit circle.

To perform the forthcoming analysis, we introduce the supporting actors of this paper. Justified by Proposition 3.13, there exists $M > 0$, such that for every $k \geq 1$ and

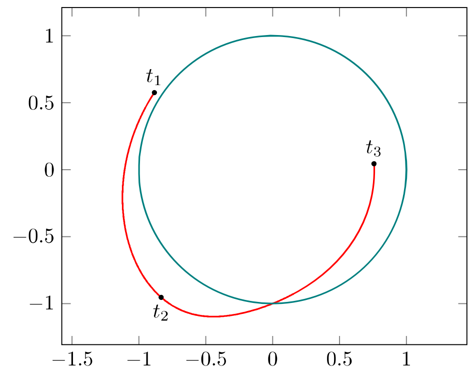

Fig. 3 In red the set $u([0, 2\pi])$, in black the points t_1, t_2, t_3 and in teal the unit circle for some $c \approx 1.427071 + i$

every small $\upsilon \geq 0$, u is analytic on the set

$$\Omega_{\upsilon,k} := \left\{ z \in \mathbb{C}\colon\ 0 \leq \Re(z) \leq 2\pi,\ \ \upsilon < |\Re(z) - \varphi_2|,\ \ |\Im(z)| \leq \frac{M}{k} \right\}. \tag{2.7}$$

To simplify notation, put $\Omega_0 \equiv \Omega_{0,1}$ and $\Omega_\upsilon \equiv \Omega_{\upsilon,1}$. For every $s \in \operatorname{clos}(\Omega_0)$, let

$$f(s) := -\frac{e^{is}u(s)^3}{c}, \tag{2.8}$$

$$p(s) := 1 - f(s), \qquad q(s) := 1 - e^{is} f(s). \tag{2.9}$$

For every small $\upsilon > 0$, and every $s \in \Omega_\upsilon$, define

$$\eta(s) := -i \ln \frac{q(s)}{p(s)}, \tag{2.10}$$

where ln stands for the principal branch of logarithm (see Lemma 4.3). Let $n \geq 3$, then for every $s \in \Omega_0$, set

$$r_n(s) := \frac{\sin \frac{s}{2} e^{i\left(\frac{n+2}{2}s - \frac{\eta(s)}{2}\right)} f(s)^{n+2}}{p(s)}, \qquad R_n(s) := -2 \arcsin(r_n(s)). \tag{2.11}$$

For every $j \in \mathbb{Z}$, put

$$d_{n,j} := \varphi_2 + \frac{2\pi j}{n+1} + \frac{2\pi \beta_n}{n+1}, \tag{2.12}$$

where

$$J_n := \left[\frac{(n+1)\varphi_2}{2\pi} \right], \qquad \beta_n := J_n - \frac{(n+1)\varphi_2}{2\pi},$$

with $[y]$ denoting the integer part of the number y. For $\delta > 0$, define the set

$$\mathrm{B}_{n,j} := \left\{ s \in \Omega_0 \colon |e_{n,j} - s| < \frac{\varepsilon_{n,j}}{(n+1)^2} \right\}, \tag{2.13}$$

where

$$e_{n,j} := d_{n,j} - \frac{\eta(d_{n,j})}{n+1}, \tag{2.14}$$

$$\varepsilon_{n,j} := \begin{cases} 3|\eta(d_{n,j})| \sup_{s\in\Omega_{\delta,n}} |\eta'(s)|, & \text{if } |\Re(s) - \varphi_2| > \delta, \\ \dfrac{3(n+1)|\eta(d_{n,j})|}{2\pi(|j|+1)}, & \text{if } |\Re(s) - \varphi_2| \le \delta. \end{cases} \tag{2.15}$$

Now, we present our main results. We start with the ones concerning the eigenvalues of $T_n(a)$ that are *far* from μ_2.

Theorem 2.1 *Let $1 \gg \epsilon > 0$, $n \in \mathbb{N}$, and $j \in \mathbb{Z}$ be, such that $d_{n,j} \in (0, 2\pi)$ and $|j|/n^{1/2+\epsilon}$ is large enough. Then, there exists a unique value $s_{n,j} \in \mathrm{B}_{n,j}$, such that $\psi(s_{n,j})$ is an eigenvalue of $T_n(a)$, and $s_{n,j}$ satisfies the equation:*

$$s = d_{n,j} - \frac{\eta(s)}{n+1} + \frac{(-1)^{j+J_n} R_n(s)}{n+1}, \tag{2.16}$$

where $|R_n(s_{n,j})| = O(n^{1/2}|j|^{-1/2} \exp(-6\pi^2 j^2/n))$. Moreover, $\psi(s_{n,j}) \ne \psi(s_{n,k})$, for every $j, k \in \mathbb{Z}$, such that $j \ne k$, $d_{n,j}, d_{n,k} \in (0, 2\pi)$, and $|j|/n^{1/2+\epsilon}$, $|k|/n^{1/2+\epsilon}$ are large enough.

Let n and j be as in Theorem 2.1 and let $s_{n,j}$ be the value in $\mathrm{B}_{n,j}$, such that $s_{n,j}$ satisfies (2.16). We call $\lambda_{n,j} \equiv \psi(s_{n,j})$ *inner eigenvalues*.

Theorem 2.2 *Let $1 \gg \epsilon > 0$, $n \in \mathbb{N}$, and let $j \in \mathbb{Z}$, such that $d_{n,j} \in (0, 2\pi)$ and $|j|/n^{1/2+\epsilon}$ is large enough. Then, there exists a unique value $s^*_{n,j} \in \mathrm{B}_{n,j}$ that satisfies the equation:*

$$s = d_{n,j} - \frac{\eta(s)}{n+1}. \tag{2.17}$$

Moreover, if $|d_{n,j} - \varphi_2| \ge \delta$, then

$$|\lambda_{n,j} - \psi(s^*_{n,j})| = O\left(\frac{1}{n^3}\right), \tag{2.18}$$

and if $\delta > |d_{n,j} - \varphi_2|$, then

$$|\lambda_{n,j} - \psi(s^*_{n,j})| = O\left(\frac{\ln \frac{n}{|j|}}{n^2 j}\right). \tag{2.19}$$

In the next theorem we derive asymptotic expansions for both $s_{n,j}$ and $\lambda_{n,j}$, where $s_{n,j}$ satisfies (2.16); for this purpose, for every $s \in \Omega_0$ we define

$$\mathfrak{v}_0(s) := s, \qquad \mathfrak{v}_1(s) := -\eta(s), \qquad \mathfrak{v}_2(s) := \eta(s)\eta'(s), \tag{2.20}$$

$$\mathfrak{l}_0(s) := \psi(s), \quad \mathfrak{l}_1(s) := \psi'(s)\mathfrak{v}_1(s), \quad \mathfrak{l}_2(s) := \psi'(s)\mathfrak{v}_2(s) + \frac{1}{2}\psi''(s)\mathfrak{v}_1(s)^2. \tag{2.21}$$

Theorem 2.3 (Asymptotic expansions of inner eigenvalues) *Let* $1 \gg \epsilon > 0$, $n \in \mathbb{N}$, *and* $j \in \mathbb{Z}$, *such that* $d_{n,j} \in (0, 2\pi)$ *and* $|j|/n^{1/2+\epsilon}$ *is large enough. Then*

$$s_{n,j} = \mathfrak{v}_0(d_{n,j}) + \frac{\mathfrak{v}_1(d_{n,j})}{n+1} + \frac{\mathfrak{v}_2(d_{n,j})}{(n+1)^2} + r_{n,j}, \tag{2.22}$$

$$\lambda_{n,j} = \mathfrak{l}_0(d_{n,j}) + \frac{\mathfrak{l}_1(d_{n,j})}{n+1} + \frac{\mathfrak{l}_2(d_{n,j})}{(n+1)^2} + R_{n,j}, \tag{2.23}$$

where

- $r_{n,j} = O(n^{-3})$ *and* $R_{n,j} = O(n^{-3})$ *if* $|d_{n,j} - \varphi_2| > \delta$, *or*
- $r_{n,j} = O\left(\ln(n/|j|)^2/(nj^2)\right)$ *and* $R_{n,j} = O\left(\ln(n/|j|)^2/(nj^2)\right)$ *if* $|d_{n,j} - \varphi_2| \leq \delta$.

For each $k = 1, 2, 3$, we denote the Taylor expansion of ψ around φ_k by

$$\psi(s) = \mu_k + \boldsymbol{\psi}_{k,1}(s - \varphi_k) + \boldsymbol{\psi}_{k,2}(s - \varphi_k)^2 + \boldsymbol{\psi}_{k,3}(s - \varphi_k)^3 + O|s - \varphi_k|^4,$$

where the coefficients $\boldsymbol{\psi}_{k,1}$, $\boldsymbol{\psi}_{k,2}$, and $\boldsymbol{\psi}_{k,3}$ are computed in Theorem 3.15.

Under the conditions of Theorem 2.3, we call the *extreme eigenvalues* $\lambda_{n,j}$, such that $\lambda_{n,j} \to \mu_k$ as $n \to \infty$, for $k = 1, 3$.

Corollary 2.4 (Asymptotic expansions of extreme eigenvalues) *Under the conditions of Theorem 2.3, let* $k = 1$ *and* $j' = j + J_n$, *or* $k = 3$ *and* $j' = n + 1 - J_n - j$. *If* $j'/n \to 0$, *then*

$$\lambda_{n,j} = \mu_k + 4\boldsymbol{\psi}_{k,2}\frac{\pi^2 j'^2}{(n+1)^2} + 2t_k^2\frac{\pi^2 j'^2}{(n+1)^3} + R_{n,j'}, \tag{2.24}$$

where $R_{n,j'} = O(j'^4/n^4)$.

In the case when $\Lambda(a)$ consists of one analytic arc, the asymptotic expansions derived in [8, Theorem 2.2] and [9, Theorem 2.4] are valid for all the eigenvalues of $T_n(a)$, as $n \to \infty$. In contrast, Theorem 2.3 does not guarantee either (2.22) or (2.23) if $d_{n,j} \to \varphi_2$ as $n \to \infty$, because the remaining terms in (2.22) and (2.23) are not small enough.

Thus, our last results pertain to the eigenvalues that approaches μ_2 as $n \to \infty$. For this purpose, for every $n \in \mathbb{N}$ we define the function:

$$\zeta_n(z) := z\frac{z^{n+1} - 1}{z - 1} \qquad (z \neq 1), \tag{2.25}$$

and for every $n \in \mathbb{N}$, every $j \in \mathbb{Z}$ and every $A > 0$, we put

$$\chi_{n,j} := \varphi_2 + \frac{j\pi}{n+1}, \tag{2.26}$$

$$\mathrm{U}_{n,j,A} := \left\{ s \in \mathbb{C}\colon |\chi_{n,j} - s| \le \frac{A|j|^2}{(n+1)^2} \right\}. \tag{2.27}$$

Theorem 2.5 *There exists $A > 0$, such that for every $n \in \mathbb{N}$ large enough, and every $j \in \mathbb{Z} \setminus \{0\}$, such that $\chi_{n,j} \in (0, 2\pi)$ and $j = o(n^{1/2})$, there exists a unique $s_{n,j} \in \mathrm{U}_{n,j,A}$ that satisfies the equation:*

$$\zeta_n(f(s)) = \zeta_n(e^{-is}). \tag{2.28}$$

Moreover, the value $\psi(s_{n,j})$ is an eigenvalue of $T_n(a)$.

Let n and j be as in Theorem 2.5, we call $\lambda_{n,j} \equiv \psi(s_{n,j})$, such that $s_{n,j}$ satisfies (2.28) and $|j|/n^{1/2} \ll 1$ *cusp eigenvalues* (observe that the index j has a different meaning than the one given in Theorem 2.1).

For every n and every j, let

$$\kappa_{n,j} := -\frac{(-1)^j e^{-i(n+1)\varphi_2} - 1}{1 - e^{i\varphi_2}}. \tag{2.29}$$

Theorem 2.6 (Asymptotic expansions of cusp eigenvalues) *For every n large enough and every $1 \le j \le n$, such that $j = o(n^{1/2})$, it follows that*

$$\lambda_{n,j} = \mu_2 + \boldsymbol{\psi}_{2,2}\frac{(j\pi)^2}{(n+1)^2} - 2\boldsymbol{\psi}_{2,2}\kappa_{n,j}\frac{(j\pi)^2}{(n+1)^3} + R_{n,j}, \tag{2.30}$$

where $R_{n,j} = O\left(j^5/n^4\right)$.

Remark 2.7 Despite the results stated in Theorems 2.3 and 2.6, there exists a small set of eigenvalues for which we have not provided an asymptotic expansion. However, numerical experiments show that (2.23) works for these eigenvalues.

3 The limit set and the generating function

Recall that $\Lambda(a)$ is the limit set defined by (2.4), where a is the Laurent polynomial given by (2.2) with parameter c in (2.5). In this section, we state several facts about $\Lambda(a)$, and derive the construction of u and ψ mentioned in (2.6). Some of the following results were proved in [10, 29].

