Eigenvectors of Hessenberg Toeplitz matrices and a problem by Dai, Geary, and Kadanoff

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The paper is devoted to the eigenvectors of Hessenberg Toeplitz matrices whose symbol has a power singularity. We describe the structure of the eigenvectors and prove an asymptotic formula which can be used to compute individual eigenvectors effectively. The symbols of our matrices are special Fisher–Hartwig symbols, and the theorem of this paper confirms and makes more precise a conjecture by Dai, Geary, and Kadanoff of 2009 in a particular case.

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1. Introduction and main results

The problem of computing eigenvalues and eigenvectors of large Toeplitz matrices has been studied by many authors for a long time, and at present one has reliable numerical algorithms which work
very well even if the order of the matrices is in the thousands. However, at least two problems cannot be solved by bare numerical computations.

First, several problems on Toeplitz matrices come from statistical physics, and in this context the matrix dimension is an astronomical number, say $10^8$, the cube root of the Avogadro number. Clearly, such matrices can hardly be tackled numerically. We remark that the question treated in this paper was eventually motivated by just a problem of statistical physics. Secondly, the spectral properties of Toeplitz matrices are heavily governed by the so-called symbol (= generating function), and often it is crucial to know the dependence of the eigenvalues or eigenvectors on the symbol, especially if the symbol involves parameters, such as the temperature in statistical physics. The concrete form of the dependence can frequently be read off from extensive numerical experiments, which might suffice for practical applications, but rigorous mathematical proofs usually represent a true challenge. In this paper, we consider the two problems alluded to for a special class of Hessenberg Toeplitz matrices. In our paper [1], we studied the eigenvalues. The present paper deals with the eigenvectors.

Given a function $a$ in the Lebesgue space $L^1$ on the complex unit circle $\mathbb{T}$, we denote by $T_n(a)$ the $n \times n$ Toeplitz matrix $(a_{j-s})_{j,s=0}^{n-1}$ constituted by the Fourier coefficients of $a$,

$$a_k = \frac{1}{2\pi} \int_{0}^{2\pi} a(e^{i\theta}) e^{-ik\theta} d\theta.$$ 

The function $a$ is usually referred to as the symbol of the matrix sequence $\{T_n(a)\}_{n=1}^\infty$. We consider symbols of the form $a(t) = t^{-1}h(t)$ ($t = e^{i\theta} \in \mathbb{T}$) with the following properties:

1. $h$ belongs to the Hardy class $H^\infty$ and $h_0 \neq 0$;
2. $h(t) = (1 - t)^\alpha f(t)$ where $\alpha \in (0, \infty) \setminus \mathbb{Z}$ and $f \in C^\infty(\mathbb{T})$ has no zeros on $\mathbb{T}$;
3. $h$ has an analytic extension into an open neighborhood $W$ of $\mathbb{T} \setminus \{1\}$ not containing the point 1;
4. $R(a)$ is a Jordan curve in $\mathbb{C}$, $\text{wind}_\lambda(a) = -1$ for each $\lambda \in D(a)$, and $a'(t) \neq 0$ for every $t \in \mathbb{T} \setminus \{1\}$.

Here $h_0$ is the zeroth Fourier coefficient of $h$, $R(a)$ is the range of $a$, and $D(a)$ is the set $\{ \lambda \in \mathbb{C} \setminus R(a) : \text{wind}_\lambda(a) \neq 0 \}$. Fig. 1 contains the ranges $R(a)$ for $a(t) = t^{-1}(1 - t)^\alpha$ and several values of $\alpha$. The arrows indicate the direction the curve $R(a)$ is traced out when the unit circle $\mathbb{T}$ is taken counterclockwise. The winding number $\text{wind}_\lambda(a)$ in property 4 is $-1$ for $0 < \alpha < 2$ and $+1$ for $2 < \alpha < 4$. $R(a)$ is no longer a Jordan curve for $\alpha > 4$.

