Eigenvalues of the Laplacian Matrices of Cycles with One Overweighted Edge

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Dedicated to Yuri Karlovich on the occasion of his 75th birthday.

Abstract We study the individual behavior of the eigenvalues of the laplacian matrices of the cyclic graph of order n, where one edge has weight $\alpha \in \mathbb{C}$, with $\operatorname{Re}(\alpha) > 1$, and all the others have weights 1. This paper is a sequel to two previous ones where we considered $\operatorname{Re}(\alpha) \in [0, 1]$ and $\operatorname{Re}(\alpha) < 0$. Now, we prove that for $\operatorname{Re}(\alpha) > 1$ and $n > \operatorname{Re}(\alpha)/\operatorname{Re}(\alpha - 1)$, one eigenvalue is greater than 4 while the others belong to $[0, 4]$ and are distributed as the function $x \mapsto 4\sin^2(x/2)$. Additionally, we prove that as n tends to ∞, the outlier eigenvalue converges exponentially to $4\operatorname{Re}(\alpha)^2/(2\operatorname{Re}(\alpha)-1)$. We give exact formulas for half of the inner eigenvalues, while for the others we justify the convergence of Newton's method and the fixed-point iteration method. We find asymptotic expansions, as n tends to ∞, both for the eigenvalues belonging to $[0, 4]$ and the outliers. We also compute the eigenvectors and their norms.

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1 Introduction

For every natural $n \geq 3$ and every α in $\mathbb{C}$, we consider the $n \times n$ complex laplacian matrix $L_{\alpha,n}$ with the following structure:

$$L_{\alpha,6} = \begin{bmatrix} 1+\overline{\alpha} & -1 & 0 & 0 & 0 & -\overline{\alpha} \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & -1 & 1+\alpha \end{bmatrix}.$$

If α is real, $L_{\alpha,n}$ is the laplacian matrix of $G_{\alpha,n}$, where $G_{\alpha,n}$ is the cyclic graph of order n, where the edge between the vertices 1 and n weighs α, and all other edges weigh 1. See [15] for the general theory on laplacian matrices. In Fig. 1, we show the case $n = 6$. The eigenvalues and eigenvectors of $L_{\alpha,n}$ are crucial to solve the heat and wave equations on $G_{\alpha,n}$. Moreover, matrices of the form $2I_n - L_{\alpha,n}$ are related to counting the paths in a cyclic graph with certain loops [5].

The matrices $L_{\alpha,n}$ can be considered as tridiagonal Toeplitz matrices with perturbations in the corners $(1, 1)$, $(1, n)$, $(n, 1)$ and (n, n). They can also be viewed as periodic Jacobi matrices. Some matrices of these classes and their applications were studied in [2–4, 6–8, 10, 11, 14, 16, 17, 19–21].

The present paper is a continuation of [12, 13]. In [12], we proved that for every α in $\mathbb{C}$ the characteristic polynomial of $L_{\alpha,n}$, defined by $D_{\alpha,n}(\lambda) := \det(\lambda I - L_{\alpha,n})$, equals the characteristic polynomial $D_{\mathrm{Re}(\alpha),n}$ of $L_{\mathrm{Re}(\alpha),n}$. This implies that the eigenvalues of $L_{\alpha,n}$ only depend on $\mathrm{Re}(\alpha)$. Therefore, to understand the behavior of the eigenvalues, it is sufficient to consider the case where $\alpha \in \mathbb{R}$ and the corresponding matrices $L_{\alpha,n}$ are real and symmetric. So, for every α in $\mathbb{C}$, the eigenvalues of $L_{\alpha,n}$ are real, and we enumerate them as follows:

$$\lambda_{\alpha,n,1} \leq \lambda_{\alpha,n,2} \leq \ldots \leq \lambda_{\alpha,n,n}.$$

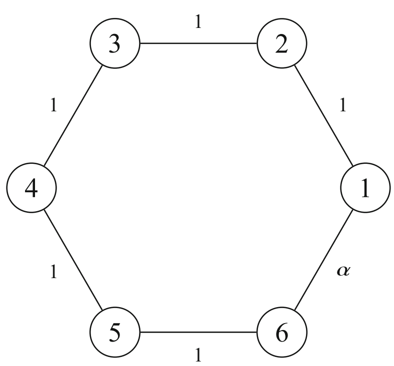

Fig. 1 Graph $G_{\alpha,6}$

It is a very well-known fact that the eigenvalues of the $n \times n$ tridiagonal Toeplitz matrix, with values $-1, 2, -1$ in the nonzero diagonals, are the numbers $g(j\pi/(n+1))$, $j = 1, \dots, n$, where g is defined by

$$g(x) := 4\sin^2\frac{x}{2} \qquad (x \in [0,\pi]). \tag{1}$$

By the Cauchy interlacing theorem (see, e.g., [18, Theorem 4.2]), the eigenvalues of $L_{\alpha,n}$ are also asymptotically distributed by g on $[0,\pi]$, as n tends to infinity. This is also a simple consequence of the theory of generalized locally Toeplitz sequences [9].

In [12], we studied the individual behavior of the eigenvalues of the matrices $L_{\alpha,n}$ for α in $(0,1)$. In that case, we showed that the eigenvalues of $L_{\alpha,n}$ belong to $[0,4]$. We solved the characteristic equation by numerical methods and derived asymptotic formulas for all eigenvalues. In [13], we considered the case where $\alpha < 0$. In that scenery, we proved that if $n > (\alpha-1)/\alpha$ then the minimal eigenvalue $\lambda_{\alpha,n,1}$ goes out of the interval $[0,4]$; moreover, the sequence $(\lambda_{\alpha,n,1})_{n>(\alpha-1)/\alpha}$ strictly decreases and converges exponentially to $4\alpha^2/(2\alpha-1)$.

In this paper, we consider the case where $\alpha > 1$ (or, more generally, $\operatorname{Re}(\alpha) > 1$). This means that the interaction between the vertices 1 and n is stronger than the interactions between the other neighbors in the cycle.

It turns out that, if n is even or if n is odd and satisfies $n > \alpha/(\alpha-1)$, then the maximal eigenvalue $\lambda_{\alpha,n,n}$ is greater than 4, while the others belong to the interval $[0,4]$ and behave similarly to the eigenvalues of $L_{\alpha,n}$ when $0 < \alpha < 1$, as discussed in [12].

We use the phrase "inner eigenvalues" for the eigenvalues belonging to the clustering set $[0,4]$, and "outlier eigenvalue" for the one that does not belong to this set. See also our general definition of outlier eigenvalue in [13].

We show that if $\alpha > 1$, then the sequence of outlier eigenvalues $(\lambda_{\alpha,n,n})_{n\geq 3}$ converges exponentially to the number $\Omega_\alpha := 4\alpha^2/(2\alpha-1)$. The major difference to the previous paper [13] is that the sequence of the outliers approaches the limit value from both directions:

$$\operatorname{sign}(\lambda_{\alpha,n,n} - \Omega_\alpha) = (-1)^n \qquad \left(n > \frac{\alpha}{\alpha-1}\right). \tag{2}$$

The main results of this paper are stated in Sect. 2, while the majority of the content is dedicated to the corresponding proofs: we represent the characteristic polynomial in convenient forms and show the localization of the eigenvalues (Sect. 3), we study the asymptotic behavior of the inner eigenvalues and guarantee their computation with the Newton method (Sect. 4), then we focus our attention on the last eigenvalue $\lambda_{\alpha,n,n}$ (Sect. 5) and analyze its asymptotic behavior separately for both odd (Sect. 6) and even values of n (Sect. 7). Finally, we calculate the norms of the eigenvectors (Sect. 8) and show some numerical experiments (Sect. 9).

2 Main Results

As will be stated in Proposition 3.1, for every $\alpha \in \mathbb{C}$ we have that $D_{\alpha,n} = D_{\mathrm{Re}(\alpha),n}$. So, unless specified otherwise, we consider $\alpha > 1$.

We begin our analysis with the localization of the eigenvalues. For this purpose, define

$$\varkappa_\alpha := \frac{\alpha - 1}{\alpha}, \tag{3}$$

$$\Omega_\alpha := \frac{4\alpha^2}{2\alpha - 1}, \qquad \text{i.e.,} \qquad \Omega_\alpha = \frac{4}{1 - \varkappa_\alpha^2}. \tag{4}$$

Notice that $0 < \varkappa_\alpha < 1$ and $\Omega_\alpha > 4$. Also, for every j in $\{1, \ldots, n\}$, we put

$$d_{n,j} := \frac{(j-1)\pi}{n}. \tag{5}$$

Theorem 2.1 (Localization of Eigenvalues) *Let $n \geq 3$. Then $\lambda_{\alpha,n,1} = 0$. For every j with $2 \leq j \leq n-1$,*

$$g\left(d_{n,j}\right) < \lambda_{\alpha,n,j} < g\left(d_{n,j+1}\right) \qquad (j \text{ odd}),$$
$$\lambda_{\alpha,n,j} = g\left(d_{n,j+1}\right) \qquad (j \text{ even}).$$

Furthermore, the localization of $\lambda_{\alpha,n,n}$ depends on n:

(1) if $n < \varkappa_\alpha^{-1}$ and n is odd, then $g(d_{n,n}) < \lambda_{\alpha,n,n} < g(\pi) = 4$;
(2) if $n = \varkappa_\alpha^{-1}$ and n is odd, then $\lambda_{\alpha,n,n} = 4$;
(3) if n is odd and $n > \varkappa_\alpha^{-1}$, then $4 < \lambda_{\alpha,n,n} < \Omega_\alpha$;
(4) if n is even, then $\Omega_\alpha < \lambda_{\alpha,n,n} \leq 4 + 2\alpha$.

According to Theorem 2.1, the eigenvalues $\lambda_{\alpha,n,j}$ with even indices j do not depend of α. This theorem also implies that the eigenvalues are asymptotically distributed as the function g on $[0, \pi]$:

$$\lim_{n\to\infty} \frac{\#\left\{j \in \{1, \ldots, n\}\colon\ \lambda_{\alpha,n,j} \leq u\right\}}{n} = \frac{\mu(\{x \in [0,\pi]\colon\ g(x) \leq u\})}{\pi}. \tag{6}$$

Here, μ is the Lebesgue measure.

Statements (3) and (4) of Theorem 2.1 mean that for n large enough, we have two different localizations of the largest eigenvalue $\lambda_{\alpha,n,n}$ depending of the parity of n.

If n is odd, then the outlier eigenvalues of $L_{\alpha,n}$ and $L_{1-\alpha,n}$ are related by $\lambda_{\alpha,n,n} = 4 - \lambda_{1-\alpha,n,1}$ (Proposition 6.1). Therefore, in the analysis of $\lambda_{\alpha,n,n}$ for odd n, we can proceed very similarly to [12].

However, for even values of n, the equation for $\lambda_{\alpha,n,n}$ has a quite different form, see Theorem 2.2.

Motivated by Theorem 2.1, we use g defined by (1) as a change of variable in the characteristic equation when $\lambda_{\alpha,n,j} \in [0, 4]$ and set

$$z_{\alpha,n,j} := \widetilde{g}^{-1}(\lambda_{\alpha,n,j}), \tag{7}$$

where $\widetilde{g} \colon [0, \pi] \to [0, 4]$ is a restriction of g.

To state the main equation for inner eigenvalues, we define the function $\eta_\alpha \colon [0, \pi] \to \mathbb{R}$ by

$$\eta_\alpha(x) := 2 \arctan\left(\varkappa_\alpha \tan \frac{x}{2}\right), \quad \text{i.e.,} \quad \eta_\alpha(x) = 2 \arctan\left(\frac{\alpha - 1}{\alpha} \tan \frac{x}{2}\right). \tag{8}$$

Since $\varkappa_\alpha$ is positive, η_α is positive, strictly increasing and takes values on $[0, \pi]$.