Here, we use the idea given in [10], which is to analyze the equation $a(z) - a(e^{i\varphi}z) = 0$. Since, for every $z, \tau \in \mathbb{C} \setminus \{0\}$:

$$a(z) - a(\tau z) = \tau^{-1}z^{-1}(1-\tau)\left(\tau(1+\tau)z^3 + c\tau z^2 - c\right), \tag{3.1}$$

studying the roots of (3.1) is equivalent to studying the roots of the polynomial:

$$\Phi(z,\tau) := \tau(1+\tau)z^3 + c\tau z^2 - c \qquad (\forall z,\tau \in \mathbb{C}). \tag{3.2}$$

Observe that, unlike (3.1), Φ does not have any singularity at $z = 0$.

Notice also that

$$a'(z) = \frac{2z^3 + cz^2 - c}{z^2} = \frac{\Phi(z,1)}{z^2}.$$

So, let t_1, t_2, t_3 be the solutions of $\Phi(z,1) = 0$ labeled, such that $|t_2| \geq |t_1| \geq |t_3|$, and denote $\mu_k := a(t_k)$ for each $k = 1, 2, 3$. Then, for each $k = 1, 2, 3$, the roots of $a(z) - \mu_k$ are the values

$$t_k, \quad t_k, \quad -c(t_k)^{-2}.$$

Next theorem says that μ_k are the branch points of $\Lambda(a)$, i.e., that $a(z) - \mu_k$ has multiple roots. The corresponding proof is in [10, Theorem 4.3].

Theorem 3.1 *For each $k = 1, 2, 3$, $\mu_k \equiv a(t_k)$ is a branch point of $\Lambda(a)$. Moreover, there exists $\varphi_2 \in (-\pi, \pi] \setminus \{0\}$, such that the roots of $a(z) - \mu_2$ are $t_2, t_2, e^{i\varphi_2} t_2$.*

For some parameter c in (2.5), the black dots in Figs. 2 and 3 show the values μ_k and t_k, respectively.

Fix φ_2 to be the value from Theorem 3.1, such that $t_2, t_2, e^{i\varphi_2} t_2$ are the roots of $a(z) - \mu_2$. Vietta's theorem for the equation $a(z) - \mu_2 = 0$ implies

$$-c = t_2^3 e^{i\varphi_2}, \qquad -\mu_2 = t_2^2(1 + 2e^{i\varphi_2}), \qquad -c = t_2(2 + e^{i\varphi_2}). \tag{3.3}$$

From the second equality in (3.3), we get

$$\mu_2 = -t_2^2(1 + 2e^{i\varphi_2}) = -t_2^2 e^{i\varphi_2}(2 + e^{-i\varphi_2}).$$

From the third equality in (3.3), $2 + e^{-i\varphi_2} = -\overline{c}/\overline{t_2}$, hence

$$\mu_2 = \frac{t_2^2 e^{i\varphi_2}\overline{c}}{\overline{t_2}} = \frac{t_2^3 e^{i\varphi_2}\overline{c}}{|t_2|^2}.$$

From the first equality in (3.3), $t_2^3 e^{i\varphi_2}\overline{c} = -|c|^2$ and $|t_2|^2 = |c|^{2/3}$, so

$$\mu_2 = -\frac{|c|^2}{|c|^{2/3}} = -|c|^{4/3}.$$

Similar procedures yield $\mu_2 = -5 - 4\cos(\varphi_2)$ as well as $|c| \geq 1$, equality holding only for $c = \pm i$ $(\varphi_2 = \pm\pi)$.

A simple analysis shows that (3.2) is an irreducible polynomial in two variables. Let $\mathcal{D}$ be the discriminant of Φ and Φ_z. Denote by Ξ the set consisting of the zeros of

$\tau \mapsto \tau(1+\tau)$ and the zeros of $\mathcal{D}$. If $\tau_0 \in \mathbb{C} \setminus \Xi$, then the equation $\Phi(z, \tau_0)$ has three distinct roots $\omega_1, \omega_2, \omega_3$. Under these conditions we have the following Lemma [1, Chapter 8, Section 2, Lemma 1].

Lemma 3.2 *There exists an open disk $V_0 \subseteq \mathbb{C}\backslash\Xi$, containing τ_0, and three analytic functions ζ_j^0, ζ_2^0, ζ_3^0 defined on V_0 with these properties:*

1. $\Phi(\zeta_j^0(\tau), \tau) = 0$ *in* V_0*;*
2. $\zeta_j^0(\tau_0) = \omega_j$*;*
3. *if* $\Phi(z, \tau) = 0$*, with* $\tau \in V_0$*, then* $z = \zeta_j^0(\tau)$ *for some* j*.*

The collection of all pairs (V_0, ζ_j^0) from Lemma 3.2, forms what is called an algebraic function corresponding to the polynomial Φ, [1, Chapter 8, Section 2].

Our next lemma is a direct application of Lemma 3.2.

Lemma 3.3 *There exists an open neighborhood U of $(-\pi, \pi)$ in $\mathbb{C}$ not containing π or $-\pi$, and analytic functions u_1, u_2, u_3 on U, such that*

1. $\Phi(u_j(s), e^{is}) = 0$ *for every* $j = 1, 2, 3$ *and every* s *in* U*;*
2. $u_j(s) \neq u_k(s)$ *for every* s *in* U *and every* $j \neq k$*.*

Proof For every $s \in (-\pi, \pi)$, we can apply Lemma 3.2 with $\tau_0 = e^{is}$, since $\tau_0 \notin \Xi$. Then, for every $s \in (-\pi, \pi)$ there exists an open disk U_s containing s and three analytic functions u_j^s defined on U_s with the properties given in Lemma 3.2, and such that $u_j^s(s^*) \neq u_k^s(s^*)$ for every $s^* \in U_s$ and every $j \neq k$. The conclusion follows by an usual analytic extension argument on the obtained pairs (U_s, u_j^s). □

Let u_k be as in Lemma 3.3, we label them, so that $u_k(0) = t_k$. Notice also that $\Phi(e^{is}u_k(s), e^{-is}) = 0$ for every $k = 1, 2, 3$ and every s in $(0, \pi)$, hence, $u_k(-s) = e^{is}u_k(s)$.

Observe that $\Phi(0, \tau) = -c \neq 0$, so, for every s in $(0, 2\pi)$, $u_k(s) \neq 0$. For every $k = 1, 2, 3$, define $z_{3,k}(s) := -ce^{-is}(u_k(s))^{-2}$.

The next step is to construct a nice parametrization function, taking as *bricks* the compositions $a \circ u_k$. Thus, our next results describe the necessary properties for this purpose.

Proposition 3.4 *Let $k = 1, 2, 3$. For every $s \in [0, \pi)$, the solutions of the equation $a(z) = a(u_k(s))$ are*

$$u_k(s), \quad e^{is}u_k(s), \quad z_{3,k}(s).$$

Proof From Lemma 3.3, $a(u_k(s)) = a(e^{is}u_k(s))$. Immediately follows that $u_k(s)$ and $e^{is}u_k(s)$ are two solutions of the equation $a(z) = a(u_k(s))$. Now, $z_{3,k}(s)$ is the third solution, since it satisfies Vietta's equations applied to $a(z) - a(u_k(s)) = 0$, in particular:

$$e^{is}u_k(s)^2 z_{3,k}(s) = -c. \tag{3.4}$$

Hence the conclusion. □

To state the remaining results of this section, we need some facts about the u_k stated in [10], and list them in the next remark.

Remark 3.5 In the proof of [10, Theorem 4.3, part (c)] were shown the following facts:

1. for every $\lambda \in \Lambda(a)$, the roots $z_1(\lambda), z_2(\lambda), z_3(\lambda)$ of $a(z) - \lambda$ satisfy

$$|z_1(\lambda)| = |z_2(\lambda)| \leq |z_3(\lambda)|,$$

   the equality holding only for $\lambda \equiv \mu_2$;
2. $|u_1(0)| = |t_1| < |z_{3,1}(0)|$ and $|u_3(0)| = |t_3| < |z_{3,3}(0)|$;
3. there exists $\epsilon > 0$, such that for every $s \in [-\epsilon, \epsilon] \setminus \{0\}$:

$$|u_1(s)| < |z_{3,1}(s)|, \quad |u_3(s)| < |z_{3,3}(s)|, \quad |u_2(s)| > |z_{3,2}(s)|;$$

4. there exists $\delta > 0$, such that for every s in a neighborhood of $(\varphi_2 - \delta, \varphi_2 + \delta) \setminus \{\varphi_2\}$, exclusively one of the u_k satisfies $u_k(\varphi_2) = t_2$ or $u_k(\varphi_2) = e^{i\varphi_2} t_2$, and $|u_k(s)| < |z_{3,k}(s)|$.

Now, we give a criterion for determining when a value $a(u_k(s))$ belongs to $\Lambda(a)$ for given $k = 1, 2, 3$ and s in $[0, \pi)$.

Theorem 3.6 *Let $s \in [0, \pi)$, and let $k = 1, 2, 3$. Then, $a(u_k(s)) \in \Lambda(a)$ if and only if $|u_k(s)| \leq |c|^{1/3}$. Furthermore, $|u_k(s)| = |c|^{1/3}$ if and only if $|u_k(s)| = |t_2|$.*

Proof From Proposition 3.4, $u_k(s)$, $e^{is} u_k(s)$ and $z_{3,k}(s)$ are the roots of $a(z) - a(u_k(s))$.

Suppose $a(u_k(s)) \in \Lambda(a)$. By (2.4), $|u_k(s)| \leq |z_{3,k}(s)|$, since $|u_k(s)| = |e^{is} u_k(s)|$. So, (3.4) gives

$$|u_k(s)|^3 \leq |u_k(s)^2 z_{3,k}(s)| = |c|.$$

Now, assume that $|u_k(s)|^3 \leq |c|$. From (3.4)

$$|u_k(s)| \leq \frac{|c|}{|u_k(s)^2|} = |z_{3,k}(s)|. \tag{3.5}$$

By (2.4) it follows that $a(u_k(s)) \in \Lambda(a)$.

The second statement follows from (3.5) and i) of Remark 3.5. □

Corollary 3.7 *For every s in $(0, \pi)$, $a(u_2(s)) \notin \Lambda(a)$.*

Proof Without loss of generality we can assume that $\varphi_2 > 0$.

By (iii) of Remark 3.5, there exists $\epsilon > 0$, such that for every $s \in [-\epsilon, \epsilon] \setminus \{0\}$

$$|u_2(s)| > |z_{3,2}(s)|.$$

If there exists $\pi > \delta > \epsilon$, such that $|u_2(\delta)| = |z_{3,2}(\delta)|$ and $|u_2(s)| > |z_{3,2}(s)|$ for every $\epsilon \leq s < \delta$, then, by (i) of Remark 3.5, necessarily $a(u_2(\delta)) = \mu_2$, so, Theorem 3.1 yields

$$\left\{u_2(\delta), e^{i\delta}u_2(\delta), z_{3,2}(\delta)\right\} = \left\{t_2, e^{i\varphi_2}t_2, t_2\right\}.$$

The previous equality is only possible if $\delta = \varphi_2$, leading to a contradiction with Remark 3.5 part (iv). Analogously, if $-\pi < \delta < -\epsilon$. Then, $|u_2(s)| > |z_{3,2}(s)|$ for every $s \neq 0$, which by (2.4), implies the conclusion. □

Proposition 3.8 *There is a unique ordered pair* $(j, k) \in \{(1, 3), (3, 1)\}$*, such that for every* s *in* $[0, \pi]$*,* $|u_j(s)| \leq |c|^{1/3}$*,* $|u_k(s)| \leq |c|^{1/3}$*, and* $|u_k(s)| = |t_2| = |c|^{1/3}$ *if and only if* $s = \varphi_2$*.*

Proof By Corollary 3.7 and (iv) of Remark 3.5, at least one of u_1 or u_3 should satisfy $u_k(\varphi_2) = t_2$ or $u_k(\varphi_2) = e^{i\varphi_2}t_2$. Suppose that $\pi > \varphi_2 > 0$ and $u_1(\varphi_2) = t_2$, the other cases follow similarly.

By (iii) of Remark 3.5, there exists $\epsilon > 0$, such that for every $s \in [-\epsilon, \epsilon]$

$$|u_1(s)| < |z_{3,1}(s)|.$$

Suppose that there exists $\pi > s^* > \epsilon$, such that $|u_1(s^*)| = |z_{3,1}(s^*)|$ and $|u_1(s)| < |z_{3,1}(s)|$ for every $s^* > s \geq \epsilon$. By (i) of Remark 3.5, $a(u_1(s^*)) = \mu_2$, so

$$\{u_1(s^*), e^{is^*}u_1(s^*), z_{3,1}(s^*)\} = \{t_2, e^{i\varphi_2}t_2, t_2\}.$$

Hence, $|u_1(s^*)| = |z_{3,1}(s^*)|$, which happens only at $s^* = \varphi_2$. By (iv) of Remark 3.5, there exists $\delta > 0$, such that $|u_1(s)| < |z_{3,1}(s)|$ for every $s \in (\varphi_2 - \delta, \varphi_2 + \delta) \setminus \{\varphi_2\}$. A similar argument shows that $|u_1(s)| < |z_{3,1}(s)|$ for every $s \in (\varphi_2 - \delta, \pi + \epsilon) \setminus \{\varphi_2\}$. We have proved that for every $s \in [0, \pi] \setminus \{\varphi_2\}$, the inequality $|u_1(s)| < |z_{3,1}(s)|$ holds, by (2.4), this means that $a(u_1(s)) \in \Lambda(a)$, and by Theorem 3.6, we obtain that $|u_1(s)| \leq |c|^{1/3}$, with equality $|u_1(s)| = |c|^{1/3}$ occurring only if $s = \varphi_2$.