Let $\lambda_{1,n}, \ldots, \lambda_{n,n}$ be the eigenvalues of $T_n(a)$ counted according to their multiplicities. Numerical experiments reveal that these clusters along $R(a)$ lying inside $D(a)$. This observation can to a certain extent be proved with mathematical rigor. See [1, 8]. In [1], we established the following. Let $W_0$ be any small open neighborhood of the origin in $\mathbb{C}$. Then for every $\lambda \in D(a) \cap (a(W) \setminus W_0)$ there exists a unique point $t_\lambda$ outside the closed unit disk such that $a(t_\lambda) = \lambda$. In particular, letting $\omega_0 := \exp(-2\pi i/n)$ and $J_n := \{ j \in \{1, \ldots, n-1\} : a(\omega_0^j) \notin W_0 \}$, we get points $t_{\lambda,j,n}$ such that $a(t_{\lambda,j,n}) = \lambda_{j,n}$ for $j \in J_n$. Formula (1.4) of [1] tells us that

$$t_{\lambda,j,n} = n^{(\alpha+1)/n} \omega_0^j \left( 1 + \log D_1(\omega_0^j) \frac{1}{n} + S_1(j, n) \right),$$

where

$$D_1(u) := \frac{c_\alpha a^2(u)}{u^2 a'(u)}, \quad c_\alpha := \frac{\pi}{f(1)\Gamma(\alpha + 1) \sin(\alpha \pi)}$$

and $S_1(j, n) = O(1/n^{a_0+1}) + O(\log n/n^2)$ uniformly in $j \in J_n$, with $a_0$ defined as $a_0 := \min\{\alpha, 1\}$. The branch of the logarithm is specified by the argument in $(-\pi, \pi]$. Formula (1.5) of [1] says that

$$\lambda_{j,n} = a(\omega_0^j) + \omega_0^j a'(\omega_0^j) \left( (\alpha + 1) \frac{\log n}{n} + \log D_1(\omega_0^j) \frac{1}{n} + S_2(j, n) \right),$$

where $S_2(j, n) = O(1/n^{a_0+1}) + O(\log n/n^2)$ uniformly in $j \in J_n$. Formula (1.5) of [1] says that
Fig. 1. The picture shows the range of the symbol $a(t) = t^{-1}(1 - t)^{\alpha}$ for different values of $\alpha$.

with $S_2(j, n) = O(1/n^{\alpha_0 + 1}) + O((\log n)^2/n^2)$ uniformly in $j \in J_n$.

To state formulas for the eigenvectors, we define the functions $b^{(j,n)}$ on $\mathbb{T}$ by

$$b^{(j,n)}(t) := 1/(h(t) - \lambda_{j,n} t)$$

and denote by $b_s^{(j,n)}$ the $s$th Fourier coefficient of $b^{(j,n)}$.

**Theorem 1.1.** If $b^{(j,n)}_{n-1} \neq 0$, then the complex vector

$$v_{j,n} := \left( b_s^{(j,n)} \right)_{s=0}^{n-1}$$

is an eigenvector of $T_n(a)$ for the eigenvalue $\lambda_{j,n}$.

The following result describes the asymptotics of $b_s^{(j,n)}$ for large $s$. 


Theorem 1.2. We have
\[ b_s^{(j,n)} = \frac{D_2(j,n)\omega_n^{-js}}{(D_1(j,n)^{n+1})^{1/n}} (1 + R_1(j,n,s)) + \frac{D_3(j,n)}{s^{\alpha+1}} (1 + R_2(j,n,s)), \] (1.4)
where \( D_1(u) \) is as above,
\[ D_2(u) := \frac{-1}{u^2a'(u)}, \quad D_3(u) := \frac{1}{c_\alpha a^2(u)}, \]
\[ R_1(j,n,s) = O(s/n^{\alpha_0+1}) + O(\log n/n) \text{ as } s,n \rightarrow \infty \text{ uniformly in } j \in \mathcal{J}_n, \text{ and } R_2(j,n,s) = O(\log n/n) + O(1/s^{\alpha_0}) \text{ as } s \rightarrow \infty, \text{ uniformly with respect to } n \text{ and } j \in \mathcal{J}_n. \]

We emphasize that we are considering only the points \( \omega_j \) belonging to the region \( W \setminus a^{-1}(W_3) \). In this region the functions \( a \) and \( a' \) remain bounded and bounded away from zero. Thus, the functions \( D_\ell, \ell = 1, 2, 3, \) are bounded and bounded away from zero in this region as well. In this sense we can think of \( D_\ell (\ell = 1, 2, 3) \) as “constants”. Since \( s < n \), the term \( R_1 \) goes to zero as \( n \rightarrow \infty \).

The eigenvector \( \mathbf{v}_{j,n} \) is not yet normalized. The following theorem shows that the norm of the eigenvector \( \mathbf{v}_{j,n} \) is rather large: it behaves like \( \sqrt{n/\log n} \).