Theorem 2.2 (Main Equation for Inner Eigenvalues) *Let j be odd with $3 \le j \le n - 1$. Then the number $z_{\alpha,n,j}$ is the unique solution in $[0, \pi]$ of the equation*

$$x = d_{n,j} + \frac{\eta_\alpha(x)}{n}. \tag{9}$$

The same Eq. (9) also holds for $z_{\alpha,n,n}$, if n is odd and $n < \varkappa_\alpha^{-1}$.

Now, we need a suitable change of variable associated to $\lambda_{\alpha,n,n}$. Thus, define $g_+ \colon [0, \infty) \to [4, \infty)$ by

$$g_+(x) := 2 + 2\cosh(x) = 4\cosh^2 \frac{x}{2} = 4 + 4\sinh^2 \frac{x}{2}. \tag{10}$$

Let also

$$N_\alpha := \max\{3, \lfloor \varkappa_\alpha^{-1} \rfloor + 1\}. \tag{11}$$

So, if $n \ge 4$ is even or $n \ge N_\alpha$ is odd, then we use (10) as a change of variable and put

$$s_{\alpha,n} := g_+^{-1}(\lambda_{\alpha,n,n}). \tag{12}$$

In Fig. 2 we have glued together g and $x \mapsto g_+(x - \pi)$ into one spline.

Theorem 2.1 says that for every $n \ge N_\alpha$, $\lambda_{\alpha,n,n}$ is in a neighborhood of Ω_α, thus we define

$$\omega_\alpha := g_+^{-1}(\Omega_\alpha) = \log(2\alpha - 1). \tag{13}$$

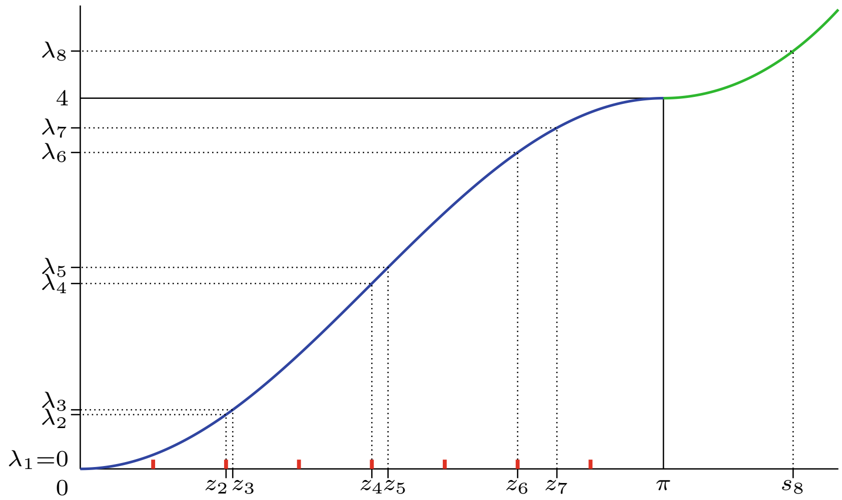

Fig. 2 Plot of g (blue), plot of $x \mapsto g_+(x-\pi)$ (green), points $z_{\alpha,n,j}$ and $s_{\alpha,n}$, and the corresponding values of $\lambda_{\alpha,n,j}$, for $\alpha = 3/2$ and $n = 8$. The red labels on the horizontal axis are $j\pi/n$

To get the main equation for the outlier eigenvalue, we define the real-valued functions $\psi_{\alpha,n}$ by

$$\psi_{\alpha,n}(x) := \begin{cases} 2\,\mathrm{arctanh}\left(\varkappa_\alpha \tanh \frac{nx}{2}\right), & \text{if } n \geq 3,\ n \text{ is odd},\ x \in [0,+\infty), \\ 2\,\mathrm{arctanh}\left(\varkappa_\alpha \coth \frac{nx}{2}\right), & \text{if } n \geq 4,\ n \text{ is even},\ x \in [\omega_\alpha,+\infty). \end{cases} \tag{14}$$

For every $n \geq 3$ and every $x \geq \omega_\alpha$,

$$\varkappa_\alpha \coth \frac{nx}{2} < \varkappa_\alpha \coth \frac{x}{2} \leq \varkappa_\alpha \coth \frac{\omega_\alpha}{2} = 1,$$

hence $\psi_{\alpha,n}$ is well defined. The two cases in (14) can be joined by elevating $\tanh(nx/2)$ to the power $(-1)^{n+1}$.

Theorem 2.3 (Main Equation for the Outlier Eigenvalue) *If n is odd and $n > \varkappa_\alpha^{-1}$, then $s_{\alpha,n}$ is the unique solution in $(0,\omega_\alpha)$ of the equation*

$$x = \psi_{\alpha,n}(x). \tag{15}$$

If n is even, then $s_{\alpha,n}$ is the unique solution in $(\omega_\alpha,+\infty)$ of the Eq. (15).

To get asymptotic expansions for the inner eigenvalues, we introduce the function $\Lambda_{\alpha,n} \colon [0,\pi] \to \mathbb{R}$ by

$$\Lambda_{\alpha,n}(x) := g(x) + \frac{g'(x)\eta_\alpha(x)}{n} + \frac{g'(x)\eta_\alpha(x)\eta_\alpha'(x) + \frac{1}{2}g''(x)\eta_\alpha(x)^2}{n^2}.$$

Then, for all $n \ge N_\alpha$ and all odd j with $3 \le j \le n-1$, we define $\lambda_{\alpha,n,j}^{\text{asympt}}$ by

$$\lambda_{\alpha,n,j}^{\text{asympt}} := \Lambda_{\alpha,n}(d_{n,j}). \tag{16}$$

Theorem 2.4 (Asymptotic Expansion of Inner Eigenvalues) *There exists $C_1(\alpha) > 0$ such that for every $n \ge N_\alpha$,*

$$\max_{\substack{3 \le j \le n-1 \\ j \text{ odd}}} \left| \lambda_{\alpha,n,j} - \lambda_{\alpha,n,j}^{\text{asympt}} \right| \le \frac{C_1(\alpha)}{n^3}. \tag{17}$$

To state the asymptotic expansion for $\lambda_{\alpha,n,n}$, we introduce the following numbers:

$$\begin{aligned} \beta_{\alpha,1} &:= \frac{16\alpha^2(\alpha-1)^2}{(2\alpha-1)^2}, \quad \beta_{\alpha,2} := -\frac{64\alpha^3(\alpha-1)^3}{(2\alpha-1)^3}, \\ \beta_{\alpha,3} &:= \frac{32\alpha^2(1-\alpha)^2(2\alpha^2-2\alpha+1)}{(2\alpha-1)^3}. \end{aligned} \tag{18}$$

Equivalently,

$$\beta_{\alpha,1} = \frac{16\varkappa_\alpha^2}{(1-\varkappa_\alpha^2)^2}, \qquad \beta_{\alpha,2} = -\frac{64\varkappa_\alpha^3}{(1-\varkappa_\alpha^2)^3}, \qquad \beta_{\alpha,3} = \frac{32\varkappa_\alpha^2(\varkappa_\alpha^2+1)}{(1-\varkappa_\alpha^2)^3}. \tag{19}$$

Now, we define $\lambda_{\alpha,n,n}^{\text{asympt}}$ by

$$\lambda_{\alpha,n,n}^{\text{asympt}} := \Omega_\alpha + (-1)^n \beta_{\alpha,1} e^{-n\omega_\alpha} + \beta_{\alpha,2} n e^{-2n\omega_\alpha} + \beta_{\alpha,3} e^{-2n\omega_\alpha}. \tag{20}$$

Of course, $e^{-n\omega_\alpha}$ can also be written as $1/(2\alpha-1)^n$.

Theorem 2.5 (Asymptotic Expansion of the Last Eigenvalue) *As $n \to \infty$, the extreme eigenvalue $\lambda_{\alpha,n,n}$ of $L_{\alpha,n}$ converges exponentially to Ω_α. More precisely, there exists $C_2(\alpha) > 0$ such that for every $n \ge N_\alpha$,*

$$\left| \lambda_{\alpha,n,n} - \lambda_{\alpha,n,n}^{\text{asympt}} \right| \le C_2(\alpha) n^2 e^{-3n\omega_\alpha}. \tag{21}$$

So, in the case when $\alpha > 1$ and n is large enough, the maximal eigenvalue goes out of $[0, 4]$ and converges rapidly to the number $\Omega_\alpha > 4$. While, the rest behaves asymptotically as the function g on $[0, \pi]$. The "right spectral gap" $\lambda_{\alpha,n,n} - \lambda_{\alpha,n,n-1}$ converges to $\Omega_\alpha - 4$.

Our last analysis focuses on the eigenvectors and their norms. Similarly to the situation with the eigenvalues, we have to separate the case $\lambda = 0$, the "trigonometric case" ($0 < \lambda \le 4$), and the "hyperbolic case" ($\lambda > 4$).

Theorem 2.6 (Eigenvectors for $\operatorname{Re}(\alpha) > 1$**)** *Let* $\alpha \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) > 1$ *and* $n \geq 3$. *Then,* $L_{\alpha,n}$ *has the following eigenvectors.*

1. $[1, \dots, 1]^\top$ *is an eigenvector associated to the eigenvalue* $\lambda_{\alpha,n,1} = 0$.
2. *For every* j, $2 \leq j \leq n-1$, *the vector* $v_{\alpha,n,j} = [v_{\alpha,n,j,k}]_{k=1}^n$ *with the following components is an eigenvector associated to* $\lambda_{\alpha,n,j}$*:*

$$v_{\alpha,n,j,k} := \sin(kz_{\alpha,n,j}) - (1-\overline{\alpha}) \sin((k-1)z_{\alpha,n,j}) + \overline{\alpha} \sin((n-k)z_{\alpha,n,j}). \tag{22}$$

   The same formula (22) *also works for* $j = n$, *if* n *is odd and* $n \leq \varkappa_{\operatorname{Re}(\alpha)}^{-1}$.
3. *If* n *is odd and* $n > \varkappa_{\operatorname{Re}(\alpha)}^{-1}$, *or* n *is even, then the vector* $v_{\alpha,n,n} = [v_{\alpha,n,n,k}]_{k=1}^n$ *with the following components is an eigenvector associated to* $\lambda_{\alpha,n,n}$*:*

$$\begin{aligned} v_{\alpha,n,n,k} := (-1)^k \big[&(-1)^n \overline{\alpha} \sinh((n-k)s_{\alpha,n}) + (1-\overline{\alpha}) \sinh((k-1)s_{\alpha,n}) \\ &+ \sinh(ks_{\alpha,n}) \big]. \end{aligned} \tag{23}$$

Finally, to present the asymptotic behavior of the norms of the eigenvectors given by (22), we need the following auxiliar function: for every x in $[0, \pi]$, we define

$$\nu_\alpha(x) := \frac{1-\operatorname{Re}(\alpha)}{2} g(x) + \frac{\operatorname{Re}(\alpha)}{2} g(\eta_{\operatorname{Re}(\alpha)}(x)) + \frac{|\alpha|^2 - \operatorname{Re}(\alpha)}{2} g(x - \eta_{\operatorname{Re}(\alpha)}(x)). \tag{24}$$

Theorem 2.7 (Norms of Eigenvectors for $\operatorname{Re}(\alpha) > 1$**)** *Let* $\alpha \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) > 1$ *and* $n \geq N_{\operatorname{Re}(\alpha)}$.