Now, from (iii) of Remark 3.5, $|u_3(s)| < |z_{3,3}(s)|$ for every $s \in [-\epsilon, \epsilon]$. Assume that there exists s' in (ϵ, π), such that $|u_3(s')| = |z_{3,3}(s')|$. By i) of Remark 3.5, necessarily

$$\{u_3(s'), e^{is'}u_3(s'), z_{3,3}(s')\} = \{t_2, e^{i\varphi_2}t_2, t_2\}.$$

The last equality is possible only if $s' = \varphi_2$. However, by (iv) of Remark 3.5, we get that $u_1(s) = u_3(s)$ in a neighborhood of φ_2, but this contradicts Lemma 3.3. Then, $|u_3(s)| < |z_{3,3}(s)|$ for all s in $[0, \pi]$. Analogously as before, from (2.4) and Theorem 3.6, $|u_3(s)| \leq |c|^{1/3}$ for every s in $[0, \pi]$.

If $\varphi_2 = \pi$, which by (2.5) occurs only when $c = \pm i$, then, the conclusion follows similarly, just that in this case $|u_1(\pi)| = |u_3(\pi)| = 1 = |c|$. See Remark 3.9 below. □

Notice that, at first, it is not guaranteed that the u_k are analytic at $s = \pi, -\pi$ $(\tau = -1)$, the principal reason being that at those points the degree of Φ drops

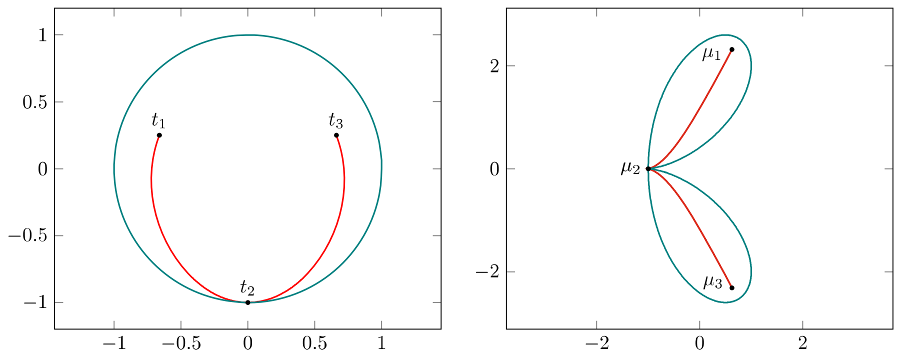

Fig. 4 For $c = i$, left: the set $u[0, 2\pi]$ in red, $\mathbb{T}$ in teal, right: the limit set $\Lambda(a)$ in red and $a(\mathbb{T})$ in teal

(see [1] for further details):

$$\Phi(z, -1) = -cz^2 - c = -c(z^2 + 1) = -c(z + i)(z - i).$$

By Corollary 3.7, Proposition 3.8, and the continuity of the roots under continuous variation of the coefficients of Φ, as $s \to \pi$, it follows that $|u_2(s)| \to \infty$, while one of $u_1(s)$ and $u_3(s)$ converges to i and the other to $-i$. Moreover, it follows that $u_1(s)$ and $e^{is}u_3(s)$ are analytic continuations of each other for s on some open neighborhood of π.

Hence, with (j, k) obtained from Proposition 3.8, if $u_k(\varphi_2) = t_2$, then we can define an analytic function u on some open neighborhood of $[0, 2\pi]$ that satisfies $u(\varphi_k) = t_k$ for each $k = 1, 2, 3$, and, restricted to $[0, 2\pi]$:

$$u(s) \equiv \begin{cases} u_k(s), & \text{if } 0 \le s \le \pi, \\ u_j(s - 2\pi), & \text{if } \pi \le s \le 2\pi. \end{cases} \tag{3.6}$$

If $e^{i\varphi_2}u_k(\varphi_2) = t_2$, then, $e^{is}u(s)$ restricted to $[0, 2\pi]$ coincides with the right-hand side of (3.6). Without loss of generality, in what follows, we suppose that u satisfies (3.6).

Now, we can properly define the generating function given by (2.6).

Remark 3.9 (cases $c = \pm i$) Notice that $i, -i, -c$ are the roots of $a(z) + 1 = 0$, and as $|c| \ge 1$, then $-1 \in \Lambda(a)$. If $c = \pm i$, then $\varphi_2 = \pi$, and

$$t_1 = \pm\frac{i}{4} + \frac{\sqrt{7}}{4}, \qquad t_2 = \mp i, \qquad t_3 = \pm\frac{i}{4} - \frac{\sqrt{7}}{4},$$
$$\mu_1 = \frac{5}{8} \pm i\frac{7\sqrt{7}}{8}, \qquad \mu_2 = -1, \qquad \mu_3 = \frac{5}{8} \mp i\frac{7\sqrt{7}}{8}.$$

Figure 4 shows the graph of $u([0, 2\pi])$ and $\Lambda(a)$ when $c = i$.

Define $w(s) := -ce^{-is}u^{-2}(s)$. By Vietta's theorem, it is clear that $u(s)$, $e^{is}u(s)$ and $w(s)$ are the roots of $a(z) - a(u(s))$.

Now, we present the main properties of ψ and u that benefit all the posterior analysis.

Proposition 3.10 $\psi([0, 2\pi]) = \Lambda(a)$.

Proof From Theorem 3.6 and Proposition 3.8, we obtain that for every $s \in [0, 2\pi]$, $|u(s)| \le |w(s)|$, hence $\psi(s) \in \Lambda(a)$, i.e., $\psi([0, 2\pi]) \subseteq \Lambda(a)$. Notice that $\psi(0) = \mu_1$ and $\psi(2\pi) = \mu_3$. Since ψ is continuous and $[0, 2\pi]$ connected, then $\psi([0, 2\pi])$ is a connected subset of the connected set $\Lambda(a)$ that has the extreme points μ_1 and μ_3, so, necessarily $\psi([0, 2\pi]) = \Lambda(a)$. □

Proposition 3.11 *The function u is one to one on* $[0, 2\pi]$.

Proof Suppose that there exist $s_1, s_2 \in [0, 2\pi] \setminus \{\varphi_2\}$, such that $u(s_1) = u(s_2)$. Then, $\psi(s_1) = \psi(s_2)$. By Proposition 3.10, $\psi(s_1) \in \Lambda(a)$, then the roots of $a(z) - \psi(s_1)$ are the same as the roots of $a(z) - \psi(s_2)$, that is

$$\{u(s_1), e^{is_1}u(s_1), w(s_1)\} = \{u(s_2), e^{is_2}u(s_2), w(s_2)\}. \tag{3.7}$$

From Proposition 3.8, $|u(s_j)| < |w(s_j)|$, so, $w(s_1) = w(s_2)$. Then, $e^{is_1}u(s_1) = e^{is_2}u(s_2)$. Therefore, $e^{i(s_1-s_2)} = 1$, and this occurs if and only if $s_1 = s_2$. □

Proposition 3.12 *The function ψ is one to one on* $[0, 2\pi]$.

Proof Suppose that there exist $s_1, s_2 \in [0, \pi] \setminus \{\varphi_2\}$, such that $\psi(s_1) = \psi(s_2)$. As in the proof of Proposition 3.11, we have (3.7). In particular, $w(s_1) = w(s_2)$. If $u(s_1) = u(s_2)$, then $s_1 = s_2$. If $u(s_1) = e^{is}u(s_2)$, then $s_1 = -s_2$, which cannot happen. Thus, ψ is one to one on $[0, \pi]$.

By Proposition 3.10 and previous calculus, ψ is one to one on $[0, \pi]$, tracing an arc contained in $\Lambda(a)$ starting at $\psi(0) = \mu_1$ and terminating at $\psi(\pi) = -1$.

Analogously, ψ is one to one on $[\pi, 2\pi]$, tracing an arc contained in $\Lambda(a)$ starting on $\psi(\pi) = -1$ and ending in $\psi(2\pi) = \mu_3$.

Therefore, $\psi[0, \pi) \cap \psi(\pi, 2\pi] = \emptyset$, so, ψ is one to one on $[0, 2\pi]$. □

For each $j \in \{1, 2\}$, denote by $[\lambda_j \sim \lambda_{j+1}]$ the analytic arc contained in $\Lambda(a)$ with endpoints λ_j and λ_{j+1}. The next propositions summarize the properties of u and ψ proved above.

Proposition 3.13 *The function u has the following properties:*

1. *u is analytic on some open neighborhood of* $[0, 2\pi]$.
2. $u(0) = t_1$, $u(\varphi_2) = t_2$, $u(2\pi) = t_3$.
3. *the restriction of u to* $[0, 2\pi]$ *is one to one.*
4. $|u(s)| < |c|^{1/3}$ *for every* $s \in [0, 2\pi]\setminus\{\varphi_2\}$, *and* $|u(\varphi_2)| = |c|^{1/3}$.

Proposition 3.14 *The function ψ has the following properties:*

1. *ψ is analytic on some open neighborhood of* $[0, 2\pi]$.
2. $\psi(0) = \mu_1$, $\psi(\varphi_2) = \mu_2$, $\psi(2\pi) = \mu_3$.
3. *The restriction of ψ to* $[0, 2\pi]$ *is one-to-one.*
4. $\psi([0, 2\pi]) = \Lambda(a)$, *in particular* $\psi([0, \varphi_2]) = [\mu_1 \sim \mu_2]$, $\psi([\varphi_2, 2\pi]) = [\mu_2 \sim \mu_3]$.

We conclude this section with the asymptotic expansions of u and ψ around the points φ_k. First we introduce some notation.

Recall that $\varphi_1 \equiv 0$ and $\varphi_3 \equiv 2\pi$. Let $k = 1, 2, 3$. Hence, $a(t_k) = \mu_k$ and $a'(t_k) = 0$. Define the numbers $\mathfrak{a}_{k,2} := a''(t_k)/2$, $\mathfrak{a}_{k,3} := a'''(t_k)/6$. From the Vietta's formulas applied to $a(z) - \mu_k$, we obtain

$$\mathfrak{a}_{k,2} = \frac{3 - t_k^2}{1 - t_k^2}, \qquad \mathfrak{a}_{k,3} = -\frac{2}{t_k(1 - t_k^2)}.$$

In particular, $\mathfrak{a}_{2,2} = 1 - e^{i\varphi_2}$ and $\mathfrak{a}_{2,3} = e^{i\varphi_2} t_2^{-1}$. Then, for every z in a small neighborhood of t_k:

$$a(z) = \mu_k + \mathfrak{a}_{k,2}(z - t_k)^2 + \mathfrak{a}_{k,3}(z - t_k)^3 + O|z - t_k|^4. \tag{3.8}$$

Recall also that $a(e^{i\varphi_k} t_k) = \mu_k$. Define the numbers $\mathfrak{b}_{k,1} := a'(e^{i\varphi_k} t_k)$, $\mathfrak{b}_{k,2} := a''(e^{i\varphi_k} t_k)/2$; clearly $\mathfrak{b}_{k,1} = 0$ and $\mathfrak{b}_{k,2} = \mathfrak{a}_{k,2}$ for $k = 1, 3$, and again by Vietta's formulas $\mathfrak{b}_{2,1} = t_2 e^{-i\varphi_2}(1 - e^{i\varphi_2})^2$ and $\mathfrak{b}_{2,2} = 1 - 2e^{-2i\varphi_2}$. Therefore, for every z in a small neighborhood of $e^{i\varphi_k} t_k$:

$$a(z) = \mu_k + \mathfrak{b}_{k,1}(z - e^{i\varphi_k} t_k)^2 + \mathfrak{b}_{k,2}(z - e^{i\varphi_k} t_k)^3 + O|z - e^{i\varphi_k} t_k|^4. \tag{3.9}$$

The proof of the following theorem is analogous to the proof of [10, Theorem 4.5].