Theorem 1.3. We have
\[ |\mathbf{v}_{j,n}|^2 = \frac{n}{2(\alpha + 1)|a'(t_{j,n})|^2} \log n \left( 1 - \frac{\text{Re} \log D_1(\omega_j)}{(\alpha + 1) \log n} + R_3(j,n) \right), \]
where \( R_3(j,n) = O(1/(\log n)^2) \) as \( n \rightarrow \infty \) uniformly in \( j \in \mathcal{J}_n \).

Dai, Geary, and Kadanoff [4] studied the symbol
\[ a(t) = \left( 2 - t - \frac{1}{t} \right)^\gamma (-t)^\beta = (-1)^{\beta+\gamma} t^{\beta-\gamma} (1 - t)^{2\gamma} \] (1.5)
for \( 0 < \gamma < -\beta < 1 \). See also papers [5, 7]. This is a special so-called Fisher–Hartwig symbol; see, e.g. [3]. On the basis of numerical experiments, they conjectured asymptotic expressions for the eigenvalues and the eigenvectors of \( T_n(a) \) as \( n \rightarrow \infty \). In the case where \( \beta = \gamma - 1 \), the function \( a \) becomes our symbol with \( h(t) = (1 - t)^{2\gamma - 1}(1 - t)^{2\gamma} \), and we may omit the factor \((1 - t)^{2\gamma - 1}\) for our purposes. In that case the conjecture of [4] about the eigenvectors is
\[ \mathbf{v}_{j,n} \approx \left( 1 + O(1/n) \right)^{n-1} \left( \frac{1}{t_{j,n}} \right) \approx n^{(\alpha+1)/n} \omega_j^{s}, \] (1.6)
as \( n \rightarrow \infty \). Formula (1.1), and Theorems 1.1 and 1.2 prove this conjecture along with precise and mathematically justified error bounds. Note that
\[ D_1^{s/n}(\omega_j) = \exp \left( \frac{s}{n} \log D_1(\omega_j) \right) = \exp \left( \frac{s}{n} O(1) \right) = 1 + O \left( \frac{s}{n} \right), \]
which shows that (1.6) corresponds to the first term on the right of (1.4) when \( s/n \) is close to zero. In the remaining cases (1.6) generates big errors. Fig. 2 shows an example in which the error produced by (1.6) exceeds the absolute value of the components \( b_s^{(j,n)} \) for the last part of the eigenvector \( \mathbf{v}_{j,n} \).
Of course, the approach of our paper depends heavily on the lucky circumstance that the Toeplitz matrices are of Hessenberg form for $O$ and large values of $s$. On the other hand, the symbols considered here encompass a lot of Toeplitz matrices which are not generated by (1.5).

Our asymptotic expansion (1.4) is valid for eigenvalues $\lambda_{j,n}$ outside a small open neighborhood $W_0$ of the origin and large values of $s$. The values of $b_s^{(j,n)}$ for small $s$, say for $s = 0, 1, \ldots, m - 1, m \ll n$, can be computed using the relation between the Fourier coefficients $b_s^{(j,n)}$ and $h_s$ of the functions $b^{(j,n)} = 1/(h(t) - \lambda_{j,n})$ and $h(t) = (1 - t)^\alpha f(t)$, respectively. The solution of the corresponding triangular linear system can be shown to be an $O(m^2) / n$ complexity problem. Choosing $m = \lfloor \sqrt{n} \rfloor$ we arrive at $O(n)$ complexity problem. The remaining components can be calculated using the asymptotic formula (1.4), which turns out to be an $O(n)$ complexity problem. Table 1 and Figs. 3 and 4 show numerical results obtained in this way. In Figs. 3 and 4, the dots were obtained with Matlab. The stars and crosses are the approximations resulting from formula (1.4) and (1.6) (times $D_2(\omega^j_n)$), respectively. The red, blue, green, and black marks correspond to the components with index $0 \bmod 4$, $1 \bmod 4$, $2 \bmod 4$, and $3 \bmod 4$, respectively. The four branches come from the factor $\omega_n^{-js}$ in the main term of (1.4), and their spiral behavior is caused by the factor $D_1^{-s/n}(\omega^j_n)$, which is not contained in (1.6).

We want to remark that the results of this paper can be easily translated to the case where the symbol is $a(t) = t(1 - t^{-1})^\alpha f(t)$.