1. *If* j *is even and* $2 \leq j \leq n-1$, *then*

$$\|v_{\alpha,n,j}\|_2 = |\alpha - 1| \sqrt{2n} \sin \frac{j\pi}{2n}. \tag{25}$$

2. *If* j *is odd and* $3 \leq j \leq n-1$, *then as* $n \to \infty$

$$\|v_{\alpha,n,j}\|_2 = \sqrt{\nu_\alpha(d_{n,j})n} + O_\alpha\left(\frac{1}{\sqrt{n}}\right), \tag{26}$$

   with $O_\alpha\left(1/\sqrt{n}\right)$ *uniformly on* j.
3. *As* $n \to \infty$,

$$\|v_{\alpha,n,n}\|_2 = \frac{|\alpha|}{2\sqrt{2\operatorname{Re}(\alpha)(\operatorname{Re}(\alpha)-1)}} e^{n\omega_{\operatorname{Re}(\alpha)}} + O_\alpha(n). \tag{27}$$

In numerical computation of the eigenvectors, it is convenient to divide the expressions given in Theorem 2.6 by the norms' approximations from Theorem 2.7.

3 The Characteristic Polynomial and Eigenvalues' Localization

Recall that we denote the characteristic polynomial $\det(\lambda I - L_{\alpha,n})$ by $D_{\alpha,n}(\lambda)$. Aditionally, $\varkappa_\alpha$, ω_α are defined by (3), (13).

For every $m \geq 0$, let T_m and U_m the m-th degree Chebyshev polynomials of the first and second kind, respectively.

By cofactor expansion, it is easy to prove the following proposition.

Proposition 3.1 (Characteristic Polynomial of $L_{\alpha,n}$ for Complex α) *For $n \geq 3$ and $\alpha \in \mathbb{C}$,*

$$D_{\alpha,n}(\lambda) = (\lambda - 2\operatorname{Re}(\alpha))U_{n-1}\left(\frac{\lambda-2}{2}\right) - 2\operatorname{Re}(\alpha)U_{n-2}\left(\frac{\lambda-2}{2}\right) + 2(-1)^{n+1}\operatorname{Re}(\alpha). \tag{28}$$

Equivalently,

$$D_{\alpha,n}(\lambda) = U_n\left(\frac{\lambda-2}{2}\right) + 2(1-\operatorname{Re}(\alpha))U_{n-1}\left(\frac{\lambda-2}{2}\right) + (1-2\operatorname{Re}(\alpha))U_{n-2}\left(\frac{\lambda-2}{2}\right) + 2(-1)^{n+1}\operatorname{Re}(\alpha). \tag{29}$$

The next proposition is similar to [12, Proposition 14], but here we use the change of variable $\lambda = t^2$ instead of $\lambda = 4 - t^2$.

For $n \geq 3$ define

$$p_n(t) := \begin{cases} U_{n-1}(t/2), & \text{if } n \text{ is even,} \\ T_n(t/2), & \text{if } n \text{ is odd,} \end{cases} \tag{30}$$

$$q_{\alpha,n}(t) := \begin{cases} (1-\alpha)\frac{t}{2}T_n\left(\frac{t}{2}\right) + \alpha\frac{t^2-4}{4}U_{n-1}\left(\frac{t}{2}\right), & \text{if } n \text{ is even,} \\ (1-\alpha)\frac{t}{2}U_{n-1}\left(\frac{t}{2}\right) + \alpha T_n\left(\frac{t}{2}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{31}$$

Proposition 3.2 *For every α in $\mathbb{R}$, every $n \geq 3$ and every t in $\mathbb{C}$,*

$$D_{\alpha,n}(t^2) = 4p_n(t)q_{\alpha,n}(t). \tag{32}$$

Proof Let $w = (t^2-2)/2$, i.e., $t^2 = 2w+2$. Then, (28) takes the following form:

$$D_{\alpha,n}(2w+2) = 2\Big((w+1-\alpha)U_{n-1}(w) - \alpha U_{n-2}(w) + (-1)^{n+1}\alpha\Big). \tag{33}$$

Let $n = 2m$. We apply $U_{2m-2}(w) = -U_{2m}(w) + 2wU_{2m-1}(w)$ on (33), obtaining

$$D_{\alpha,2m}(2w+2) = 2\Big(\alpha U_{2m}(w) + (w+1-\alpha-2\alpha w)U_{2m-1}(w) - \alpha\Big).$$

Now, we use the identities

$$\begin{aligned} U_{2m-1}(w) &= 2U_{m-1}(w)T_m(w), \\ U_{2m}(w) &= 2wU_{m-1}(w)T_m(w) + 2T_m^2(w) - 1, \\ U_{2m}(w) - U_{2m-1}(w) + 1 &= 2(w^2-1)U_{m-1}^2(w), \end{aligned}$$

deriving

$$D_{\alpha,2m}(2w+2) = 4(w+1)U_{m-1}(w)\Big((1-\alpha)T_m(w) + \alpha(w-1)U_{m-1}(w)\Big).$$

Considering the relations

$$T_{2m}\left(\frac{t}{2}\right) = T_m\left(\frac{t^2-2}{2}\right), \quad U_{2m+1}\left(\frac{t}{2}\right) = tU_m\left(\frac{t^2-2}{2}\right),$$

we obtain that the characteristic polynomial is the product of the polynomials (30) and (31).

If $n = 2m+1$, the analysis is similar. □

Remark 3.3 If $n \geq 3$ is odd, then the polynomial $q_{\alpha,n}$ coincides with the polynomial $q_{1-\alpha,n}$ written in [13].

We will apply the following elementary identities:

$$T_n\left(\sin\frac{x}{2}\right) = (-1)^{\frac{n}{2}}\cos\frac{nx}{2}, \quad U_n\left(\sin\frac{x}{2}\right) = (-1)^{\frac{n}{2}}\frac{\cos\frac{(n+1)x}{2}}{\cos\frac{x}{2}} \quad (n \text{ is even}), \tag{34}$$

$$T_n\left(\sin\frac{x}{2}\right) = (-1)^{\frac{n-1}{2}}\sin\frac{nx}{2}, \quad U_n\left(\sin\frac{x}{2}\right) = (-1)^{\frac{n-1}{2}}\frac{\sin\frac{(n+1)x}{2}}{\cos\frac{x}{2}} \quad (n \text{ is odd}). \tag{35}$$

Then, using the change of variable $t = 2\sin(x/2)$ in (30) and (31) yields

$$p_n\left(2\sin\frac{x}{2}\right) = \begin{cases} (-1)^{\frac{n}{2}+1}\frac{\sin\frac{nx}{2}}{\cos\frac{x}{2}}, & \text{if } n \text{ is even}, \\ (-1)^{\frac{n-1}{2}}\sin\frac{nx}{2}, & \text{if } n \text{ is odd}. \end{cases} \tag{36}$$

$$q_{\alpha,n}\left(2\sin\frac{x}{2}\right)=\begin{cases}(-1)^{\frac{n}{2}}\left((1-\alpha)\sin\frac{x}{2}\cos\frac{nx}{2}+\alpha\cos\frac{x}{2}\sin\frac{nx}{2}\right), & \text{if } n \text{ is even,}\\ \frac{(-1)^{\frac{n-1}{2}}}{\cos\frac{x}{2}}\left((1-\alpha)\sin\frac{x}{2}\cos\frac{nx}{2}+\alpha\cos\frac{x}{2}\sin\frac{nx}{2}\right), & \text{if } n \text{ is odd.}\end{cases} \tag{37}$$

So, (32) becomes

$$D_{\alpha,n}(g(x))=(-1)^{n+1}\frac{4\sin\frac{x}{2}\sin\frac{nx}{2}}{\cos\frac{x}{2}}\left((1-\alpha)\cos\frac{nx}{2}+\alpha\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\cos\frac{x}{2}\right). \tag{38}$$

After the change of variable $t=2\cosh(x/2)$, formula (32) transforms to

$$D_{\alpha,n}(g_+(x))=4\cosh\frac{x}{2}\frac{\sinh\frac{nx}{2}}{\sinh\frac{x}{2}}\left((1-\alpha)\cosh\frac{nx}{2}+\alpha\frac{\sinh\frac{x}{2}\sinh\frac{nx}{2}}{\cosh\frac{x}{2}}\right)$$
$$(n \text{ is even}),$$
$$D_{\alpha,n}(g_+(x))=4\cosh\frac{x}{2}\cosh\frac{nx}{2}\left((1-\alpha)\frac{\sinh\frac{nx}{2}}{\sinh\frac{x}{2}}+\alpha\frac{\cosh\frac{nx}{2}}{\cosh\frac{x}{2}}\right)\quad (n \text{ is odd}). \tag{39}$$

Proposition 3.4 (Trivial Eigenvalues of $L_{\alpha,n}$) *For every $n \geq 3$ and every even j with $0 \leq j \leq n-1$, the number $g(j\pi/n)$ is an eigenvalue of $L_{\alpha,n}$.*

Proof These eigenvalues come from the factor p_n in the decomposition (32). Indeed, the change of variable $\lambda=(2\sin(x/2))^2$ yields the factor $p_n\,(2\sin(x/2))$. According to (36), this expression vanishes for $x=2k\pi/n$, where k is an integer and $0 \leq k \leq (n-1)/2$. □

Lemma 3.5 *If n is even, then $\lim_{t\to+\infty} q_{\alpha,n}(t)=+\infty$.*

Proof From the recurrent definition of Chebyshev polynomials, the leading term of $T_n(t/2)$ is $(1/2)t^n$, and the leading term of $U_{n-1}(t/2)$ is t^{n-1}. Therefore, by (31), the leading term of $q_{\alpha,n}(t)$ is $(1/4)t^{n+1}$. So, the leading coefficient is strictly positive, which implies the result. □

For every j with $1 \leq j \leq n$, we define

$$I_{n,j} := \left(\frac{(j-1)\pi}{n}, \frac{j\pi}{n}\right)=(d_{n,j}, d_{n,j+1}). \tag{40}$$

Proof of Theorem 2.1 For $1 \leq j \leq n-1$, the proof is similar to the proof of [12, Theorem 1]. In particular, for odd j, we use Proposition 3.4.

1. If n is odd and satisfies $n < \varkappa_\alpha^{-1}$, then, using (37), it is easy to see that $q_{\alpha,n}(2\sin(x/2))$ changes of sign in $I_{n,n}$. Indeed, $q_{\alpha,n}(2\sin(d_{n,n}/2)) = -1$, and

$$\lim_{x\to\pi^-} q_{\alpha,n}\left(2\sin\frac{x}{2}\right) = -(n(1-\alpha)+\alpha) = (\alpha-1)(n-\varkappa_\alpha^{-1}).$$

2. If n is odd and satisfies $n = \varkappa_\alpha^{-1}$, then $q_{\alpha,n}(2) = (1-\alpha)n+\alpha$, hence $\lambda = 4$ is an eigenvalue of $L_{\alpha,n}$.
3. If $n \geq 3$ is odd and $n > \varkappa_\alpha^{-1}$, then $q_{\alpha,n}(t)$ takes values of opposite signs at the ends of the interval $[2, r_\alpha + r_\alpha^{-1}]$ where $r_\alpha := \sqrt{2\alpha-1}$:

$$q_{\alpha,n}(2) = (1-\alpha)n+\alpha < 0, \quad q_{\alpha,n}(r_\alpha + r_\alpha^{-1}) = \frac{r_\alpha^2+1}{2} r_\alpha^{-n} > 0.$$

   Then,

$$4 < \lambda_{\alpha,n,n} < \left(r_\alpha + \frac{1}{r_\alpha}\right)^2 = \Omega_\alpha.$$

4. For every even $n \geq 4$, $q_{\alpha,n}$ changes its sign in the interval $[r_\alpha + r_\alpha^{-1}, +\infty)$. Indeed, $\lim_{t\to+\infty} q_{\alpha,n}(t) = +\infty$ by Lemma 3.5, whereas

$$q_{\alpha,n}(r_\alpha + r_\alpha^{-1}) = -\frac{1}{4}(r_\alpha^2+1)\left(r_\alpha^{n+1} + r_\alpha^{-(n+1)}\right) < 0.$$

   Moreover, by the Gershgorin disks theorem (see, e.g., [18, Theorem 2.1]), the eigenvalues are bounded from above by $4+2\alpha$. Thus,

$$\Omega_\alpha = \left(r_\alpha + \frac{1}{r_\alpha}\right)^2 \leq \lambda_{\alpha,n,n} \leq 4+2\alpha.$$

Items 1, 2, and 3 could also be derived from [13, Lemmas 3.3, 3.4], taking into account Remark 3.3. □

4 Inner Eigenvalues

In this section we deal with the inner eigenvalues. The proofs of the upcoming propositions are very similar, if not identical, to the proofs given in [12, 13]. Recall that $\varkappa_\alpha$, η_α are defined by (3), (8).