Theorem 3.15 *Let $k = 1, 2, 3$. Then, as $s \to \varphi_k$,*

$$u(s) = t_k + \mathfrak{u}_{k,1}(s - \varphi_k) + \mathfrak{u}_{k,2}(s - \varphi_k)^2 + O|s - \varphi_k|^3, \tag{3.10}$$

$$\psi(s) = \mu_k + \boldsymbol{\psi}_{k,1}(s - \varphi_k) + \boldsymbol{\psi}_{k,2}(s - \varphi_k)^2 + \boldsymbol{\psi}_{k,3}(s - \varphi_k)^3 + O|s - \varphi_k|^4, \tag{3.11}$$

where

$$\mathfrak{u}_{k,1} \equiv -i\frac{t_k}{2}, \qquad \mathfrak{u}_{k,2} \equiv -\frac{t_k}{4(3 - t_k^2)}, \qquad (k = 1, 3),$$

$$\mathfrak{u}_{2,1} \equiv -i t_2, \qquad \mathfrak{u}_{2,2} \equiv -\frac{t_2}{2}\left(2 + i\cot\frac{\varphi_2}{2}\right),$$

$$\boldsymbol{\psi}_{k,1} \equiv 0, \qquad \boldsymbol{\psi}_{k,2} \equiv -\frac{(3 - t_k^2)t_k^2}{4(1 - t_k^2)}, \qquad \boldsymbol{\psi}_{k,3} \equiv 0, \qquad (k = 1, 3),$$

$$\boldsymbol{\psi}_{2,1} \equiv 0, \qquad \boldsymbol{\psi}_{2,2} \equiv -t_2^2(1 - e^{i\varphi_2}), \qquad \boldsymbol{\psi}_{2,3} \equiv 3i t_2^2.$$

Proof Consider the Taylor expansion of u around φ_k of the form (3.10); substitute it in (3.8) and apply the chain rule. Then, as $s \to \varphi_k$:

$$a(u(s)) = \mu_k + \mathfrak{a}_{k,2}\mathfrak{u}_{k,1}^2(s - \varphi_k)^2 + (2\mathfrak{a}_{k,2}\mathfrak{u}_{k,1}\mathfrak{u}_{k,2} + \mathfrak{a}_{k,3}\mathfrak{u}_{k,1}^3)(s - \varphi_k)^3 + O|s - \varphi_k|^4. \tag{3.12}$$

Therefore, $\boldsymbol{\psi}_{k,1}$ is zero. From (3.12) we get that $\boldsymbol{\psi}_{k,2} = \mathfrak{a}_{k,2}\mathfrak{u}_{k,1}^2$ and $\boldsymbol{\psi}_{k,3} = 2\mathfrak{a}_{k,2}\mathfrak{u}_{k,1}\mathfrak{u}_{k,2} + \mathfrak{a}_{k,3}\mathfrak{u}_{k,1}^3$, hence, we only need to find the values $\mathfrak{u}_{k,j}$. For this purpose, we consider the expansion:

$$e^{is} = e^{i\varphi_k}\left(1 + i(s-\varphi_k) - \frac{(s-\varphi_k)^2}{2} + O|s-\varphi_k|^3\right). \tag{3.13}$$

Multiply (3.13) and (3.10), then substitute it in (3.9) and apply the chain rule. Then, as $s \to \varphi_k$:

$$\begin{aligned} a(e^{is}u(s)) = \mu_k &+ \mathfrak{b}_{k,1}e^{i\varphi_k}(\mathfrak{u}_{k,1} + it_k)(s-\varphi_k) \\ &+ \left(\mathfrak{b}_{k,1}e^{i\varphi_2}\left(\mathfrak{u}_{k,2} + i\mathfrak{u}_{k,1} - \frac{t_k}{2}\right) + \mathfrak{b}_{k,2}e^{2i\varphi_k}(\mathfrak{u}_{k,1} + it_2)^2\right)(s-\varphi_k)^2 \\ &+ O|s-\varphi_k|^3. \end{aligned} \tag{3.14}$$

Recall that for every $s \in [0, 2\pi]$, $\psi(s) \equiv a(u(s)) = a(e^{is}u(s))$; therefore, the values $\mathfrak{u}_{k,j}$ and hence $\boldsymbol{\psi}_{k,j}$, are now easily obtained by comparing the coefficients of the Taylor expansions around φ_k of (3.12) and (3.14). □

Remark 3.16 (Algorithm for constructing the limit set) Given (2.2) with parameter c in (2.5), Theorem 3.6, and Propositions 3.10, 3.11, 3.12, justify the following numerical way of constructing the corresponding limit set: Estimate the roots t_1, t_2, and t_3 of $\Phi(z, 1)$, which, after the evaluation by the symbol a, returns the branch points μ_1, μ_2, and μ_3. After that, take N arbitrary points $0 < \sigma_1 < \cdots < \sigma_N < \pi$, and, for each σ_j, select the only two solutions $u_k(\sigma_j)$ of $\Phi(z, e^{i\sigma_j}) = 0$ that satisfy the inequality $|u_k(\sigma_j)| \leq |c|^{1/3}$. Compute both $a(u_k(\sigma_j))$; these (distinct) values will then belong to $\Lambda(a)$. After this procedure, it is feasible to provide a *nice* approximation of the limit set using polynomial interpolation at the generated points.

4 Supporting actors

Recall that $\Omega_{\nu,k}$, f, p and q, η, , r_n and R_n are defined by (2.7), (2.8), (2.9), (2.10), and (2.11), respectively. In this section, we provide some properties of these functions.

By Proposition 3.13, $|f(s)| \leq 1$ for every s in $[0, 2\pi]$, attaining the equality only at $s = \varphi_2$. Figure 5 shows the graph of f restricted to $[0, 2\pi]$.

Define the values

$$\begin{gathered} \mathfrak{f}_{k,0} := -\frac{1-t_k^2}{2}, \qquad \mathfrak{f}_{k,1} := i\,\frac{1-t_k^2}{4}, \qquad \mathfrak{f}_{k,2} := \frac{t_k^2(1-t_k^2)}{8(3-t_k^2)}, \qquad (k = 1, 3), \\ \mathfrak{f}_{2,0} := 1, \qquad \mathfrak{f}_{2,1} := -2i, \qquad \mathfrak{f}_{2,2} := -\frac{1}{2}\left(7 + 3i\cot\frac{\varphi_2}{2}\right). \end{gathered} \tag{4.1}$$

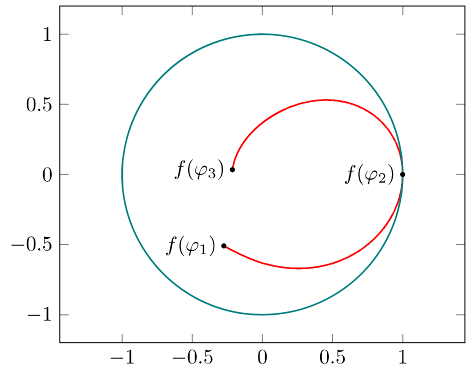

Fig. 5 In red the set $f([0, 2\pi])$, in black the points $f(\varphi_k)$ and in teal the unit circle for $c \approx -1.91873 - i0.66558$

Proposition 4.1 *There exists some open neighborhood of* $[0, 2\pi]$*, where* f *is analytic. Moreover, for every* $k = 1, 2, 3$*, as* $s \to \varphi_k$*:*

$$f(s) = \mathfrak{f}_{k,0} + \mathfrak{f}_{k,1}(s - \varphi_k) + \mathfrak{f}_{k,2}(s - \varphi_k)^2 + O|s - \varphi_k|^3. \tag{4.2}$$

Furthermore, as $s \to \varphi_2$

$$f(s) = \exp\left(\mathfrak{f}_{2,1}(s - \varphi_2) + \left(\mathfrak{f}_{2,2} - \frac{1}{2}\mathfrak{f}_{2,1}^2\right)(s - \varphi_2)^2 + O|s - \varphi_2|^3\right). \tag{4.3}$$

Proof By Proposition 3.13, f is analytic on some open neighborhood of $[0, 2\pi]$. From (3.10), as $s \to \varphi_k$:

$$u(s)^3 = t_k^3 + 3t_k^2 u_{k,1}(s - \varphi_k) + 3(t_k u_{k,1}^2 + t_k^2 u_{k,2})(s - \varphi_k)^2 + |s - \varphi_k|^3.$$

Multiplying the last expression by (3.13), (4.2) follows.

Notice that $f(s) = \exp(\ln(f(s)))$ for s in a neighborhood of φ_2. Applying the Mercator series to $s \mapsto \ln(f(s))$, as $s \to \varphi_2$:

$$\ln(f(s)) = \mathfrak{f}_{2,1}(s - \varphi_2) + \left(\mathfrak{f}_{2,2} - \frac{1}{2}\mathfrak{f}_{2,1}^2\right)(s - \varphi_2)^2 + O|s - \varphi_2|^3.$$

Hence, (4.3) is proven after taking the exponential in both sides of the equality above. □

Let $k = 1, 2, 3$. Define $\mathfrak{p}_{k,0} := 1 - \mathfrak{f}_{k,0}$, $\mathfrak{p}_{k,1} := -\mathfrak{f}_{k,1}$, $\mathfrak{p}_{k,2} := -\mathfrak{f}_{k,2}$, $\mathfrak{q}_{k,0} := 1 - e^{i\varphi_k}\mathfrak{f}_{k,0}$, $\mathfrak{q}_{k,1} := -e^{i\varphi_k}(i\mathfrak{f}_{k,0} + \mathfrak{f}_{k,1})$, and $\mathfrak{q}_{k,2} = e^{i\varphi_k}(\mathfrak{f}_{k,0}/2 - i\mathfrak{f}_{k,1} - \mathfrak{f}_{k,2})$.

Proposition 4.2 p *and* q *are analytic on some open neighborhood of* $[0, 2\pi]$*. Moreover, for every* $k = 1, 2, 3$*, as* $s \to \varphi_k$*:*

$$\begin{aligned} p(s) &= \mathfrak{p}_{k,0} + \mathfrak{p}_{k,1}(s - \varphi_k) + \mathfrak{p}_{k,2}(s - \varphi_k)^2 + O|s - \varphi_k|^3, \\ q(s) &= \mathfrak{q}_{k,0} + \mathfrak{q}_{k,1}(s - \varphi_k) + \mathfrak{q}_{k,2}(s - \varphi_k)^2 + O|s - \varphi_k|^3. \end{aligned} \tag{4.4}$$

Proof By Proposition 4.1, p and q are analytic on some open neighborhood of $[0, 2\pi]$. The expansions in the conclusion easily follow from (3.13) and (4.2). □

Lemma 4.3 *There exist V_1, V_2 simply connected open neighborhoods of $[0, \varphi_2)$ and $(\varphi_2, 2\pi]$, respectively, neither of them including φ_2, such that η restricted to each V_ℓ is a well-defined analytic function. Moreover, for every $\delta > 0$ small enough, η and all its derivatives are bounded functions on Ω_δ.*

Proof By Proposition 4.2, there exist two simply connected open sets V_1 and V_2, such that

- $[0, \varphi_2) \subseteq V_1$ and $(\varphi_2, 2\pi] \subseteq V_2$;
- $\varphi_2 \notin V_\ell$ for $\ell = 1, 2$;
- $\Omega_\delta \subset V_1 \cup V_2$;
- both p and q are analytic on each V_ℓ;
- for every $\ell = 1, 2$ and every $s \in V_\ell$ (possibly for a smaller value of M than in Ω_δ (2.7))

$$|1 - q(s)|, |1 - p(s)| < 1.$$

Then, for every $s \in V_1$, $p(s)$ and $q(s)$ belong to the open ball centered at 1 of radius 1, hence $\Re(q(s)), \Re(p(s)) > 0$. This implies that $q(s)/p(s) \notin \{\omega \in \mathbb{C} \colon \Re(\omega) \leq 0,\ \Im(\omega) = 0\}$. Hence, η is a well-defined analytic function on V_1. Similarly follows that η is a well-defined analytic function on V_2. Furthermore, η is analytic on the closure of Ω_δ, hence, η as well as all its derivatives are bounded on Ω_δ. □

Let V_1 and V_2 be given by Lemma 4.3.

Proposition 4.4 *For every $k = 1, 3$, as $s \to \varphi_k$:*

$$\eta(s) = \frac{1 - t_k^2}{3 - t_k^2}(s - \varphi_k) + O\left((s - \varphi_k)^2\right). \tag{4.5}$$

Furthermore, as $s \to \varphi_2$

$$\eta(s) = i \ln |s - \varphi_2| + O(1), \qquad \eta'(s) = -\frac{i}{|s - \varphi_2|} + O(1). \tag{4.6}$$

Proof Let $k = 1, 3$. Lemma 4.3, implies that as $s \to \varphi_k$:

$$\eta(s) = -i \ln\left(1 + \frac{\mathfrak{p}_{k,1}}{\mathfrak{q}_{k,0}}(s - \varphi_k) + O\left((s - \varphi_k)^2\right)\right).$$

Hence, (4.5) is derived by the application of the Mercator series to the right-hand side of previous expression.

Now, let $k = 2$. It follows easily that $\Re(q(s)/p(s)) > 0$ for every s in $[0, 2\pi]\setminus\{\varphi_2\}$. This fact and Lemma 4.3 together imply that there exist open neighborhoods $(0, \varphi_2) \subseteq$

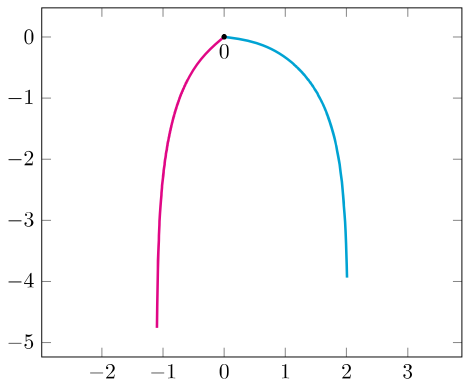

Fig. 6 For some $c \in \Gamma$ and $\delta > 0$, in blue and red the sets $\eta([0, \varphi_2 - \delta))$ and $\eta((\varphi_2 + \delta, 2\pi])$, respectively

$V_1' \subseteq V_1$ and $(\varphi_2, 2\pi) \subseteq V_2' \subseteq V_2$, such that restricted to them, η is analytic and $-\pi \le \arg(q(s)/p(s)) \le \pi$. Hence, as $s \to \varphi_2$

$$\eta(s) = -i \ln \left| \frac{q(s)}{p(s)} \right| + \arg \frac{q(s)}{p(s)} = -i \ln \left| \frac{q(s)}{p(s)} \right| + O(1).$$

By (4.4), we get $\eta(s) = i \ln |s - \varphi_2| + O(1)$ and $\eta'(s) = -i/|s - \varphi_2| + O(1)$, as $s \to \varphi_2$. □

Figure 6 illustrates the graph of η.