**Table 1**

The table shows the maximum error obtained with our formulas (1.6), (1.4), (1.7) for the components of the $[n/4]$th eigenvector of $T_n(t^{-1}(1 - t)^{1/4})$ for different values of $n$. The data were obtained by comparison with the solutions given by Matlab, taking into account only the components with index greater than $[\sqrt{n}]$, that is $s > \lfloor \sqrt{n} \rfloor$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>256</th>
<th>512</th>
<th>1024</th>
<th>2048</th>
<th>4096</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.6)</td>
<td>$4.6 \times 10^{-2}$</td>
<td>$4.3 \times 10^{-2}$</td>
<td>$3.8 \times 10^{-2}$</td>
<td>$3.2 \times 10^{-2}$</td>
<td>$2.4 \times 10^{-2}$</td>
</tr>
<tr>
<td>(1.4)</td>
<td>$4.8 \times 10^{-3}$</td>
<td>$2.7 \times 10^{-3}$</td>
<td>$1.5 \times 10^{-3}$</td>
<td>$8.5 \times 10^{-4}$</td>
<td>$4.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>(1.7)</td>
<td>$1.7 \times 10^{-3}$</td>
<td>$7.3 \times 10^{-4}$</td>
<td>$3.6 \times 10^{-4}$</td>
<td>$1.6 \times 10^{-4}$</td>
<td>$7.7 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Theorems 1.2 and 1.3 exhibit the first terms of complete asymptotic expansions. Proceeding as in our proofs, see paper [2], it is in principle possible to get more terms, but the expenses are immense. We can, for example, show that if $0 < \alpha < 1$, then

$$b_s^{(j, n)} = \frac{D_2(\omega_n^j)\omega_n^{-js}}{(D_1(\omega_n^j)n^{\alpha+1})^{1/n}} \left[ 1 + \frac{d_\alpha s}{a(\omega_n^j)n^{\alpha+1}} + \frac{\omega_n^j D_2'(\omega_n^j)}{D_2(\omega_n^j)n}((\alpha + 1) \log n + \log D_1(\omega_n^j)) \right]$$

$$- \frac{d_\alpha \omega_n^j D_2'(\omega_n^j)}{a(\omega_n^j)D_2(\omega_n^j)n^{\alpha+1}} + R_4(j, n, s) + \frac{D_3(\omega_n^j)}{s^{\alpha+1}} \left[ 1 + \frac{d_\alpha}{a(\omega_n^j)s^{\alpha}} + R_5(j, n, s) \right] ,$$

(1.7)
From Jacobi’s theorem, for which see e.g. [6, Chapter I, Section 4], we have
\[ A \cdot \text{adj} A = \det A \cdot I_n, \]
where adj\( A \) stands for the adjugate of \( A \) and \( I_n \) for the \( n \times n \) identity matrix.