Proof of Theorem 2.2 If $\lambda \in (0,4)$, we use the change of variable $\lambda = g(x)$, with $x \in (0,\pi)$. So, $D_{\alpha,n}(g(x))$ transforms into (38). Equivalently, we apply (32) with

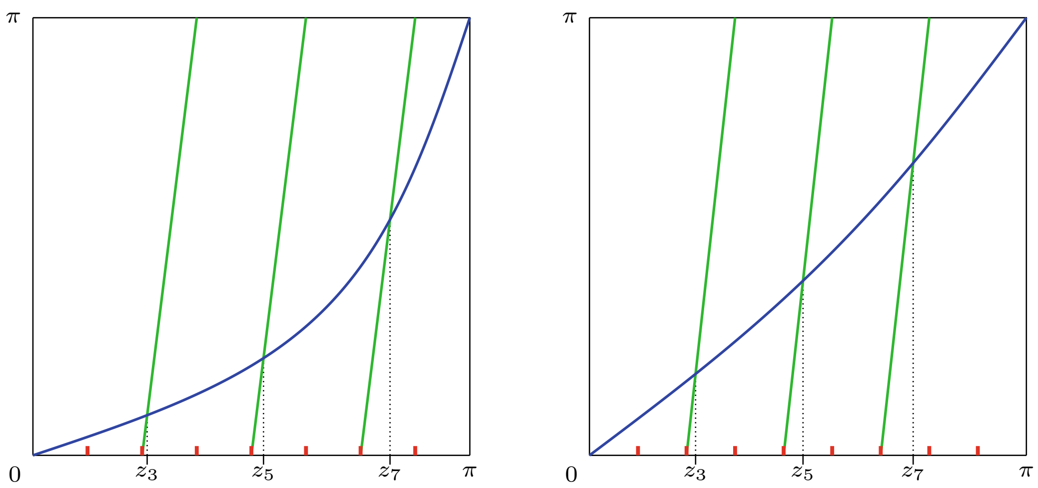

Fig. 3 Plot of η_α (blue) and the left-hand side of (42) (green) for $\alpha = 3/2$, $n = 8$ (left) and $\alpha = 4$, $n = 9$ (right)

$t = 2\sin(x/2)$. Then, $D_{\alpha,n}(g(x)) = 0$ reduces to $q_{\alpha,n}(2\sin(x/2)) = 0$, which is equivalent to

$$\tan\frac{nx}{2} = \varkappa_\alpha \tan\frac{x}{2}. \tag{41}$$

In particular, for odd j with $3 \le j \le n-1$, the solution $z_{\alpha,n,j}$ belonging to $I_{n,j}$ satisfies (41). □

Equation (9) from Theorem 2.2 can be rewritten in the form

$$nx - (j-1)\pi = \eta_\alpha(x). \tag{42}$$

Figure 3 shows η_α and the left-hand side of (42) for a couple of examples.
The first two derivatives of η_α are

$$\eta'_\alpha(x) = \frac{2\varkappa_\alpha}{1 + \varkappa_\alpha^2 + (1-\varkappa_\alpha^2)\cos(x)}, \tag{43}$$

$$\eta''_\alpha(x) = \frac{2\varkappa_\alpha(1-\varkappa_\alpha^2)\sin(x)}{(1 + \varkappa_\alpha^2 + (1-\varkappa_\alpha^2)\cos(x))^2}. \tag{44}$$

Proposition 4.1 and Theorem 4.2 follow directly from the properties of η_α, similarly to [12, Propositions 21 and 22].

Proposition 4.1 *Each derivative of η_α is a bounded function on $(0,\pi)$. In particular,*

$$\sup_{0<x<\pi} |\eta'_\alpha(x)| = \varkappa_\alpha^{-1}, \qquad \sup_{0<x<\pi} |\eta''_\alpha(x)| \le \frac{\varkappa_\alpha^{-2}-1}{2}.$$

Recall that N_α is defined by (11), and that for every j, the numbers $d_{n,j}$, $z_{n,j}$ are defined by (5) and (7), respectively.

Theorem 4.2 *Let $n \ge N_\alpha$, j be odd, $3 \le j \le n-1$. Then, the function $x \mapsto d_j + \eta_\alpha(x)/n$ is a contraction on $\operatorname{cl}(I_{n,j})$, and its fixed point is $z_{\alpha,n,j}$.*

In [12, Proposition 24], we proved some simple facts about the convergence of Newton's method for convex functions. Now we are going to state without proofs some similar facts for concave functions (see also [12, Remark 27]). Assume that $a, b \in \mathbb{R}$ with $a < b$; f is differentiable and $f' > 0$ on $[a,b]$; there exists c in $[a,b]$ such that $f(c) = 0$; $y^{(0)}$ is a point in $[a,b]$ and the sequence $(y^{(m)})_{m=0}^{\infty}$ is defined (when possible) by the recurrence relation

$$y^{(m+1)} = y^{(m)} - \frac{f\left(y^{(m)}\right)}{f'\left(y^{(m)}\right)}. \tag{45}$$

Proposition 4.3 (Linear Convergence of Newton's Method for Concave Functions) *If f is concave on $[a,b]$, $a \le y^{(0)} \le c$, then $y^{(m)}$ belongs to $[a,c]$ for every $m \ge 0$, the sequence $(y^{(m)})_{m=0}^{\infty}$ increases and converges to c, with*

$$c - y^{(m)} \le (b-a)\left(1 - \frac{f'(b)}{f'(a)}\right)^m. \tag{46}$$

For every $n \ge 4$ and every j odd with $3 \le j \le n$, we define $h_{\alpha,n,j}\colon \operatorname{cl}(I_{n,j}) \to \mathbb{R}$ by

$$h_{\alpha,n,j}(x) := nx - (j-1)\pi - \eta_\alpha(x).$$

Theorem 4.4 (Convergence of Newton's Method Applied to $h_{\alpha,n,j}$) *Let $n \ge N_\alpha$, j be odd, $3 \le j \le n-1$ and $y_{\alpha,n,j}^{(0)} = d_{n,j}$. Define the sequence $(y_{\alpha,n,j}^{(m)})_{m=0}^{\infty}$ by the recursive formula*

$$y_{\alpha,n,j}^{(m)} := y_{\alpha,n,j}^{(m-1)} - \frac{h_{\alpha,n,j}\left(y_{\alpha,n,j}^{(m-1)}\right)}{h'_{\alpha,n,j}\left(y_{\alpha,n,j}^{(m-1)}\right)} \quad (m \ge 1). \tag{47}$$

Then $(y_{\alpha,n,j}^{(m)})_{m=0}^{\infty}$ *is well defined and converges to* $z_{\alpha,n,j}$*, and the convergence is at least linear:*

$$z_{\alpha,n,j} - y_{\alpha,n,j}^{(m)} \leq \frac{\pi}{n}\left(\frac{\varkappa_\alpha^{-2}-1}{\varkappa_\alpha^{-1}n-1}\right)^m. \tag{48}$$

Moreover, if $n \geq 2N_\alpha$*, then the convergence is quadratic, and*

$$z_{\alpha,n,j} - y_{\alpha,n,j}^{(m)} \leq \frac{\pi}{n}\left(\frac{\pi\varkappa_\alpha^{-2}}{2n^2}\right)^{2^m-1}. \tag{49}$$

Proof Formulas (43) and (44) for η'_α and η''_α imply that $h'_{\alpha,n,j} > 0$ and $h''_{\alpha,n,j} < 0$ on $\mathrm{cl}(I_{n,j})$. Moreover, $y_{\alpha,n,j}^{(0)} = d_{n,j} < z_{\alpha,n,j} < d_{n,j+1}$. So, the assumptions of Proposition 4.3 are satisfied. Here are rough estimates of the derivatives of $h_{\alpha,n,j}$ at the extremes of $I_{n,j}$:

$$n - \varkappa_\alpha = h'_{\alpha,n,j}(0) \geq h'_{\alpha,n,j}(d_{n,j}) \geq h'_{\alpha,n,j}(d_{n,j+1}) \geq h'_{\alpha,n,j}(\pi) = n - \frac{1}{\varkappa_\alpha}.$$

Therefore,

$$1 - \frac{h'_{\alpha,n,j}(d_{n,j+1})}{h'_{\alpha,n,j}(d_{n,j})} \leq 1 - \frac{n-\varkappa_\alpha^{-1}}{n-\varkappa_\alpha} = \frac{\varkappa_\alpha^{-2}-1}{n\varkappa_\alpha^{-1}-1},$$

and we obtain (48).

Finally, if $n \geq 2N_\alpha$, then

$$\frac{\pi}{n}\cdot\frac{\max\limits_{0\leq x\leq\pi}|h''_{\alpha,n,j}(x)|}{2\min\limits_{0\leq x\leq\pi}|h'_{\alpha,n,j}(x)|} \leq \frac{\pi\max\limits_{0\leq x\leq\pi}|\eta''_\alpha(x)|}{2n\left(n-\max\limits_{0\leq x\leq\pi}|\eta'_\alpha(x)|\right)} \leq \frac{\pi(\varkappa_\alpha^{-2}-1)}{4n(n-\varkappa_\alpha^{-1})} \leq \frac{\pi\varkappa_\alpha^{-2}}{2n^2} < 1,$$

which implies the quadratic convergence with upper estimate (49); see, e.g., [1, Sect. 2.2] or [12, Proposition 26]. □

Proposition 4.5 *There exists* $C_1(\alpha) > 0$ *such that for every* $n \geq 3$ *and every* j *odd with* $3 \leq j \leq n-1$*,*

$$z_{\alpha,n,j} = d_{n,j} + \frac{\eta_\alpha(d_{n,j})}{n} + \frac{\eta_\alpha(d_{n,j})\,\eta'_\alpha(d_{n,j})}{n^2} + r_{\alpha,n,j},$$

where $|r_{\alpha,n,j}| \leq \frac{C_1(\alpha)}{n^3}$.

Proof of Theorem 2.4 Substituting (4.5) into g and using Taylor expansion of g around $d_{n,j}$, we obtain the asymptotic expansion (16) with error bound (17). □

5 Transformation of the Characteristic Equation for the Last Eigenvalue

Recall that $\varkappa_\alpha$, N_α, ω_α are defined by (3), (11), (13), respectively.