Lemma 4.5 *For every $\delta > 0$ small enough, there exists $C(\delta) > 0$, such that for every s in Ω_δ*

$$\left| \frac{e^{-i\frac{\eta(s)}{2}}}{p(s)} \right| \le C(\delta). \tag{4.7}$$

Moreover, as $s \to \varphi_2$

$$\left| \frac{e^{-i\frac{\eta(s)}{2}}}{p(s)} \right| = O\left(\frac{1}{\sqrt{|s - \varphi_2|}} \right). \tag{4.8}$$

Proof By Lemma 4.3, and the fact that p is uniformly bounded by below on Ω_δ, we obtain (4.7).

A direct calculus using Propositions 4.2 and 4.4 yield (4.8). □

Straightforward computations gives that the derivative of r_n is

$$r_n'(s) = \frac{f^{n+1} e^{i\left(\frac{n+2}{2}s - \frac{\eta}{2}\right)}}{p} \left(\frac{f \cos\frac{s}{2}}{2} + i \frac{n + 2 - \eta'}{2} f \sin\frac{s}{2} + (n+2) f' \sin\frac{s}{2} \right) - \frac{p'}{p} r_n. \tag{4.9}$$

Proposition 4.6 *Let $\delta > 0$ be small enough, then, there exists $\Delta > 0$, such that for every $n \in \mathbb{N}$ large enough:*

$$\sup_{s \in \Omega_{\delta,n}} |r_n(s)| = O\left(e^{-n\Delta}\right), \qquad \sup_{s \in \Omega_{\delta,n}} |r_n'(s)| = O\left(n e^{-n\Delta}\right), \tag{4.10}$$

$$\sup_{s\in\Omega_{\delta,n}} |R_n(s)| = O\left(e^{-n\Delta}\right), \qquad \sup_{s\in\Omega_{\delta,n}} |R_n'(s)| = O\left(ne^{-n\Delta}\right). \tag{4.11}$$

Proof By Proposition 3.13, there exists $\Delta \equiv \Delta(\delta) > 0$, such that $|f(s)| < e^{-\Delta}$ for every s in Ω_δ. Therefore, for every s in $\Omega_{\delta,n}$

$$|f(s)|^n < e^{-n\Delta}.$$

Moreover, since $|\Im(s)| \leq M/n$ for every s in $\Omega_{\delta,n}$, we conclude that $\exp(i(n+1)s/2)$ is uniformly bounded on $\Omega_{\delta,n}$. By Proposition 3.13 and Lemmas 4.5, 4.3, all the other terms in (2.11) and (4.9) are bounded. Thus, we get (4.10). Now, for (4.11), on R_n apply the Taylor polynomial expansion of $x \mapsto \arcsin(x)$ around 0 combined, with (4.10). ◻

Proposition 4.7 *For every $n \geq 3$, on $\Omega_{0,n}$, as $s \to \varphi_2$*

$$\begin{aligned} |r_n(s)| &= O\left(\frac{\exp\left(-\frac{3}{2}(n+1)(\Re(s)-\varphi_2)^2\right)}{|\Re(s)-\varphi_2|^{1/2}}\right), \\ |r_n'(s)| &= O\left(\frac{\exp\left(-\frac{3}{2}(n+1)(\Re(s)-\varphi_2)^2\right)}{|\Re(s)-\varphi_2|^{3/2}}\right), \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} |R_n(s)| &= O\left(\frac{\exp\left(-\frac{3}{2}(n+1)(\Re(s)-\varphi_2)^2\right)}{|\Re(s)-\varphi_2|^{1/2}}\right), \\ |R_n'(s)| &= O\left(\frac{\exp\left(-\frac{3}{2}(n+1)(\Re(s)-\varphi_2)^2\right)}{|\Re(s)-\varphi_2|^{3/2}}\right). \end{aligned} \tag{4.13}$$

Proof Let $n \geq 3$. From Proposition 3.4, it follows that as $s \to \varphi_2$

$$|f(s)|^{n+1} = O\left(\exp\left(-\frac{3}{2}(n+1)(\Re(s)-\varphi_2)^2\right)\right). \tag{4.14}$$

By (4.4), as $s \to \varphi_2$

$$\eta'(s) = O\left(\frac{1}{\Re(s)-\varphi_2}\right). \tag{4.15}$$

Now (4.12) is obtained from (4.14), (4.15) and (4.8). The computation for (4.13) follows in similar manner. ◻

5 Inner eigenvalues

In this section, we prove Theorems 2.1, 2.2 and 2.3.

Let $n \geq 3$, and recall that $T_n(a)$ is the $n \times n$ Toeplitz matrix generated by (2.2) with parameter c in (2.5).

Widom's formula [7, Theorem 2.8], applied to the case at hand, says that for given s in Ω_0, since $u(s)$, $e^{is}u(s)$ and $w(s)$ are the roots of $a(z) - \psi(s)$, the characteristic polynomial takes the form:

$$\det(\psi(s) - T_n(a)) = \frac{u^{n+1}w^{n+1}}{(u - e^{is}u)(w - e^{is}u)} + \frac{(e^{is}u)^{n+1}w^{n+1}}{(e^{is}u - u)(w - u)} + \frac{u^{n+1}(e^{is}u)^{n+1}}{(u - w)(e^{is}u - w)}. \tag{5.1}$$

We start by reducing (5.1) to the *main equation* (2.16).

Theorem 5.1 *For every $n \geq 3$, and for every $s \in \Omega_0$*

$$\det(\psi(s) - T_n(a)) = \frac{u(s)^n w(s)^n e^{i\frac{ns+\eta(s)}{2}}}{q(s)\sin\frac{s}{2}}\left(\sin\frac{(n+1)s + \eta(s)}{2} + r_n(s)\right). \tag{5.2}$$

Proof Let $n \geq 3$ and $s \in \Omega_0$. Put $D_n(s) := \det(\psi(s) - T_n(a))$. We can then rewrite (5.1) as follows

$$\begin{aligned} D_n(s) &= \frac{u^{n+1}w^{n+1}}{(u - e^{is}u)(w - e^{is}u)}\left(1 - \frac{e^{i(n+1)s}(w - e^{is}u)}{(w - u)} + \frac{e^{i(n+1)s}(u - e^{is}u)u^{n+1}}{(w - u)w^{n+1}}\right) \\ &= \frac{u^n w^n}{(1 - e^{is})(1 - e^{is}\frac{u}{w})}\left(1 - \frac{e^{i(n+1)s}(1 - e^{is}\frac{u}{w})}{1 - \frac{u}{w}} + \frac{e^{i(n+1)s}(1 - e^{is})u^{n+2}}{(1 - \frac{u}{w})w^{n+2}}\right). \end{aligned}$$

Since $w(s) \equiv -ce^{-is}u^{-2}(s)$, we have that $f(s) = u(s)/w(s)$, hence

$$\begin{aligned} D_n(s) &= \frac{u^n w^n}{q(1 - e^{is})}\left(1 - \frac{e^{i(n+1)s}q}{p} + \frac{e^{i(n+1)s}(1 - e^{is})f^{n+2}}{p}\right) \\ &= \frac{u^n w^n}{2ie^{i\frac{s}{2}}q\sin\frac{s}{2}}\left(e^{i(n+1)s}e^{i\eta} - 1 + \frac{2ie^{i\frac{s}{2}}\sin\frac{s}{2}e^{i(n+1)s}f^{n+2}}{p}\right) \\ &= \frac{u^n w^n e^{i\frac{ns+\eta}{2}}}{q\sin\frac{s}{2}}\left(\sin\frac{(n+1)s+\eta}{2} + \frac{e^{i\frac{s}{2}}\sin\frac{s}{2}e^{i\frac{(n+1)s}{2} - i\frac{\eta}{2}}f^{n+2}}{p}\right). \end{aligned}$$

Finally, we just rewrite the right-hand side using (2.11). □

Let $k = 1, 3$. From, (3.10), (4.2), (4.4), (4.5), for every n sufficiently big:

$$\lim_{s \to \varphi_k} \det(\psi(s) - T_n(a)) = \frac{(-1)^n c^n}{t_k^n \mathfrak{q}_{k,0}}\left(n + 1 + \frac{1 - t_k^2}{3 - t_k^2}\right) \neq 0.$$

Hence, μ_1 and μ_3 are not eigenvalues of $T_n(a)$.

Recall that $d_{n,j}$, $\mathrm{B}_{n,j}$, and $e_{n,j}$, are defined by (2.12), (2.13), and (2.14), respectively. For every $n \geq 3$ and every $j \in \mathbb{Z}$, we define $H_{n,j} \colon \Omega_0 \to \mathbb{C}$ by

$$H_{n,j}(s) := d_{n,j} - \frac{\eta(s)}{n+1} + \frac{(-1)^{j'} R_n(s)}{n+1},$$

where $j' := j + J_n$.

Proposition 5.2 *Let $n \geq 4$, and $s \in \Omega_0$ with $s \neq \varphi_k$ for each $k = 1, 2, 3$. If $\det(\psi(s) - T_n(a)) = 0$, then there exists j in $\mathbb{Z}$, such that*

$$s = H_{n,j}(s). \tag{5.3}$$

Proof For every s in Ω_0 with $s \neq \varphi_k (k = 1, 2, 3)$, as $u(s)$ cannot take the value zero:

$$\frac{u^n(s)\, w^n(s)\, e^{i\frac{ns+\eta(s)}{2}}}{\sin\frac{s}{2}\, q(s)} \neq 0.$$

Hence, (5.2) is equal to zero if $\sin\frac{(n+1)s+\eta(s)}{2} + r_n(s) = 0$, and this equation is equivalent to (5.3). □

To solve (5.3), Propositions 4.6 and 4.7 suggest analyzing two cases: when $|d_{n,j} - \varphi_2| > \delta$ for some $\delta > 0$, and when $d_{n,j} \to \varphi_2$. We, therefore, establish sufficient conditions on n, j, and $d_{n,j}$ for (5.3) to admit a solution, and then prove Theorem 2.1.

Denote by $\|\cdot\|_{\mathrm{A}}$ the supremum norm of functions defined on the set $\mathrm{A} \subseteq \mathbb{C}$.

Lemma 5.3 *Let $\delta > 0$, $n \in \mathbb{N}$ large enough, and $j \in \mathbb{Z}$, such that $d_{n,j} \in \Omega_{0,n}$. If $|d_{n,j} - \varphi_2| > \delta$, then there exists $\Delta \equiv \Delta(\delta) > 0$, such that for every $s \in \mathrm{B}_{n,j}$:*

$$\|R_n\|_{\mathrm{B}_{n,j}} = O(e^{-n\Delta}), \quad \|R_n'\|_{\mathrm{B}_{n,j}} = O(ne^{-n\Delta}). \tag{5.4}$$

If $|d_{n,j} - \varphi_2| \leq \delta$, then

$$\|R_n\|_{\mathrm{B}_{n,j}} = O\left(\frac{n^{1/2}\exp\left(-\frac{6\pi^2 j^2}{n+1}\right)}{|j|^{1/2}}\right), \quad \|R_n'\|_{\mathrm{B}_{n,j}} = O\left(\frac{n^{3/2}\exp\left(-\frac{6\pi^2 j^2}{n+1}\right)}{|j|^{3/2}}\right). \tag{5.5}$$

Proof (5.4) and (5.5) are easily derived from (4.11) and (4.13), respectively. □

Theorem 5.4 *For every $n \in \mathbb{N}$ and every $j \in \mathbb{Z}$, such that $d_{n,j} \in \Omega_{0,n}$ and $|j|/n^{1/2+\epsilon}$ is large enough, $H_{n,j}$ is a contraction on $\mathrm{B}_{n,j}$.*

Proof Let $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, such that $d_{n,j} \in \Omega_{0,n}$, and let $0 < \delta \ll 1$.

Step 1 ($H_{n,j}[\mathrm{B}_{n,j}] \subseteq \mathrm{B}_{n,j}$). Take $s \in \mathrm{B}_{n,j}$, then

$$|e_{n,j} - H_{n,j}(s)| \leq \frac{\left|\eta(d_{n,j}) - \eta(s)\right|}{n+1} + \frac{|R_n(s)|}{n+1} \leq \frac{\left|\eta(d_{n,j}) - \eta(s)\right|}{n+1} + \frac{\|R_n\|_{\mathrm{B}_{n,j}}}{n+1}.$$

By the mean value theorem applied to η restricted to $\mathrm{B}_{n,j}$, we find that

$$|e_{n,j} - H_{n,j}(s)| \le \|\eta'\|_{\mathrm{B}_{n,j}} \frac{|d_{n,j} - s|}{n+1} + \frac{\|R_n\|_{\mathrm{B}_{n,j}}}{n+1}.$$

Recall that $d_{n,j} = e_{n,j} - \eta(d_{n,j})/(n+1)$ and $|e_{n,j} - s| < \varepsilon_{n,j}/(n+1)^2$. Therefore

$$\begin{aligned} |e_{n,j} - H_{n,j}(s)| &\le \|\eta'\|_{\mathrm{B}_{n,j}} \frac{\left|e_{n,j} - s - \frac{\eta(d_{n,j})}{n+1}\right|}{n+1} + \frac{\|R_n\|_{\mathrm{B}_{n,j}}}{n+1} \\ &\le \frac{\varepsilon_{n,j}}{(n+1)^2} \frac{\|\eta'\|_{\mathrm{B}_{n,j}}}{n+1} + \frac{\|\eta'\|_{\mathrm{B}_{n,j}} |\eta(d_{n,j})|}{(n+1)^2} + \frac{\|R_n\|_{\mathrm{B}_{n,j}}}{n+1}. \end{aligned} \tag{5.6}$$

Suppose that $|d_{n,j} - \varphi_2| \ge \delta$. From Lemma 4.3, η and η' are bounded on Ω_δ, in particular, $\|\eta'\|_{\mathrm{B}_{n,j}} \le \|\eta'\|_{\Omega_\delta} < \infty$. Hence, if n is large enough, by (2.15) and (5.4):

$$|e_{n,j} - H_{n,j}(s)| \le \frac{\varepsilon_{n,j}}{(n+1)^2} \left(\frac{\|\eta'\|_{\mathrm{B}_{n,j}}}{n+1} + \frac{1}{3} \right) + \frac{\|R_n\|_{\mathrm{B}_{n,j}}}{n+1} < \frac{\varepsilon_{n,j}}{(n+1)^2}.$$

Now, suppose that $|d_{n,j} - \varphi_2| < \delta$. From (4.6), $\|\eta'\|_{\mathrm{B}_{n,j}} \le nK_1/|j|$ for some $K_1 > 0$. Then, if $|j|/n^{1/2+\epsilon}$ is large enough, (5.5) and (5.6) applied to (5.6) yield

$$|e_{n,j} - H_{n,j}(s)| \le \frac{\varepsilon_{n,j}}{(n+1)^2} \left(\frac{K_1}{|j|} + \frac{1}{3} \right) + \frac{\|R_n\|_{\mathrm{B}_{n,j}}}{n+1} < \frac{\varepsilon_{n,j}}{(n+1)^2}.$$

Step 2 ($|H_{n,j}(s) - H_{n,j}(t)| \le L|s-t|$ for some $0 < L < 1$). Let $s, t \in \mathrm{B}_{n,j}$. By the mean value theorem applied separately to η and R_n restricted to $\mathrm{B}_{n,j}$, we obtain

$$\begin{aligned} \left|H_{n,j}(s) - H_{n,j}(t)\right| &\le \frac{|\eta(s) - \eta(t)|}{n+1} + \frac{|R_n(s) - R_n(t)|}{n+1} \\ &\le \frac{\|\eta'\|_{\mathrm{B}_{n,j}}}{n+1}|s-t| + \|R_n'\|_{\mathrm{B}_{n,j}} \frac{|s-t|}{n+1}. \end{aligned}$$

Suppose $|d_{n,j} - \varphi_2| \ge \delta$. From Lemmas 4.3 and 5.3, there exists $K_2 > 0$, such that if n is sufficiently large, then there exists $1 > L_1 > 0$ for which

$$\left|H_{n,j}(s) - H_{n,j}(t)\right| \le \frac{K_2}{n+1}|s-t| + K_2 e^{-n\Delta}|s-t| \le \frac{2K_2}{n+1}|s-t| \le L_1|s-t|.$$

Now, suppose $|d_{n,j} - \varphi_2| < \delta$. From (4.6) and (5.5), there exists $K_3 > 0$ such that if $|j|/n^{1/2+\epsilon}$ is large enough, then there exists $1 > L_2 > 0$ for which

$$\left|H_{n,j}(s) - H_{n,j}(t)\right| \le \frac{K_3}{|j|}|s-t| + \frac{K_3 n^{1/2} \exp\left(-\frac{6\pi^2 j^2}{n+1}\right)}{(2\pi|j|)^{3/2}}|s-t| \le L_2|s-t|.$$

Steps 1 and 2 combined yield the conclusion. □

Proof of Theorem 2.1 By Theorem 5.4 and the Banach fixed point theorem, for every n and j, such that $d_{n,j} \in \mathrm{B}_{n,j}$ and $|j|/n^{1/2+\epsilon}$ is large enough, there exists a unique point $s_{n,j} \in \mathrm{B}_{n,j}$, such that $H_{n,j}(s_{n,j}) = s_{n,j}$. From Proposition 5.2 we obtain that $\psi(s_{n,j})$ is an eigenvalue of $T_n(a)$.

Let j, k be different integers, such that $d_{n,j}, d_{n,k} \in (0, 2\pi)$, and $|j|/n^{1/2+\epsilon}$, $|k|/n^{1/2+\epsilon}$ are large enough. If $H_{n,j}(s_{n,j}) = H_{n,k}(s_{n,k})$, then $j = k$, i.e., $s_{n,j} \neq s_{n,k}$ for $j \neq k$. Furthermore, Proposition 3.14 implies that ψ is one to one on some open neighborhood of $(0, 2\pi)$. Hence, for n large enough $\psi(s_{n,j}) \neq \psi(s_{n,k})$. □

Now, we show that if we solve the reduced version of (2.16), namely, (2.17), then the resulting solutions $s^*_{n,j}$, and the corresponding values $\psi(s^*_{n,j})$, approximate $s_{n,j}$ and $\lambda_{n,j}$, respectively.

For every $n \geq 4$ and every j, define $H^*_{n,j} \colon \Omega_0 \to \mathbb{C}$ by

$$H^*_{n,j}(s) = d_{n,j} - \frac{\eta(s)}{n+1}.$$

The proof of next theorem is similar to the proof of Theorem 5.4, so we omit it.

Theorem 5.5 *For every $n \in \mathbb{N}$ and every $j \in \mathbb{Z}$, such that $d_{n,j} \in \Omega_{0,n}$ and $|j|/n^{1/2+\epsilon}$ is large enough, $H^*_{n,j}$ is a contraction on $\mathrm{B}_{n,j}$.*

Proof of Theorem 2.2 Theorem 5.5 and the Banach fixed point theorem, combined imply that for every n and every j, such that $d_{n,j} \in \Omega_{0,n}$, and $|j|/n^{1/2+\epsilon}$ is large enough, there exists a unique point $s^*_{n,j} \in \mathrm{B}_{n,j}$, such that $H^*_{n,j}(s^*_{n,j}) = s^*_{n,j}$, i.e., $s^*_{n,j}$ satisfies (2.17).

Since $H_{n,j}(s_{n,j}) = s_{n,j}$ and $H^*_{n,j}(s^*_{n,j}) = s^*_{n,j}$, we get that

$$|s_{n,j} - s^*_{n,j}| \leq \frac{|\eta(s_{n,j}) - \eta(s^*_{n,j})|}{n+1} + \frac{|R_n(s_{n,j})|}{n+1}.$$

By the mean value theorem applied to η restricted to $B_{n,j}$,

$$\begin{aligned} |s_{n,j} - s^*_{n,j}| &\leq \frac{|\eta(s_{n,j}) - \eta(d_{n,j})|}{n+1} + \frac{|\eta(d_{n,j}) - \eta(s^*_{n,j})|}{n+1} + \frac{|R_n(s_{n,j})|}{n+1} \\ &\leq \frac{2\|\eta'\|_{\mathrm{B}_{n,j}} \varepsilon_{n,j}}{n+1} + \frac{|R_n(s_{n,j})|}{n+1}. \end{aligned}$$

Suppose $|d_{n,j} - \varphi_2| \geq \delta$ and n is large enough. From (2.15), (5.4), and Lemma 4.3, we obtain

$$|s_{n,j} - s^*_{n,j}| \leq \frac{6|\eta(d_{n,j})| \|\eta'\|^2_{\mathrm{B}_{n,j}}}{(n+1)^3} + O\left(\frac{e^{-n\Delta}}{n}\right) = O\left(\frac{1}{n^3}\right).$$

Proposition 3.14 implies that ψ' is uniformly bounded on some open set containing $[0, \pi]$. Then, we use the Taylor expansion of ψ around $s_{n,j}$, thus, for some ξ:

$$\psi(s^*_{n,j}) = \psi(s_{n,j}) + \psi'(\xi)(s_{n,j} - s^*_{n,j}) = \psi(s_{n,j}) + O\left(\frac{1}{n^3}\right).$$

So (2.18) follows.

Now, suppose $\delta > |d_{n,j} - \varphi_2|$ and $|j|/n^{1/2+\epsilon}$ is large enough. From (2.15), (4.6) and (5.5), we get

$$|s_{n,j} - s^*_{n,j}| \le \frac{3|\eta(d_{n,j})|\|\eta'\|_{\mathrm{B}_{n,j}}}{\pi(n+1)^2|j|} + O\left(\frac{n^{1/2}\exp\left(-\frac{6\pi^2 j^2}{n+1}\right)}{n|j|^{1/2}}\right) = O\left(\frac{\ln\frac{n}{|j|}}{nj^2}\right).$$

Then, we apply the Taylor expansion of ψ around $s_{n,j}$, thus, for some ξ

$$\psi(s^*_{n,j}) = \psi(s_{n,j}) + \psi'(\xi)(s_{n,j} - s^*_{n,j}).$$

Notice that $\psi'(\varphi_2) = 0$; therefore, $\psi'(\xi) = \psi'(\varphi_2) + O(|j|/n) = O(|j|/n)$. Then (2.19) is derived. □

Now, we are prepared to obtain the asymptotic formulas for inner eigenvalues. First, recall that $\mathfrak{v}_k$ and $\mathfrak{l}_k$ are defined by (2.20) and (2.21), respectively.

Proof of Theorem 2.3 From Theorem 5.4, if $|j|/n^{1/2+\epsilon}$ is large enough, then $s_{n,j} \in \mathrm{B}_{n,j}$, hence

$$s_{n,j} = e_{n,j} + O\left(\frac{\varepsilon_{n,j}}{n^2}\right) = d_{n,j} - \frac{\eta(d_{n,j})}{n+1} + O\left(\frac{\varepsilon_{n,j}}{n^2}\right). \tag{5.7}$$

Moreover, $s_{n,j}$ satisfies (5.3). Then, substituting (5.7) in $H_{n,j}$ and expanding η by the Taylor formula around $d_{n,j}$:

$$\begin{aligned} s_{n,j} &= d_{n,j} - \frac{\eta\left(d_{n,j} + \frac{\eta(d_{n,j})}{n+1} + O\left(\frac{\varepsilon_{n,j}}{n^2}\right)\right)}{n+1} + O\left(\frac{\|R_n\|_{\mathrm{B}_{n,j}}}{n}\right) \\ &= d_{n,j} - \frac{\eta(d_{n,j})}{n+1} - \frac{\eta(d_{n,j})\eta'(d_{n,j})}{(n+1)^2} + O\left(\frac{|\eta'(d_{n,j})|\varepsilon_{n,j}}{n^3}\right) \\ &\quad + O\left(\frac{|\eta(d_{n,j})|^2\|\eta''\|_{\mathrm{B}_{n,j}}}{n^3}\right) + O\left(\frac{\|\eta''\|_{\mathrm{B}_{n,j}}|\eta(d_{n,j})|\varepsilon_{n,j}}{n^4}\right) \\ &\quad + O\left(\frac{\varepsilon^2_{n,j}\|\eta''\|_{\mathrm{B}_{n,j}}}{n^5}\right) + O\left(\frac{\|R_n\|_{\mathrm{B}_{n,j}}}{n}\right). \end{aligned}$$

If $|d_{n,j} - \varphi_2| > \delta$, by (5.4), (2.15), and Lemma 4.3, we obtain

$$s_{n,j} = d_{n,j} - \frac{\eta(d_{n,j})}{n+1} - \frac{\eta(d_{n,j})\eta'(d_{n,j})}{(n+1)^2} + O\left(\frac{1}{n^3}\right).$$

Now, if $\delta \geq |d_{n,j} - \varphi_2|$, from (4.6), (5.5) and (2.15):

$$s_{n,j} = d_{n,j} - \frac{\eta(d_{n,j})}{n+1} - \frac{\eta(d_{n,j})\eta'(d_{n,j})}{(n+1)^2} + O\left(\frac{\left(\ln \frac{n}{|j|}\right)^2}{nj^2}\right).$$

So, we get (2.22).

Finally, substitute (2.22) in ψ, and expand ψ by Taylor around $d_{n,j}$ to arrive at (2.23). □

Proof of Corollary 2.4 Let $k = 1$, and $j = -J_n + j'$ for some sufficiently small $j' \geq 1$. Then, $d_{n,j'} = 2\pi j/(n+1)$. Hence, from Theorem 5.5, (2.15), and (4.5), we get

$$s_{n,j} = e_{n,j} + O\left(\frac{\varepsilon_{n,j}}{n^2}\right) = \frac{2\pi j}{n+1} - \frac{1-t_1^2}{3-t_1^2}\frac{2\pi j}{(n+1)^2} + O\left(\frac{j^2}{n^3}\right). \tag{5.8}$$

To derive (2.24), substitute (5.8) in (3.11). If $k = 3$ and $j = n+1-J_n-j'$, the proof is similar. □

6 Cusp eigenvalues

If in Theorem 2.3 we suppose that $|j|/n^{1/2} \to 0$, then the asymptotic expansion (2.23) is not justified. Thus, in this section, we derive the eigenvalue asymptotic expansions when $|j|/n^{1/2} \ll 1$. In particular, we prove Theorems 2.5 and 2.6.