Proof of Theorem 2.3 If $\lambda \in (4,\infty)$, we make the change of variable $\lambda = g_+(x)$ with $x \in (0,\infty)$. In other words, we use (32) with $t = 2\cosh(x/2)$. Then, $p_n(2\cosh(x/2)) \neq 0$, and equation $D_{\alpha,n}(g_+(x)) = 0$ is equivalent to $q_{\alpha,n}(2\cosh(x/2)) = 0$, which takes the following form:

$$\tanh\frac{nx}{2} = \frac{1}{\varkappa_\alpha}\tanh\frac{x}{2} \quad (n \text{ is odd}), \tag{50}$$

$$\tanh\frac{nx}{2} = \varkappa_\alpha \coth\frac{x}{2} \quad (n \text{ is even}). \tag{51}$$

By Theorem 2.1, if n is odd and $n > \varkappa_\alpha$, then (50) has a unique solution on $(0,\omega_\alpha)$. If n is even and $n \geq 4$, then (51) has a unique solution on (ω_α,∞). We apply arctanh to both sides of the Eqs. (50) and (51), and rewrite them as (15). □

6 Last Eigenvalue with Odd n

In this section, we suppose that n is odd and $n \geq N_\alpha$, and we study the behavior of $\lambda_{\alpha,n,n}$ and $s_{\alpha,n}$ which are related by (12), i.e., $\lambda_{\alpha,n,n} = g_+(s_{\alpha,n})$.

The main idea of this section is to exploit the symmetry between the last eigenvalue of $L_{\alpha,n}$ and the first eigenvalue of $L_{1-\alpha,n}$. Since $\alpha > 1$, the "dual" parameter $\alpha' := 1-\alpha$ satisfies $\alpha' < 0$, and the matrices $L_{\alpha',n}$ with $\alpha' < 0$ were studied in [13].

As we showed in [13, proof of Theorem 2.2], $\lambda_{\alpha',n,1}$ can be computed as $g_-(s_{\alpha',n})$ where $g_-(x) := -4\sinh^2(x/2)$ and $s_{\alpha',n}$ is the unique solution of

$$\tanh\left(\frac{nx}{2}\right) = \varkappa_{\alpha'}\tanh\left(\frac{x}{2}\right). \tag{52}$$

Proposition 6.1 *Let n be odd such that $n \geq N_\alpha$. Then $\lambda_{\alpha,n,n} = 4 - \lambda_{1-\alpha,n,1}$.*

Proof Let $\alpha' := 1-\alpha$. Notice that $\varkappa_{\alpha'} = \varkappa_\alpha^{-1}$. Therefore, Eqs. (50) and (52) coincide. They have the same solutions:

$$s_{\alpha,n} = s_{\alpha',n}. \tag{53}$$

Finally,

$$4 - \lambda_{\alpha',n,1} = 4 - g_-(s_{\alpha',n}) = g_+(s_{\alpha',n}) = g_+(s_{\alpha,n}) = \lambda_{\alpha,n,n}.$$

□

Let n be odd, and recall that $\psi_{\alpha,n}$ is defined by (14); another useful representation is

$$\psi_{\alpha,n}(x) := \operatorname{arctanh}\left(\varkappa_{1-\alpha}^{-1} \tanh \frac{nx}{2}\right).$$

It follows that $\psi_{\alpha,n}$ equals the function $\varphi_{1-\alpha,n}$ given in [13, (2.6)]. Therefore, the properties of (14) and (56) are the ones developed in [13, Propostions 5.1 and 5.3]. In particular, the first two derivatives of $\psi_{\alpha,n}$ are

$$\psi'_{\alpha,n}(x) = \frac{2n\varkappa_\alpha}{(1-\varkappa_\alpha^2)\cosh(nx) + 1 + \varkappa_\alpha^2}, \tag{54}$$

$$\psi''_{\alpha,n}(x) = -\frac{2n^2\varkappa_\alpha(1-\varkappa_\alpha^2)\sinh(nx)}{((1-\varkappa_\alpha^2)\cosh(nx) + 1 + \varkappa_\alpha^2)^2}. \tag{55}$$

Define

$$\ell_{\alpha,n} := \frac{2}{n}\operatorname{arccosh}\sqrt{\frac{n\varkappa_\alpha - \varkappa_\alpha^2}{1-\varkappa_\alpha^2}} = \frac{2}{n}\operatorname{arccosh}\sqrt{\frac{n\alpha(\alpha-1)-(\alpha-1)^2}{2\alpha-1}}.$$

Proposition 6.2 *Let n be odd such that $n \geq N_\alpha$. Then $\psi_{\alpha,n}$ has the following properties.*

1. $\psi'_{\alpha,n} > 0$ *and* $\psi''_{\alpha,n} < 0$ *on* $[0, +\infty)$.
2. $\psi'_{\alpha,n}(\ell_{\alpha,n}) = 1$*; moreover,* $\psi'_{\alpha,n} > 1$ *on* $[0, \ell_{\alpha,n})$ *and* $\psi'_{\alpha,n} < 1$ *on* $(\ell_{\alpha,n}, +\infty)$.
3. $\lim_{x\to+\infty} \psi_{\alpha,n}(x) = \omega_\alpha$.
4. $s_{\alpha,n}$ *is the unique fixed point of* $\psi_{\alpha,n}$ *in* $(0, +\infty)$.
5. $\psi_{\alpha,n}(x) > x$ *for every* x *in* $(0, \ell_{\alpha,n}]$.
6. $\ell_{\alpha,n} < \psi_{\alpha,n}(\ell_{\alpha,n}) < s_{\alpha,n}$.

For every odd $n \geq N_\alpha$, we define $f_{\alpha,n}\colon [0, +\infty) \to \mathbb{R}$ by

$$f_{\alpha,n}(x) := x - \psi_{\alpha,n}(x) = x - 2\operatorname{arctanh}\left(\varkappa_\alpha \tanh \frac{nx}{2}\right). \tag{56}$$

Figure 4 shows $f_{\alpha,n}$.

The following theorem contains more detailed information than its analog [13, Theorem 5.4].

Theorem 6.3 (Convergence of Newton's Method Applied to $f_{\alpha,n}$ for Odd n) *Let $n \geq N_\alpha$ and n be odd. Then the sequence $(y_{\alpha,n}^{(m)})_{m=0}^\infty$ defined by*

$$y_{\alpha,n}^{(0)} := \omega_\alpha, \qquad y_{\alpha,n}^{(m)} := y_{\alpha,n}^{(m-1)} - \frac{f_{\alpha,n}\left(y_{\alpha,n}^{(m-1)}\right)}{f'_{\alpha,n}\left(y_{\alpha,n}^{(m-1)}\right)} \quad (m \geq 1),$$

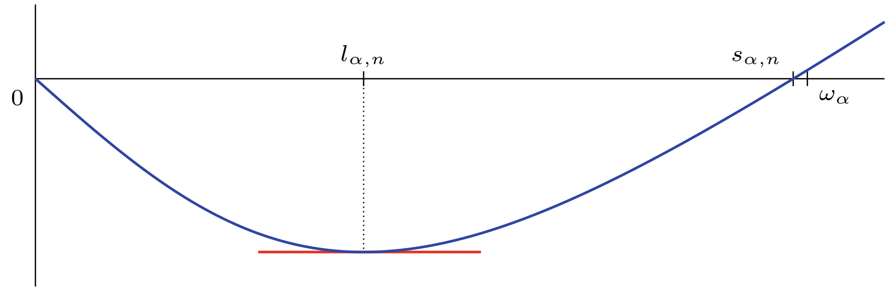

Fig. 4 Plot of $f_{\alpha,n}$ (blue) and tangent line to the graph of $f_{\alpha,n}$ at $\ell_{\alpha,n}$ (red), for $\alpha = 3/2$ and $n = 7$

takes values in $[s_{\alpha,n}, \omega_\alpha]$ *and converges to* $s_{\alpha,n}$*. The convergence is at least linear.*

Moreover, if n *is odd and large enough, then the convergence is quadratic, i.e., there exists* $Q_{\alpha,n}$ *in* $(0, 1/2)$ *such that for every* $m \geq 1$*,*

$$0 \leq y_{\alpha,n}^{(m)} - s_{\alpha,n} \leq \omega_\alpha Q_{\alpha,n}^{2^m - 1}. \tag{57}$$

Proof By Proposition 6.2, $f'_{\alpha,n} > 0$ and $f''_{\alpha,n} > 0$ on $[\psi_{\alpha,n}(\ell_{\alpha,n}), \omega_\alpha]$. So, [13, Proposition 4.3] implies that the points $y_{\alpha,n}^{(m)}$ belong to the segment $[\psi_{\alpha,n}(\ell_{\alpha,n}), \omega_\alpha]$ (which is contained in $[s_{\alpha,n}, \omega_\alpha]$), and the convergence is at least linear.

It is easy to see that for n large enough, the dependence $n \mapsto \ell_{\alpha,n}$ is decreasing. Let n_0 be such a number that $\ell_{\alpha,n} \leq \ell_{\alpha,n_0}$ for every $n \geq n_0$.

Take $b_\alpha := \ell_{\alpha,n_0}$. Then for every $n > n_0$,

$$\ell_{\alpha,n} < b_\alpha < s_{\alpha,n} < \omega_\alpha.$$

Let $J_\alpha := [b_\alpha, \omega_\alpha]$. Since $\psi'_{\alpha,n}(b_\alpha) \to 0$ and $\sup_{J_\alpha} |\psi''| \to 0$ as $n \to \infty$, we choose n_1 such that for every $n > n_1$,

$$\psi'_{\alpha,n}(b_\alpha) < \frac{1}{2}, \qquad \sup_{J_\alpha} |\psi''_{\alpha,n}| < \frac{1}{2\omega_\alpha}.$$

Then, for $n > n_1$ and for every x in J_α,

$$\frac{1}{2} < f'_{\alpha,n}(b_\alpha) \leq f'_{\alpha,n}(x), \qquad |f''_{\alpha,n}(x)| < \frac{1}{2\omega_\alpha},$$

and

$$Q_{\alpha,n} := (\omega_\alpha - b_\alpha) \cdot \frac{\sup_{J_\alpha} |f''_{\alpha,n}|}{2 \inf_{J_\alpha} |f'_{\alpha,n}|} < \frac{1}{2}.$$

In fact, $Q_{\alpha,n}$ tends rapidly to 0 as n tends to ∞, but we have not found simple estimates. □

Define

$$\gamma_{1,\alpha} := \frac{4\varkappa_\alpha}{1-\varkappa_\alpha^2}, \qquad \gamma_{2,\alpha} := \frac{4\varkappa_\alpha(1+\varkappa_\alpha^2)}{(1-\varkappa_\alpha^2)^2}. \tag{58}$$

Theorem 6.4 (Asymptotic Expansion of $s_{\alpha,n}$ Where n Is Odd) *As n is odd and tends to infinity,*

$$s_{\alpha,n} = \omega_\alpha - \gamma_{1,\alpha} e^{-n\omega_\alpha} - \gamma_{1,\alpha}^2 n e^{-2n\omega_\alpha} + \gamma_{2,\alpha} e^{-2n\omega_\alpha} + O(n^2 e^{-3n\omega_\alpha}). \tag{59}$$

Proof Let $\alpha' := 1-\alpha$. In [13, Theorem 5.9], we proved that

$$s_{\alpha',n} = \omega_{\alpha'} - \gamma_{1,\alpha'} e^{-n\omega_{\alpha'}} - \gamma_{1,\alpha'}^2 n e^{-2n\omega_{\alpha'}} + \gamma_{2,\alpha'} e^{-2n\omega_{\alpha'}} + O(n^2 e^{-3n\omega_{\alpha'}}),$$

where

$$\omega_{\alpha'} = \log(1-2\alpha') = \log(2\alpha-1) = \omega_\alpha, \qquad \gamma_{1,\alpha'} = \gamma_{1,\alpha}, \qquad \gamma_{2,\alpha'} = \gamma_{2,\alpha}.$$

Now the result follows from (53). □

The asymptotic expansion of $\lambda_{\alpha,n,n}$ will be derived at the end of Sect. 7.