Recall that ζ_n, $\chi_{n,j}$, $\mathrm{U}_{n,j,A}$, and $\kappa_{n,j}$ are defined by (2.25), (2.26), (2.27), and (2.29), respectively. Let $\mathfrak{f}_j \equiv \mathfrak{f}_{2,j}$ be the values defined by (4.1).

We start our analysis by reducing (5.1) to (2.28) (cf. (2.17)). Let $n \geq 4$, $1/2 \gg \alpha > 0$ and $M > 0$. We define the set

$$\Pi_n := \left\{s \in \mathbb{C}\colon\ 0 < |\Re(s) - \varphi_2| \leq \frac{M}{n^{\frac{1}{2}+\alpha}},\quad |\Im(s)| \leq \frac{M}{n}\right\}.$$

Define also $F_n\colon \Pi_n \to \mathbb{C}$ by

$$F_n(s) := \mathfrak{f}_1(s - \varphi_2)\left(\zeta_n(f(s)) - \zeta_n(e^{-is})\right). \tag{6.1}$$

Theorem 6.1 *For every $n \geq 3$, and every $s \in \Pi_n$*

$$\det(\psi(s) - T_n(a)) = \frac{u(s)^n w(s)^{n+1} e^{i(n+1)s}}{u(s)e^{is} - w(s)} \frac{F_n(s)}{\mathfrak{f}_1(s - \varphi_2)}. \tag{6.2}$$

Proof Let $D_n(\psi(s)) \equiv \det(\psi(s) - T_n(a))$. By Widom's formula (5.1),

$$\begin{aligned} D_n(\psi(s)) &= \frac{u^{n+1}e^{i(n+1)s}}{u-w}\left(\frac{u^{n+1}}{ue^{is}-w} - \frac{w^{n+1}}{ue^{is}-u}\right) + \frac{u^{n+1}w^{n+1}}{(u-e^{is}u)(w-e^{is}u)} \\ &= \frac{u^{n+1}e^{i(n+1)s}}{ue^{is}-w}\left(\frac{u^{n+1}-w^{n+1}}{u-w} - \frac{w^{n+1}}{ue^{is}-u}\right) + \frac{u^{n+1}w^{n+1}}{(u-e^{is}u)(w-e^{is}u)} \\ &= \frac{u^{n+1}w^{n+1}e^{i(n+1)s}}{u(ue^{is}-w)}\left(\zeta_n(f(s)) - \zeta_n(e^{-is})\right). \end{aligned}$$

The conclusion follows from (6.1). □

Notice that, for every $n \geq 3$, from the properties of u, in (6.2), the factor

$$\frac{u(s)^n w(s)^{n+1} e^{i(n+1)s}}{u(s)e^{is} - w(s)} \tag{6.3}$$

is both bounded from above and away from zero on Π_n. Hence, from (6.2), a straightforward computation yields

$$\lim_{s\to\varphi_2} \det(\psi(s) - T_n(a)) = \frac{(n+1)t_2^{2n}e^{i(n+1)\varphi_2}}{e^{i\varphi_2}-1} \neq 0.$$

So μ_2 is not an eigenvalue of $T_n(a)$.

We proceed to explore the asymptotic behavior of F_n as $s \to \varphi_2$, for n large.

To simplify notation, we relabel $x \equiv s - \varphi_2$, $\mathfrak{f}_1 \equiv \mathfrak{f}_{2,1}$, $\mathfrak{f}_2 \equiv \mathfrak{f}_{2,2}$, where $\mathfrak{f}_{2,1}$ and $\mathfrak{f}_{2,2}$ are given by (4.1).

Proposition 6.2 *For every $n \in \mathbb{N}$ large enough, and for every x, such that $\varphi_2 + x \in \Pi_n$,*

$$f(\varphi_2 + x)^{n+1} = e^{\mathfrak{f}_1(n+1)x}\left(1 + \left(\mathfrak{f}_2 - \frac{1}{2}\mathfrak{f}_1^2\right)(n+1)x^2 + \frac{\left(\mathfrak{f}_2 - \frac{1}{2}\mathfrak{f}_1^2\right)^2}{2}(n+1)^2x^4\right) + O(nx^3) + O(n^3x^6).$$

Proof By (4.3)

$$f(\varphi_2 + x)^{n+1} = \exp\left(\mathfrak{f}_1(n+1)x + \left(\mathfrak{f}_2 - \frac{1}{2}\mathfrak{f}_1^2\right)(n+1)x^2 + O(n|x|^3)\right).$$

Now, we factorize the term $\exp(\mathfrak{f}_1(n+1)x)$ and expand by Taylor the rest, giving us

$$f(\varphi_2 + x)^{n+1} = e^{\mathfrak{f}_1(n+1)x}\left(1 + \left(\mathfrak{f}_2 - \frac{1}{2}\mathfrak{f}_1^2\right)(n+1)x^2 + \frac{1}{2}\left(\mathfrak{f}_2 - \frac{1}{2}\mathfrak{f}_1^2\right)^2(n+1)^2x^4 + O(n|x|^3) + O(n^2|x|^5) + O(n^2|x|^6) + O(n^3|x|^6)\right).$$

We derive the conclusion taking into consideration that $|x| \leq M/n^{\frac{1}{2}+\alpha}$. □

Proposition 6.3 *For every $n \in \mathbb{N}$ large enough, and for every x, such that $\varphi_2+x \in \Pi_n$,*

$$\zeta_n(f(\varphi_2+x)) = \frac{e^{\mathfrak{f}_1(n+1)x}-1}{\mathfrak{f}_1 x} + \left(1-\frac{\mathfrak{f}_2}{\mathfrak{f}_1^2}\right)\left(e^{\mathfrak{f}_1(n+1)x}-1\right) + \frac{\left(\mathfrak{f}_2-\frac{1}{2}\mathfrak{f}_1^2\right)}{\mathfrak{f}_1}(n+1)xe^{\mathfrak{f}_1(n+1)x} + \frac{\left(\mathfrak{f}_2-\frac{1}{2}\mathfrak{f}_1^2\right)^2}{2\mathfrak{f}_1}(n+1)^2x^3e^{\mathfrak{f}_1(n+1)x} + O|nx^2|.$$

Proof By (4.2)

$$\frac{f(\varphi_2+x)}{f(\varphi_2+x)-1} = \frac{1}{\mathfrak{f}_1 x} + 1 - \frac{\mathfrak{f}_2}{\mathfrak{f}_1^2} + O|x|.$$

Combining last equality with the expansion given in Proposition 6.2 yield the conclusion. □

Proposition 6.4 *For every $n \in \mathbb{N}$ large enough, and for every x, such that $\varphi_2+x \in \Pi_n$:*

$$\zeta_n(e^{-i(\varphi_2+x)}) = \frac{e^{-i(n+1)\varphi_2}e^{-i(n+1)x}-1}{1-e^{i\varphi_2}} + O|x|.$$

Proof The conclusion follows after plugging the expansion $e^{-ix} = 1 + O|x|$ in $\zeta_n(e^{-i(\varphi_2+x)})$. □

From Propositions 6.3 and 6.4, for every $n \in \mathbb{N}$ large enough, and for every $x \to 0$, such that $\varphi_2 + x \in \Pi_n$, we obtain

$$F_n(\varphi_2+x) = e^{\mathfrak{f}_1(n+1)x} - 1 + \mathfrak{F}_1 x\left(e^{\mathfrak{f}_1(n+1)x}-1\right) + \mathfrak{F}_2(n+1)x^2e^{\mathfrak{f}_1(n+1)x} + \frac{\mathfrak{F}_2^2}{2}(n+1)^2x^4e^{\mathfrak{f}_1(n+1)x} + \mathfrak{f}_1 x\kappa_n(x) + O(n|x^3|), \tag{6.4}$$

where

$$\mathfrak{F}_1 := \mathfrak{f}_1 - \frac{\mathfrak{f}_2}{\mathfrak{f}_1}, \qquad \mathfrak{F}_2 := \mathfrak{f}_2 - \frac{1}{2}\mathfrak{f}_1^2, \qquad \kappa_n(x) := -\frac{e^{-i(n+1)\varphi_2}e^{-i(n+1)x}-1}{1-e^{i\varphi_2}}.$$

The last ingredient we need to prove Theorem 2.5, is the Taylor expansion of F_n around $\chi_{n,j}$. To this end, let $n \in \mathbb{N}$ be large enough and $j \in \mathbb{Z}$. Consider (6.4) and its derivatives with respect to x, and $|\theta| \le A|j|^2$, for some $A > 0$; then

$$F_n\left(\chi_{n,j} - \frac{\theta}{(n+1)^2}\right) = F_n(\chi_{n,j}) - F_n'(\chi_{n,j})\frac{\theta}{(n+1)^2} + \frac{1}{2}F_n''(\chi_{n,j})\frac{\theta^2}{(n+1)^4} + O\left(F_n'''(y)\frac{|j|^6}{n^6}\right). \tag{6.5}$$

Straightforward computations yield that

$$\begin{aligned} F_n\left(\chi_{n,j}\right) &= \mathfrak{F}_2\frac{(j\pi)^2}{n+1} + \mathfrak{f}_1\kappa_{n,j}\frac{j\pi}{n+1} + \frac{\mathfrak{F}_2^2}{2}\frac{(j\pi)^4}{(n+1)^2} + O\left(\frac{j^3}{n^2}\right), \\ F_n'\left(\chi_{n,j}\right)\frac{\theta}{(n+1)^2} &= \frac{\mathfrak{f}_1\theta}{n+1} + \mathfrak{F}_2\mathfrak{f}_1\frac{(j\pi)^2\theta}{(n+1)^2} + O\left(\frac{j^3}{n^2}\right) + O\left(\frac{j^6}{n^3}\right), \\ F_n''\left(\chi_{n,j}\right)\frac{\theta^2}{(n+1)^4} &= \mathfrak{f}_1^2\frac{\theta^2}{(n+1)^2} + O\left(\frac{j^6}{n^3}\right) + O\left(\frac{j^8}{n^4}\right), \\ F_n'''(\xi)\frac{|j|^6}{(n+1)^6} &= O\left(\frac{j^6}{n^3}\right). \end{aligned}$$

Hence, (6.5) becomes:

$$\begin{aligned} F_n\left(\chi_{n,j} - \frac{\theta}{(n+1)^2}\right) = {} & \mathfrak{F}_2\frac{(j\pi)^2}{n+1} + \mathfrak{f}_1\kappa_{n,j}\frac{j\pi}{n+1} + \frac{\mathfrak{F}_2^2}{2}\frac{(j\pi)^4}{(n+1)^2} \\ & - \frac{\mathfrak{f}_1\theta}{n+1} - \mathfrak{F}_2\mathfrak{f}_1\frac{(j\pi)^2\theta}{(n+1)^2} + \mathfrak{f}_1^2\frac{\theta^2}{2(n+1)^2} \\ & + O\left(\frac{j^3}{n^2}\right) + O\left(\frac{j^6}{n^3}\right) + O\left(\frac{j^8}{n^4}\right). \end{aligned} \tag{6.6}$$

For every $n \ge 4$ and every j in $\mathbb{Z}$, set

$$\theta_{n,j} := \frac{\mathfrak{F}_2(j\pi)^2}{\mathfrak{f}_1} + \kappa_{n,j}\pi j = \frac{3}{4}\left(\cot\frac{\varphi_2}{2} - i\right)\pi^2 j^2 + \kappa_{n,j}\pi j. \tag{6.7}$$

From (6.6),

$$\begin{aligned} F_n\left(\chi_{n,j} - \frac{\theta_{n,j}}{(n+1)^2}\right) &= \frac{\mathfrak{f}_1^2\kappa_{n,j}^2(j\pi)^2}{2(n+1)^2} + O\left(\frac{j^3}{n^2}\right) + O\left(\frac{j^6}{n^3}\right) + O\left(\frac{j^8}{n^4}\right) \\ &= O\left(\frac{j^3}{n^2}\right) + O\left(\frac{j^6}{n^3}\right) + O\left(\frac{j^8}{n^4}\right). \end{aligned}$$

Thus, if we suppose that $1 \le |j| \ll n^{1/2}$, then

$$F_n\left(\chi_{n,j} - \frac{\theta_{n,j}}{(n+1)^2}\right) = O\left(\frac{j^3}{n^2}\right) + O\left(\frac{j^6}{n^3}\right). \tag{6.8}$$

Proof of Theorem 2.5 Let $n \in \mathbb{N}$ and $j \in \mathbb{Z} \setminus \{0\}$, such that $\chi_{n,j} \in (0, 2\pi)$ and $|j|/n^{1/2} \ll 1$.

Recall that the factor (6.3) is both bounded from above and away from zero on Π_n. Therefore, if a value s in Π_n satisfies (2.28), then (6.2) equals zero, implying that $\psi(s)$ is an eigenvalue of $T_n(a)$.