7 Last Eigenvalue for Even *n*

In this section, we study the behavior of the last eigenvalue $\lambda_{\alpha,n,n}$ supposing that n is even and $n \geq 4$. More precisely, we analyze the behavior of $s_{\alpha,n}$, defined by $\lambda_{\alpha,n,n} = g_+(s_{\alpha,n})$. Thus, in this section we suppose that n is even.

Define $r_\alpha := g_+{}^{-1}(4+2\alpha) = 2\operatorname{arcsinh}(\sqrt{\alpha/2})$. By Theorem 2.1, part 4, $s_{\alpha,n}$ is the unique solution of (51) in $(\omega_\alpha, r_\alpha)$.

Recall that $\psi_{\alpha,n}$ is defined by (14):

$$\psi_{\alpha,n}(x) = 2\operatorname{arctanh}\left(\varkappa_\alpha \coth\frac{nx}{2}\right) \qquad (x \geq \omega_\alpha).$$

Note that for $x \geq \omega_\alpha$,

$$\varkappa_\alpha \coth\frac{nx}{2} < \varkappa_\alpha \coth\frac{x}{2} \leq \varkappa_\alpha \coth\frac{\omega_\alpha}{2} = 1,$$

therefore $\psi_{\alpha,n}$ is well defined. A straightforward computation gives

$$\psi'_{\alpha,n}(x) = -\frac{2n\varkappa_\alpha}{(1-\varkappa_\alpha^2)\cosh(nx) - 1 - \varkappa_\alpha^2}, \tag{60}$$

$$\psi''_{\alpha,n}(x) = \frac{2n^2\varkappa_\alpha(1-\varkappa_\alpha^2)\sinh(nx)}{((1-\varkappa_\alpha^2)\cosh(nx) - 1 - \varkappa_\alpha^2)^2}. \tag{61}$$

Proposition 7.1 *Let n be even such that $n \geq 4$. Then $\psi_{\alpha,n}$ has the following properties.*

1. *$\psi'_{\alpha,n} < 0$ and $\psi''_{\alpha,n} > 0$ on $[\omega_\alpha, +\infty)$. So, $\psi_{\alpha,n}$ is a strictly decreasing convex function.*
2. *$\lim_{x\to\infty} \psi_{\alpha,n}(x) = \omega_\alpha$.*
3. *$s_{\alpha,n}$ is the unique fixed point of $\psi_{\alpha,n}$ and $\omega_\alpha < s_{\alpha,n} < r_\alpha$.*

Proof For every $x \geq \omega_\alpha$, due to the increasing property of cosh and the condition $n \geq 4$,

$$(1-\varkappa_\alpha^2)\cosh(nx) > (1-\varkappa_\alpha^2)\cosh(\omega_\alpha) = (1-\varkappa_\alpha^2)\frac{1+\varkappa_\alpha^2}{1-\varkappa_\alpha^2} = 1+\varkappa_\alpha^2.$$

Hence, the denominators of the fractions in the right-hand sides of (60) and (61) are strictly positive, and we get statement 1.

By definition of ω_α and $\varkappa_\alpha$,

$$\tanh\frac{\omega_\alpha}{2} = \frac{1-e^{-\omega_\alpha}}{1+e^{-\omega_\alpha}} = \frac{1-\frac{1}{2\alpha-1}}{1+\frac{1}{2\alpha-1}} = \frac{\alpha-1}{\alpha} = \varkappa_\alpha. \tag{62}$$

This equality implies statement 2. Finally, statement 3 is consequence of Theorems 2.1 and 2.3. □

We define $f_{\alpha,n}\colon [\omega_\alpha, \infty) \to \mathbb{R}$,

$$f_{\alpha,n}(x) := x - \psi_{\alpha,n}(x) = x - 2\operatorname{arctanh}\left(\varkappa_\alpha \coth\frac{nx}{2}\right). \tag{63}$$

We use the same notation $f_{\alpha,n}$ for two different functions, depending on the parity of n. Figure 5 shows $f_{\alpha,n}$.

Proposition 7.2 *For every even n with $n \geq 4$, $f'_{\alpha,n} > 1$ and $f''_{\alpha,n} < 0$ on $[\omega_\alpha, r_\alpha]$. Moreover, $s_{\alpha,n}$ is its only root in $(\omega_\alpha, r_\alpha)$.*

Proof Follows from Proposition 7.1. □

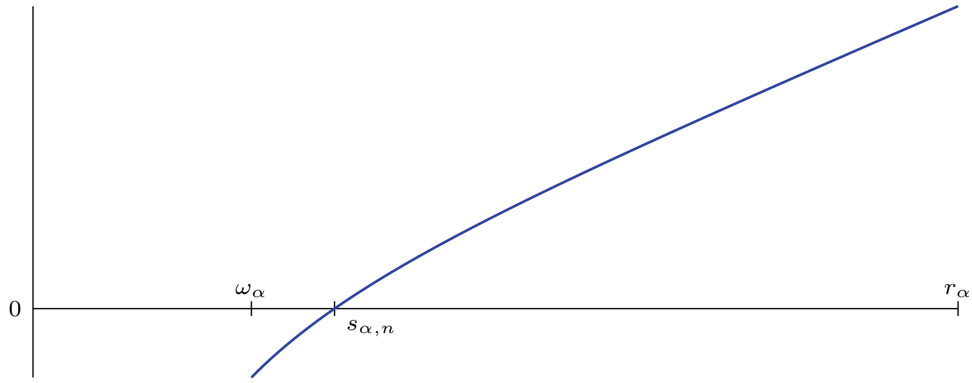

Fig. 5 Plot of $f_{\alpha,n}$ (blue), for $\alpha = 6/5$ and $n = 4$

Theorem 7.3 (Convergence of Newton's Method Applied to $f_{\alpha,n}$ for Even n) *Let $n \ge N_\alpha$ be even. Then the sequence $(y_{\alpha,n}^{(m)})_{m=0}^{\infty}$ defined by*

$$y_{\alpha,n}^{(0)} := \omega_\alpha, \qquad y_{\alpha,n}^{(m)} := y_{\alpha,n}^{(m-1)} - \frac{f_{\alpha,n}\left(y_{\alpha,n}^{(m-1)}\right)}{f'_{\alpha,n}\left(y_{\alpha,n}^{(m-1)}\right)} \quad (m \ge 1),$$

takes values in $[\omega_\alpha, s_{\alpha,n}]$ and converges to $s_{\alpha,n}$.

Moreover, if n is even and large enough, then the convergence is quadratic, i.e., there exists $Q_{\alpha,n}$ in $(0, 1)$ such that for every $m \ge 1$,

$$0 \le s_{\alpha,n} - \omega_\alpha \le r_\alpha Q_{\alpha,n}^{2^m-1}. \tag{64}$$

Proof By Propositions 7.2 and 4.3, the sequence $(y_{\alpha,n}^{(m)})_{m\ge 1}$ takes values in $[\omega_\alpha, s_{\alpha,n}]$ and converges at least linearly. Define

$$Q_{\alpha,n} := (r_\alpha - \omega_\alpha) \cdot \frac{\sup\limits_{[\omega_\alpha, r_\alpha]} |f''_{\alpha,n}|}{2 \inf\limits_{[\omega_\alpha, r_\alpha]} |f'_{\alpha,n}|}.$$

It follows from (61) that $\sup_{[\omega_\alpha, r_\alpha]} |f''_{\alpha,n}| \to 0$ as $n \to \infty$. Therefore, there exists n_0 such that $Q_{\alpha,n} < 1/2$, for every $n \ge n_0$. In fact, $Q_{\alpha,n}$ tends rapidly to 0 as n tends to ∞, but we have not found simple estimates. □

Lemma 7.4 *Let m, n be even such that $n > m \ge 4$. Then $s_{\alpha,m} > s_{\alpha,n}$.*

Proof Recall that $s_{\alpha,n}$ and $s_{\alpha,m}$ are the solutions of (15), respectively for n and m. In this lemma, we prefer to deal with the equivalent Eq. (51). Then

$$\tanh\frac{n s_{\alpha,m}}{2} \tanh\frac{s_{\alpha,m}}{2} > \tanh\frac{m s_{\alpha,m}}{2} \tanh\frac{s_{\alpha,m}}{2} = \frac{\alpha-1}{\alpha} = \tanh\frac{n s_{\alpha,n}}{2} \tanh\frac{s_{\alpha,n}}{2}.$$

This implies $s_{\alpha,m} > s_{\alpha,n}$, since $x \mapsto \tanh\frac{nx}{2}\tanh\frac{x}{2}$ is a strictly increasing function on $[\omega_\alpha, \infty)$. □

Proposition 7.5 *Let n be even such that $n \geq 4$. Then*

$$0 \leq s_{\alpha,n} - \omega_\alpha \leq C_3(\alpha) e^{-n\omega_\alpha}, \tag{65}$$

where $C_3(\alpha) = \frac{(4+2\alpha)\alpha}{\alpha-1}$.

Proof By the mean value theorem applied to $x \mapsto \coth(x/2)$ on $[\omega_\alpha, s_{\alpha,n}]$, there exists $\xi \in (\omega_\alpha, s_{\alpha,n})$ such that

$$\coth\frac{s_{\alpha,n}}{2} - \coth\frac{\omega_\alpha}{2} = -\frac{1}{2\sinh^2\frac{\xi}{2}}(s_{\alpha,n} - \omega_\alpha),$$

i.e.,

$$s_{\alpha,n} - \omega_\alpha = 2\sinh^2\frac{\xi}{2}\left(\coth\frac{\omega_\alpha}{2} - \coth\frac{s_{\alpha,n}}{2}\right).$$

Now we apply the increasing property of sinh, identity (62), and the fact that $s_{\alpha,n}$ satisfies (51):

$$\begin{aligned} s_{\alpha,n} - \omega_\alpha &\leq 2\sinh^2\frac{r_\alpha}{2}\left(\coth\frac{\omega_\alpha}{2} - \coth\frac{s_{\alpha,n}}{2}\right) = \frac{2\sinh^2\frac{r_\alpha}{2}}{\varkappa_\alpha}\left(1 - \tanh\frac{n s_{\alpha,n}}{2}\right) \\ &\leq \frac{4\cosh^2\frac{r_\alpha}{2}}{\varkappa_\alpha} e^{-n s_{\alpha,n}} \leq \frac{4\cosh^2\frac{r_\alpha}{2}}{\varkappa_\alpha} e^{-n\omega_\alpha} = \frac{g_+(r_\alpha)}{\varkappa_\alpha} e^{-n\omega_\alpha}. \end{aligned}$$

The last expression simplifies to $C_3(\alpha)e^{-n\omega_\alpha}$. □

Recall that $\gamma_{1,\alpha}$ and $\gamma_{2,\alpha}$ are defined by (58).