Now, from (6.6), if n is large enough and $j \in \mathbb{Z}$ satisfies $1 \le |j| \ll n^{1/2}$, then

$$F_n\left(\chi_{n,j} - \frac{\theta}{(n+1)^2}\right) = \mathfrak{F}_2\frac{(j\pi)^2}{n+1} + \mathfrak{f}_1\kappa_{n,j}\frac{j\pi}{n+1} - \frac{\mathfrak{f}_1\theta}{n+1} + r_{n,j}(\theta), \tag{6.9}$$

where $\theta = O(j^2)$ and $|r_{n,j}(\theta)| = O\left(|\theta|^2/n^2\right) = O(j^4/n^2)$. We define the auxiliary functions:

$$\nu(\theta) := -\frac{\mathfrak{f}_1\theta}{n+1}, \qquad \rho(\theta) := \mathfrak{F}_2\frac{(j\pi)^2}{n+1} + \mathfrak{f}_1\kappa_{n,j}\frac{j\pi}{n+1} + r_{n,j}(\theta).$$

From (6.9), there exists $A > 0$, such that for every $\theta \in \mathbb{C}$ with $|\theta| \le Aj^2/|\mathfrak{f}_1|$, we get. $|\rho(\theta)| < Aj^2/(n+1)$. Thus, for $|\theta| = Aj^2/|\mathfrak{f}_1|$:

$$|\nu(\theta)| = \frac{Aj^2}{n+1} > |\rho(\theta)|,$$

so

$$|\nu(\theta)| + |(\rho+\nu)(\theta)| > |\rho(\theta)| = |(\rho+\nu)(\theta) - \nu(\theta)|.$$

By Rouche's theorem, ν and $\rho + \nu$ have the same number of roots inside the ball $\{\omega\colon |\omega| \le Aj^2/|\mathfrak{f}_1|\}$. On this set, there exists only one root of $\rho + \nu$, since ν has only zero as a root. In other words, there exists a unique value $s_{n,j}$ in $\mathrm{U}_{n,j,A}$, such that $\psi(s_{n,j})$ is an eigenvalue of $T_n(a)$. □

We are prepared to conclude our analysis and proof Theorem 2.6.

Proof of Theorem 2.6 From Theorem 2.5 and (6.8), we have that

$$F_n\left(\varphi_2 + \frac{j\pi}{n+1} - \frac{\theta}{(n+1)^2}\right) = 0$$

if and only if θ is of the form (6.7). Then

$$s_{n,j} = \chi_{n,j} - \frac{\theta_{n,j}}{(n+1)^2} + O\left(\frac{j^4}{n^3}\right). \tag{6.10}$$

Finally, for (2.30), just substitute (6.10) in (3.11):

$$\psi(s_{n,j}) = \mu_2 + \boldsymbol{\psi}_{2,2}\frac{(j\pi)^2}{(n+1)^2} - 2\boldsymbol{\psi}_{2,2}\kappa_{n,j}\frac{(j\pi)^2}{(n+1)^3}$$

$$+\left(\boldsymbol{\psi}_{2,3}-i\boldsymbol{\psi}_{2,2}\mathfrak{F}_2\right)\frac{(j\pi)^3}{(n+1)^3}+O\left(\frac{j^5}{n^4}\right).$$

Direct computation shows that $\boldsymbol{\psi}_{2,3}-i\boldsymbol{\psi}_{2,2}\mathfrak{F}_2=0$. Hence, we get (2.24). □

Remark 6.5 In Theorem 2.6, we can use negative integers j; however, the expansion (2.30) is the same if we choose j or $-j$.

7 Numerical experiments

In this section, we show some numerical tests of the asymptotic formulas we obtained for the eingevalues $\lambda_{n,j}$ of $T_n(a)$. These experiments demonstrate the computational advantages of our formulas for even not so large n.

We compute u on a uniform mesh of $[0, 2\pi]$, see Remark 3.16. Then, with these points we construct a polynomial approximation of u. Finally, we make approximations of all the other functions that appear in (2.23).

For every $n \geq 4$ and every j, we introduce the following notation:

- $\lambda_{n,j}^{\text{Gen}}$ denotes the eigenvalues of $T_n(a)$ computed using a general eigenvalue algorithm from Sagemath, depending of n, we use from 100 to 1000 digits of precision;
- $\lambda_{n,j}^{\text{Inner}}$ denotes the approximation resulting from (2.23);
- $\lambda_{k,n,j}^{\text{Ext}}$ denotes the approximation resulting from (2.24);
- $\lambda_{n,j}^{\text{Cusp}}$ the approximation resulting from (2.30);
- define the error values

$$E_{n,j}^{\text{Inner}} := |\lambda_{n,j}^{\text{Gen}}-\lambda_{n,j}^{\text{Inner}}|, \qquad E_{k,n,j}^{\text{Ext}} := |\lambda_{n,j}^{\text{Gen}}-\lambda_{k,n,j}^{\text{Ext}}|,$$
$$E_{n,j}^{\text{Cusp}} := |\lambda_{n,j}^{\text{Gen}}-\lambda_{n,j}^{\text{Cusp}}|;$$

- define the relative error values

$$\text{RE}_{k,n,j}^{\text{Ext}} := \frac{|\lambda_{n,j}^{\text{Gen}}-\lambda_{k,n,j}^{\text{Ext}}|}{|\lambda_{n,j}^{\text{Gen}}-\mu_k|},$$
$$\text{RE}_{n,j}^{\text{Cusp}} := \frac{|\lambda_{n,j}^{\text{Gen}}-\lambda_{n,j}^{\text{Cusp}}|}{|\lambda_{n,j}^{\text{Gen}}-\mu_2|}.$$

Figure 7 plots the 10-base logarithm of $E_{n,j}^{\text{Inner}}$ for $c=i$, different values of n, and every $\lambda_{n,j}$, see Remark 3.9.

Observe in Fig. 7, that as $\lambda_{n,j}$ approaches μ_2 the error $E_{n,j}^{\text{Cusp}}$ increases drastically. On the other hand, Fig. 8 shows that our approximations $\lambda_{n,j}^{\text{Cusp}}$ are much better for these eigenvalues.

We test (2.24) for $c=i$, some values of n, $k=1$, and $j=1,2$, the corresponding error $E_{k,n,j}^{\text{Ext}}$, and normalized error $\text{NE}_{k,n,j}^{\text{Ext}} \equiv n^4 E_{k,n,j}^{\text{Ext}}/j^4$ are shown in Table 1. The

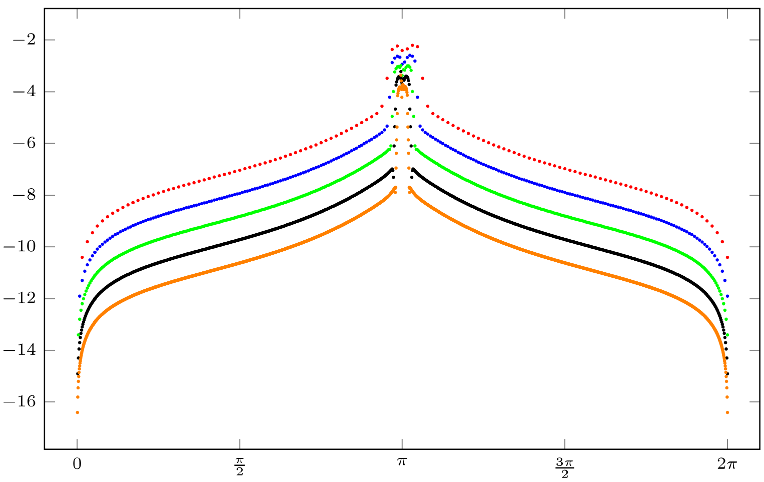

Fig. 7 Ten-base logarithm of the individual absolute error $E_{n,j}^{\mathrm{Inner}}$ for $c = i$ and $n = 128, 256, 512, 1024, 2048$, in red, blue, green, black, and orange, respectively

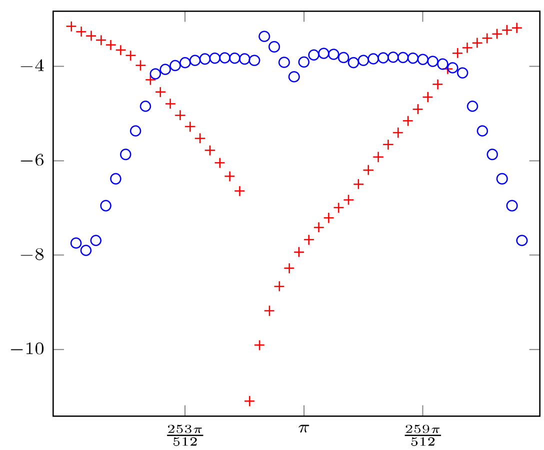

Fig. 8 Ten-base logarithm of the individual absolute errors $E_{n,j}^{\mathrm{Inner}}$ (blue) and $E_{n,j}^{\mathrm{Cusp}}$ (red) for $c = i$ and $n = 2048$, and the $\lambda_{n,j}$ closest to μ_2

relative error $\mathrm{RE}_{k,n,j}^{\mathrm{Ext}}$, and normalized relative error $\mathrm{RNE}_{k,n,j}^{\mathrm{Ext}} \equiv n^4\,\mathrm{RE}_{k,n,j}^{\mathrm{Ext}}\,/\,j^4$ are shown in Table 2.

We test (2.30) for $c = i$, some values of n and $j = 1, 2$, the corresponding error $E_{k,n,j}^{\mathrm{Cusp}}$, and normalized error $\mathrm{NE}_{k,n,j}^{\mathrm{Cusp}} \equiv n^4 E_{k,n,j}^{\mathrm{Cusp}}/\,j^5$ are shown in Table 3. The relative error $\mathrm{RE}_{k,n,j}^{\mathrm{Cusp}}$, and normalized relative error $\mathrm{RNE}_{k,n,j}^{\mathrm{Cusp}} \equiv n^4\,\mathrm{RE}_{k,n,j}^{\mathrm{Cusp}}\,/\,j^5$ are shown in Table 4.

Table 1 For $c = i$, some n and $j = 1, 2$, $E^{\text{Ext}}_{1,n,j}$ and $\text{NE}^{\text{Ext}}_{1,n,j}$

n	$E^{\text{Ext}}_{1,n,1}$	$\text{NE}^{\text{Ext}}_{1,n,1}$	$E^{\text{Ext}}_{1,n,2}$	$\text{NE}^{\text{Ext}}_{1,n,2}$
128	8.38×10^{-8}	22.5	1.45×10^{-6}	24.3
256	5.33×10^{-9}	22.9	9.22×10^{-8}	24.7
512	3.36×10^{-9}	23.1	5.81×10^{-9}	24.8
1024	2.11×10^{-11}	23.2	3.65×10^{-10}	25.1
2048	1.32×10^{-12}	23.2	2.28×10^{-11}	25.1

Table 2 For $c = i$, some n and $j = 1, 2$, $\text{RE}^{\text{Cusp}}_{1,n,j}$ and $\text{RNE}^{\text{Cusp}}_{1,n,j}$

n	$\text{RE}^{\text{Ext}}_{1,n,1}$	$\text{RNE}^{\text{Ext}}_{1,n,1}$	$\text{RE}^{\text{Ext}}_{1,n,2}$	$\text{RNE}^{\text{Ext}}_{1,n,2}$
128	7.58×10^{-5}	1.24	3.28×10^{-4}	1.34
256	1.91×10^{-5}	1.25	8.26×10^{-5}	1.35
512	4.79×10^{-6}	1.26	2.07×10^{-5}	1.36
1024	1.20×10^{-6}	1.26	5.19×10^{-6}	1.36
2048	23.2×10^{-7}	1.26	1.30×10^{-6}	1.36

Table 3 For $c = i$, some n and $j = 1, 2$, $E^{\text{Cusp}}_{1,n,j}$ and $\text{NE}^{\text{Cusp}}_{1,n,j}$

n	$E^{\text{Cusp}}_{1,n,1}$	$\text{NE}^{\text{Cusp}}_{1,n,1}$	$E^{\text{Cusp}}_{1,n,2}$	$\text{NE}^{\text{Cusp}}_{1,n,2}$
128	5.29×10^{-7}	142	8.54×10^{-6}	71.6
256	5.31×10^{-8}	142	5.19×10^{-7}	69.7
512	2.07×10^{-9}	142	3.20×10^{-8}	68.7
1024	1.29×10^{-10}	142	1.98×10^{-9}	68.2
2048	8.07×10^{-12}	142	1.24×10^{-10}	67.9

Table 4 Error and normalized errors $\text{RE}^{\text{Cusp}}_{1,n,j}$ and $\text{RNE}^{\text{Cusp}}_{1,n,j}$, respectively, for $j = 1, 2$

n	$\text{RE}^{\text{Cusp}}_{1,n,1}$	$\text{RNE}^{\text{Cusp}}_{1,n,1}$	$\text{RE}^{\text{Cusp}}_{1,n,2}$	$\text{RNE}^{\text{Cusp}}_{1,n,2}$
128	4.46×10^{-4}	7.30	1.83×10^{-3}	3.75
256	1.10×10^{-4}	7.25	4.34×10^{-4}	3.59
512	2.76×10^{-5}	7.22	1.07×10^{-4}	3.51
1024	6.88×10^{-6}	7.21	2.65×10^{-5}	3.47
2048	1.72×10^{-6}	7.20	6.58×10^{-6}	3.45

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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