Lemma 7.6 (Asymptotic Expansion of $\psi_{\alpha,1}$) *As t tends to infinity,*

$$\psi_{\alpha,1}(t) = \omega_\alpha + \gamma_{1,\alpha}e^{-t} + \gamma_{2,\alpha}e^{-2t} + O(e^{-3t}). \tag{66}$$

Proof The proof is analogous to the proof of [13, Lemma 5.8]. Since $\coth(t/2) = \frac{1+e^{-t}}{1-e^{-t}}$,

$$\psi_{\alpha,1}(t) = \sigma(e^{-t}), \quad \text{where} \quad \sigma(u) := 2\operatorname{arctanh}\left(\varkappa_\alpha\frac{1+u}{1-u}\right).$$

We start with the Taylor–Maclaurin expansion of the rational function $u \mapsto (1+u)/(1-u)$ around 0:

$$\frac{1+u}{1-u} = 1 + \frac{2u}{1-u} = 1 + 2u + 2u^2 + O(u^3).$$

Then, we apply the Taylor expansion of arctanh around $\varkappa_\alpha$:

$$\operatorname{arctanh}(\varkappa_\alpha + y) = \operatorname{arctanh}(\varkappa_\alpha) + \frac{y}{1-\varkappa_\alpha^2} + \frac{\varkappa_\alpha\, y^2}{(1-\varkappa_\alpha^2)^2} + O(y^3).$$

In the last expansion, we substitute $y = 2\varkappa_\alpha(u + u^2 + O(u^3))$ and use the relation $O(y) = O(u)$:

$$\begin{aligned}\sigma(u) &= 2\operatorname{arctanh}\left(\varkappa_\alpha + 2\varkappa_\alpha(u + u^2 + O(u^3))\right)\\ &= 2\operatorname{arctanh}(\varkappa_\alpha) + \frac{4\varkappa_\alpha}{1-\varkappa_\alpha^2}\left(u + u^2 + O(u^3)\right)\\ &\quad + \frac{8\varkappa_\alpha^3}{(1-\varkappa_\alpha^2)^2}\left(u + u^2 + O(u^3)\right)^2 + O(u^3).\end{aligned}$$

Simplifying and taking into account that $\tanh(\omega_\alpha/2) = \varkappa_\alpha$, we obtain the Taylor–Maclaurin expansion of σ around 0:

$$\sigma(u) = \omega_\alpha + \gamma_{1,\alpha}u + \gamma_{2,\alpha}u^2 + O(u^3).$$

Finally, we put $u = e^{-t}$ and obtain (66). □

Theorem 7.7 (Asymptotic Expansion of $s_{\alpha,n}$) *As n is even and tends to infinity,*

$$s_{\alpha,n} = \omega_\alpha + \gamma_{1,\alpha}e^{-n\omega_\alpha} - \gamma_{1,\alpha}^2 n e^{-2n\omega_\alpha} + \gamma_{2,\alpha}e^{-2n\omega_\alpha} + O(n^2 e^{-3n\omega_\alpha}). \tag{67}$$

Proof The proof is analogous to the proof of [13, Theorem 5.9].

By formula (65) from Proposition 7.5, we have an asymptotic expansion of $s_{\alpha,n}$ with one exact term:

$$s_{\alpha,n} = \omega_\alpha + O(e^{-n\omega_\alpha}). \tag{68}$$

Therefore,

$$e^{-ns_{\alpha,n}} = e^{-n\omega_\alpha + O(ne^{-n\omega_\alpha})} = e^{-n\omega_\alpha}(1 + O(ne^{-n\omega_\alpha})) = e^{-n\omega_\alpha} + O(ne^{-2n\omega_\alpha}). \tag{69}$$

This also implies a rough upper bound for $e^{-ns_{\alpha,n}}$:

$$e^{-ns_{\alpha,n}} = O(e^{-n\omega_\alpha}). \tag{70}$$

The main idea of the following proof is to combine (68) with (15) and Lemma 7.6. We apply the asymptotic expansion (66) with two exact terms and with $ns_{\alpha,n}$ instead of t:

$$s_{\alpha,n} = \psi_{\alpha,n}(s_{\alpha,n}) = \psi_{\alpha,1}(ns_{\alpha,n}) = \omega_\alpha + \gamma_{1,\alpha} e^{-ns_{\alpha,n}} + O(e^{-2ns_{\alpha,n}}).$$

We simplify this expression using (69) and (70):

$$\begin{aligned} s_{\alpha,n} &= \omega_\alpha + \gamma_{1,\alpha} e^{-n\omega_\alpha} + O(ne^{-2n\omega_\alpha}) + O(e^{-2n\omega_\alpha}) \\ &= \omega_\alpha + \gamma_{1,\alpha} e^{-n\omega_\alpha} + O(ne^{-2n\omega_\alpha}). \end{aligned}$$

Now, we use this expansion to improve (69):

$$\begin{aligned} e^{-ns_{\alpha,n}} &= e^{-n\omega_\alpha} e^{-\gamma_{1,\alpha} ne^{-n\omega_\alpha} + O(n^2 e^{-2n\omega_\alpha})} \\ &= e^{-n\omega_\alpha} \left(1 - \gamma_{1,\alpha} ne^{-2n\omega_\alpha} + O(n^2 e^{-2n\omega_\alpha})\right) \\ &= e^{-n\omega_\alpha} - \gamma_{1,\alpha} ne^{-2n\omega_\alpha} + O(n^2 e^{-3n\omega_\alpha}). \end{aligned}$$

Next, we combine this expansion with (66):

$$\begin{aligned} s_{\alpha,n} &= \psi_{\alpha,n}(s_{\alpha,n}) = \psi_{\alpha,1}(ns_{\alpha,n}) = \omega_\alpha + \gamma_{1,\alpha} e^{-ns_{\alpha,n}} + \gamma_{2,\alpha} e^{-2ns_{\alpha,n}} + O(e^{-3ns_{\alpha,n}}) \\ &= \omega_\alpha + \gamma_{1,\alpha} \left(e^{-n\omega_\alpha} - \gamma_{1,\alpha} ne^{-2n\omega_\alpha} + O(n^2 e^{-3n\omega_\alpha})\right) \\ &\quad + \gamma_{2,\alpha} \left(e^{-n\omega_\alpha} - \gamma_{1,\alpha} ne^{-2n\omega_\alpha} + O(n^2 e^{-3n\omega_\alpha})\right)^2 + O(e^{-3n\omega_\alpha}). \end{aligned}$$

Simplifying this expression we get (67). □

In the next corollary, we join the asymptotic expansions (59) and (67).

Corollary 7.8 *As n tends to infinity,*

$$s_{\alpha,n} = \omega_\alpha + (-1)^n \gamma_{1,\alpha} e^{-n\omega_\alpha} - \gamma_{1,\alpha}^2 ne^{-2n\omega_\alpha} + \gamma_{2,\alpha} e^{-2n\omega_\alpha} + O(n^2 e^{-3n\omega_\alpha}). \quad (71)$$

Proof If n is odd, then (59) equals (71). If n is even, then (67) equals (71). □

Proof of Theorem 2.5 We expand g_+ by Taylor formula around ω_α:

$$g_+(\omega_\alpha + x) = g_+(\omega_\alpha) + g_+'(\omega_\alpha)x + \frac{g_+''(\omega_\alpha)}{2}x^2 + O(x^3).$$

Then we substitute the expansion (71) of $s_{\alpha,n}$ and simplify:

$$
\begin{aligned}
\lambda_{\alpha,n,n} &= g_+(s_{\alpha,n}) \\
&= g_+\Big(\omega_\alpha + (-1)^n \gamma_{1,\alpha} e^{-n\omega_\alpha} - \gamma_{1,\alpha}^2 n e^{-2n\omega_\alpha} + \gamma_{2,\alpha} e^{-2n\omega_\alpha} + O(n^2 e^{-3n\omega_\alpha})\Big) \\
&= g_+(\omega_\alpha) + (-1)^n \gamma_{1,\alpha} g_+'(\omega_\alpha) e^{-n\omega_\alpha} - \gamma_{1,\alpha}^2 g_+'(\omega_\alpha) n e^{-2n\omega_\alpha} \\
&\quad + \left(\gamma_{\alpha,2} g_+'(\omega_\alpha) + \frac{\gamma_{1,\alpha}^2 g_+''(\omega_\alpha)}{2}\right) e^{-2n\omega_\alpha} + O\left(n^2 e^{-3n\omega_\alpha}\right).
\end{aligned}
$$

Recall that $g_+(\omega_\alpha) = \Omega_\alpha$. Hence, we obtain (20) and (21), with the following coefficients:

$$
\beta_{\alpha,1} = g_+'(\omega_\alpha)\gamma_{1,\alpha}, \ \ \beta_{\alpha,2} = -g_+'(\omega_\alpha)\gamma_{1,\alpha}^2, \ \ \beta_{\alpha,3} = g_+'(\omega_\alpha)\gamma_{\alpha,2} + \frac{1}{2} g_+''(\omega_\alpha)\gamma_{1,\alpha}^2.
$$

Calculate the derivatives of g_+ at ω_α:

$$
\begin{aligned}
g_+'(\omega_\alpha) &= 2\sinh(\omega_\alpha) = \frac{4\alpha(1-\alpha)}{1-2\alpha} = \frac{4\varkappa_\alpha}{1-\varkappa_\alpha^2}, \\
g_+''(\omega_\alpha) &= 2\cosh(\omega_\alpha) = \frac{2(2\alpha^2-2\alpha+1)}{2\alpha-1} = \frac{2(\varkappa_\alpha^2+1)}{1-\varkappa_\alpha^2}.
\end{aligned}
$$

Combining with formulas (58), we write $\beta_{\alpha,1}$, $\beta_{\alpha,2}$, and $\beta_{\alpha,3}$ as (18) or (19). □

8 Norm of Eigenvectors

We recall that, due to Proposition 3.1, $\lambda_{\alpha,n,j} = \lambda_{\mathrm{Re}(\alpha),n,j}$ for every α in $\mathbb{C}$, every $n \geq 3$ and every $1 \leq j \leq n$. Nevertheless, it turns out that if $\mathrm{Im}(\alpha) \neq 0$, then the eigenvectors associated to $L_{\alpha,n}$ usually have complex components. So, in this section we suppose that α is a complex number such that $\mathrm{Re}(\alpha) > 1$. To simplify subindices, we put

$$
\varkappa_\alpha := \varkappa_{\mathrm{Re}(\alpha)}, \quad N_\alpha := N_{\mathrm{Re}(\alpha)}, \quad \omega_\alpha := \omega_{\mathrm{Re}(\alpha)}, \quad \Omega_\alpha := \Omega_{\mathrm{Re}(\alpha)},
$$

$$
\eta_\alpha := \eta_{\mathrm{Re}(\alpha)}, \quad z_{\alpha,n,j} := z_{\mathrm{Re}(\alpha),n,j}, \quad s_{\alpha,n} := s_{\mathrm{Re}(\alpha),n}.
$$

Proof of Theorem 2.6 Formulas (22), (23) are consequences of [12, Proposition 8]. □

Recall that ν_α is defined by (24). For every x in $[0,\pi]$, we define

$$\xi_\alpha(x) := \frac{|\alpha-1|^2}{2} g(x)\cos(\eta_\alpha(x)) + \frac{|\alpha|^2}{2} g(\eta_\alpha(x))\cos(x) + \frac{|\alpha|^2-\operatorname{Re}(\alpha)}{2}\left(g(x)+g(\eta_\alpha(x))-g(x+\eta_\alpha(x))\right).$$

Proposition 8.1 (Exact Formulas for the Inner Eigenvectors) *Let $n \geq 3$ and $2 \leq j \leq n-1$. If j is even, then $\|v_{\alpha,n,j}\|_2$ is given by (25). If j is odd, then*

$$\|v_{\alpha,n,j}\|_2^2 = n\nu_\alpha(z_{\alpha,n,j}) + \frac{\sin(\eta_\alpha(z_{\alpha,n,j}))}{\sin(z_{\alpha,n,j})}\xi_\alpha(z_{\alpha,n,j}). \tag{72}$$

Proof These formulas are similar to [12, (66), (69)] and are proved in the same manner. □

We will use several identities for hyperbolic functions:

$$\sinh(x)\pm\sinh(y) = 2\sinh\left(\frac{x\pm y}{2}\right)\cosh\left(\frac{x\mp y}{2}\right), \tag{73}$$

$$2\sinh(x)\sinh(y) = \cosh(x+y)-\cosh(x-y), \tag{74}$$

$$2\sinh^2(x) = \cosh(2x)-1, \tag{75}$$

$$\sum_{k=1}^{n}\cosh(2kx+y) = \frac{\sinh(nx)\cosh((n+1)x+y)}{\sinh(x)}. \tag{76}$$

For every $n \geq N_\alpha$, define

$$u_1(\alpha,n) := \frac{\lambda_{\alpha,n,n}}{2}\left(\frac{\sinh(2ns_{\alpha,n})}{2\sinh(s_{\alpha,n})}-n\right),$$

$$u_2(\alpha,n) := \begin{cases} 2|\alpha|^2\cosh^2\frac{(n-1)s_{\alpha,n}}{2}\, w(\alpha,n), & \text{if } n \text{ is even,}\\ 2|\alpha|^2\sinh^2\frac{(n-1)s_{\alpha,n}}{2}\, w(\alpha,n), & \text{if } n \text{ is odd,}\end{cases}$$

$$u_3(\alpha,n) := \begin{cases} -4\operatorname{Re}(\alpha)\cosh\frac{(n-1)s_{\alpha,n}}{2}\cosh\frac{ns_{\alpha,n}}{2}\cosh\frac{s_{\alpha,n}}{2}\, w(\alpha,n), & \text{if } n \text{ is even,}\\ -4\operatorname{Re}(\alpha)\sinh\frac{(n-1)s_{\alpha,n}}{2}\sinh\frac{ns_{\alpha,n}}{2}\cosh\frac{s_{\alpha,n}}{2}\, w(\alpha,n), & \text{if } n \text{ is odd,}\end{cases}$$

where $w(\alpha,n) := \frac{\sinh(ns_{\alpha,n})}{\sinh(s_{\alpha,n})} + (-1)^{n+1}n$.

Proposition 8.2 (Exact Formula for the Norm of the Last Eigenvector) *Let $n \geq N_\alpha$. Then*

$$\|v_{\alpha,n,n}\|_2^2 = u_1(\alpha,n)+u_2(\alpha,n)+u_3(\alpha,n). \tag{77}$$

Proof Let n be even. Then, from (23) and (73),

$$\begin{aligned}
v_{\alpha,n,n,k} &= (-1)^k \left(\overline{\alpha}\sinh((n-k)s_{\alpha,n}) + (1-\overline{\alpha})\sinh((k-1)s_{\alpha,n}) + \sinh(ks_{\alpha,n})\right) \\
&= (-1)^k \big(\sinh((k-1)s_{\alpha,n}) + \sinh(ks_{\alpha,n}) \\
&\qquad\qquad + \overline{\alpha}\left(\sinh((n-k)s_{\alpha,n}) - \sinh((k-1)s_{\alpha,n})\right)\big) \\
&= (-1)^k \bigg(2\sinh\frac{(2k-1)s_{\alpha,n}}{2}\cosh\frac{s_{\alpha,n}}{2} \\
&\qquad\qquad + 2\overline{\alpha}\sinh\frac{(n+1-2k)s_{\alpha,n}}{2}\cosh\frac{(n-1)s_{\alpha,n}}{2}\bigg).
\end{aligned}$$

Taking the squared absolute value and applying (74) and (75), yields

$$\begin{aligned}
|v_{\alpha,n,n,k}|^2 &= 4\cosh^2\frac{s_{\alpha,n}}{2}\sinh^2\frac{(2k-1)s_{\alpha,n}}{2} \\
&\quad + 4|\alpha|^2\cosh^2\frac{(n-1)s_{\alpha,n}}{2}\sinh^2\frac{(n+1-2k)s_{\alpha,n}}{2} \\
&\quad + 8\,\mathrm{Re}(\alpha)\cosh\frac{(n-1)s_{\alpha,n}}{2}\cosh\frac{s_{\alpha,n}}{2}\times \\
&\qquad\qquad \times\sinh\frac{(n+1-2k)s_{\alpha,n}}{2}\sinh\frac{(2k-1)s_{\alpha,n}}{2},
\end{aligned}$$

i.e., after a simplification,

$$\begin{aligned}
|v_{\alpha,n,n,k}|^2 &= \frac{\lambda_{\alpha,n,n}}{2}\left(\cosh(2ks_{\alpha,n} - s_{\alpha,n}) - 1\right) \\
&\quad + 2|\alpha|^2\cosh^2\frac{(n-1)s_{\alpha,n}}{2}\left(\cosh(2ks_{\alpha,n} - (n+1)s_{\alpha,n}) - 1\right) \\
&\quad + 4\,\mathrm{Re}(\alpha)\cosh\frac{(n-1)s_{\alpha,n}}{2}\cosh\frac{s_{\alpha,n}}{2}\times \\
&\qquad\qquad \times\left(\cosh\frac{ns_{\alpha,n}}{2} - \cosh\left(2ks_{\alpha,n} - \frac{n+2}{2}s_{\alpha,n}\right)\right).
\end{aligned}$$

Formula (77) is obtained summing the previous expression over k, considering the identity (76) in each term.

The proof is similar for odd n. □

Lemma 8.3 *As n tends to infinity,*

$$u_1(\alpha, n) = \frac{\mathrm{Re}(\alpha)}{4(\mathrm{Re}(\alpha) - 1)} e^{2n\omega_\alpha} + O(ne^{n\omega_\alpha}),$$

$$u_2(\alpha, n) = \frac{|\alpha|^2}{8\,\mathrm{Re}(\alpha)(\mathrm{Re}(\alpha) - 1)} e^{2n\omega_\alpha} + O(ne^{n\omega_\alpha}),$$

$$u_3(\alpha, n) = -\frac{\mathrm{Re}(\alpha)}{4(\mathrm{Re}(\alpha) - 1)} e^{2n\omega_\alpha} + O(ne^{n\omega_\alpha}).$$

Proof Proceed similarly to the proof of [13, Lemma 6.3]. □

Proof of Theorem 2.7 Formulas (26), (25) follow similarly to the proofs of [13, (2.20), (2.21)]. To prove (27), we apply Proposition 8.2 and Lemma 8.3. Finally, we take the square root and obtain (25). □

Remark 8.4 Using Theorems 2.6 and 2.7, it is possible to show that for n large enough, the inner components of the normalized eigenvector $v_{\alpha,n,n}/\|v_{\alpha,n,n}\|_2$ are very small:

$$\frac{1}{\|v_{\alpha,n,n}\|_2} v_{\alpha,n,n,k} = O_\alpha(e^{-k\omega_\alpha} + e^{-(n+1-k)\omega_\alpha}) = O_\alpha(e^{-\min\{k,n+1-k\}\omega_\alpha}).$$

Figure 6 shows the components of the normalized eigenvectors $v_{\alpha,n,n}/\|v_{\alpha,n,n}\|_2$ for some α and n, and Fig. 7 shows the logarithms of the absolute values of their components.

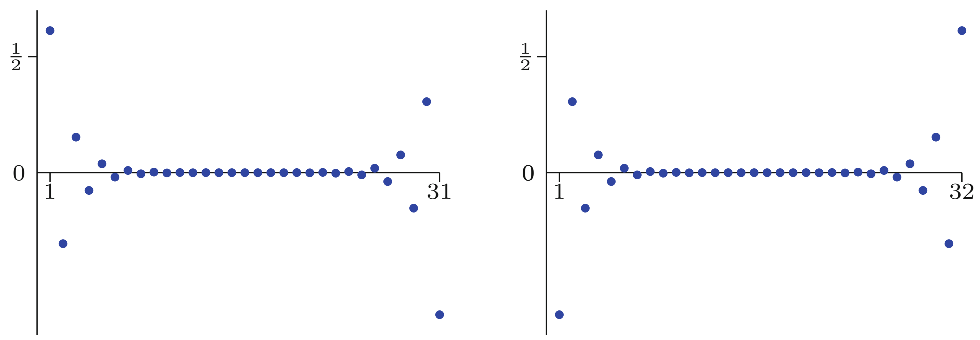

Fig. 6 Components of the eigenvectors $\frac{v_{\alpha,n,n}}{\|v_{\alpha,n,n}\|_2}$ for $\alpha = \frac{3}{2}$, $n = 31$ (left) and $n = 32$ (right)

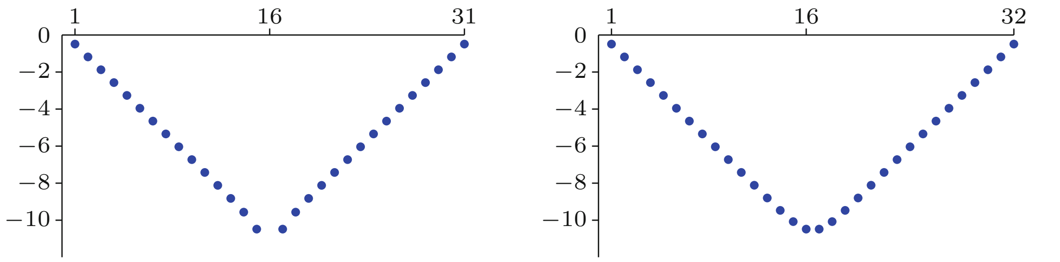

Fig. 7 Values of $\log|w_{\alpha,n,n,k}|$ where $w_{\alpha,n,n} := \frac{v_{\alpha,n,n}}{\|v_{\alpha,n,n}\|_2}$, for $\alpha = \frac{3}{2}$, $n = 31$ (left) and $n = 32$ (right). On the left picture, we skip the component with $k = 16$ because $w_{3/2,31,31,16}$ is very close to zero

9 Numerical Tests

With the help of SageMath, we have verified numerically (for many values of parameters) the representations (32), (39), (38) for the characteristic polynomial, and all the other exact formulas appearing in this paper.

We introduce the following notation for different approximations of the eigenvalues and eigenvectors.

- $\lambda^{\text{gen}}_{\alpha,n,j}$ are the eigenvalues computed with machine precision ($\approx$ 16 decimal digits), using a general eigenvalue algorithm from Sagemath.

All other computations are performed with 3322 binary digits ($\approx$ 1000 decimal digits).

- $z^{\text{N}}_{\alpha,n,j}$ is the numerical solution of the equation $h_{\alpha,n,j}(x) = 0$ computed by Newton's method, see Theorem 4.4.
- Similarly, $s^{\text{N}}_{\alpha,n}$ is the solution of $f_{\alpha,n}(x) = 0$ computed by Newton's method, see Theorems 6.3 and 7.3.
- $\lambda^{\text{N}}_{\alpha,n,j}$ is computed as $g(z^{\text{N}}_{\alpha,n,j})$ or $g(d_{n,j})$ or $g_+(s^{\text{N}}_{\alpha,n})$, depending on the case.
- $\lambda^{\text{bisec}}_{\alpha,n,j}$ is similar to $\lambda^{\text{N}}_{\alpha,n,j}$, but now we solve the corresponding equations by the bisection method.
- Using $z^{\text{bisec}}_{\alpha,n,j}$ we compute $v_{\alpha,n,j}$ by (22) and normalize it.
- Using $s^{\text{bisec}}_{\alpha,n}$ we compute $v_{\alpha,n,1}$ by (23) and normalize it.
- $\lambda^{\text{asympt}}_{\alpha,n,j}$ is the approximation given by (16) and (21).

We have constructed a large series of examples including all rational values α in $(1, 5]$ with denominators ≤ 4 and all n with $3 \leq n \leq 256$. In all these examples, we have obtained

$$\max_{1\leq j\leq n} \|L_{\alpha,n} v_{\alpha,n,j} - \lambda^{\text{bisec}}_{\alpha,n,j} v_{\alpha,n,j}\|_2 < 10^{-994}, \qquad \max_{1\leq j\leq n} |\lambda^{\text{gen}}_{\alpha,n,j} - \lambda^{\text{bisec}}_{\alpha,n,j}| < 10^{-13},$$