**Proof of Theorem 1.1.** The first column of the matrix \( \text{adj} T_n(a - \lambda_{j,n}) \) is given by
\[
\mathbf{c} := \left( \begin{array}{c}
M_{11}[T_n(a - \lambda_{j,n})] \\
-M_{12}[T_n(a - \lambda_{j,n})] \\
M_{13}[T_n(a - \lambda_{j,n})] \\
\vdots \\
(-1)^{n+1}M_{1n}[T_n(a - \lambda_{j,n})]
\end{array} \right).
\]
Let \( A \) be any \( n \times n \) matrix. For index sets \( \beta, \gamma \subset \{1, 2, \ldots, n\} \) we denote by \( A(\beta, \gamma) \) the sub-matrix of \( A \) that lies in the rows indexed by \( \beta \) and the columns indexed by \( \gamma \). For example
\[ M_{12}(A) = \det (A([2, 3, \ldots, n], [1, 3, 4, \ldots, n])). \]
From Jacobi’s theorem, for which see e.g. [6, Chapter I, Section 4], we have
\[
\det T_{n+1}^{-1}(h - \lambda_{j,n}t)(\beta', \gamma') = (-1)^{\delta} \frac{\det T_{n+1}(h - \lambda_{j,n}t)(\gamma', \beta)}{\det T_{n+1}(h - \lambda_{j,n}t)}, \tag{2.1}
\]
where \( \beta' = \{1, 2, \ldots, n+1\} \setminus \beta, \gamma' = \{1, 2, \ldots, n+1\} \setminus \gamma, \) and \( \delta = \sum_{s \in \beta \cup \gamma} s. \) From [1, p. 6] we know that \( T_{n+1}^{-1}(h - \lambda_{j,n}t) = T_{n+1}(b(j,n)). \) Note that
\[
T_{n+1}(h - \lambda_{j,n}t) = \left( \begin{array}{cccccc}
h_0 & 0 & 0 & \cdots & 0 & 0 \\
h_1 - \lambda_{j,n} & h_0 & 0 & \cdots & 0 & 0 \\
h_2 & h_1 - \lambda_{j,n} & h_0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 & 0 \\
h_n & h_{n-1} & h_{n-2} & \cdots & h_{1 - \lambda_{j,n}} & h_0
\end{array} \right)
\]
and
\[
T_n(a - \lambda_{j,n}) = \left( \begin{array}{cccccc}
h_1 - \lambda_{j,n} & h_0 & 0 & \cdots & 0 & 0 \\
h_2 & h_1 - \lambda_{j,n} & h_0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 & 0 \\
h_n & h_{n-1} & h_{n-2} & \cdots & h_{1 - \lambda_{j,n}} & h_0
\end{array} \right).
\]
Thus, taking $\beta = \{1, 2, \ldots, s - 1, s + 1, \ldots, n\}$ and $\gamma = \{3, 4, \ldots, n + 1\}$ in (2.1) we obtain
\[
\det \begin{pmatrix} b^{(j,n)}_{s-1} & b^{(j,n)}_{s-2} \\ b^{(j,n)}_n & b^{(j,n)}_{n-1} \end{pmatrix} = \frac{(-1)^{\delta}}{h_0^{n+1}} \det \left( T_n(a - \lambda_{j,n})^{(\gamma, \beta)} \right) = \frac{(-1)^{\delta}}{h_0^{n+1}} M_{15} [T_n(a - \lambda_{j,n})].
\]

From Lemma 2.1 of [1] we know that $\lambda_{j,n}$ is an eigenvalue of $T_n(a)$ if and only if $b^{(j,n)}_n$ is zero. Thus, $M_{15} = (-1)^{\delta} h_0^{n+1} b^{(j,n)}_n b^{(j,n)}_{n-1}$. If $b^{(j,n)}_{n-1} \neq 0$, we may remove the common constants in the entries of the column $c$ and arrive at the conclusion that $T_n(a) v^{(j,n)}_n = \lambda_{j,n} v^{(j,n)}_n$. As $b^{(j,n)}_0 = 1 / h_0 \neq 0$, it follows that $v^{(j,n)}_n$ is not the zero vector. Hence $v^{(j,n)}_n$ is an eigenvector for $\lambda_{j,n}$. $\square$

**Proof of Theorem 1.2.** Combining Theorem 1.1 of [1] and Lemma 2.1 of [1] we get
\[
b^{(j,n)}_s = \frac{D_2(t_{j,n})}{t_{j,n}^s} + \frac{1}{c_0 s^{2\alpha + \beta} \lambda_{j,n}^2} + Q_1(j, n, s),
\]
(2.2)

$Q_1(j, n, s) = O(1 / s^{\alpha + \alpha_0 + 1})$ as $s \to \infty$, uniformly with respect to $n$ and $j \in J_n$. See Fig. 5.

From (1.1) we infer that $t_{j,n} = \omega_{\alpha}^{(j)}(1 + O(\log n / n))$ uniformly in $j \in J_n$. Hence $a'(t_{j,n}) = a'(\omega_{\alpha}^{(j)})(1 + O(\log n / n))$ and thus $D_2(t_{j,n}) = D_2(\omega_{\alpha}^{(j)})(1 + Q_2(j, n))$ with $Q_2(j, n) = O(\log n / n)$ uniformly in $j \in J_n$.

Inserting this in (2.2) we obtain
\[
b^{(j,n)}_s = \frac{D_2(\omega_{\alpha}^{(j)})}{t_{j,n}^s} (1 + Q_2(j, n)) + \frac{1}{c_0 s^{2\alpha + \beta} \lambda_{j,n}^2} + Q_1(j, n, s).
\]
(2.3)

**Fig. 5.** The black dots are the $\log_{10}$ of the absolute value of $b^{(32,128)}_s$ calculated with Matlab. The red circles and green stars are the $\log_{10}$ of the absolute value of the first and second terms of (2.2), respectively. The blue crosses are the $\log_{10}$ of the absolute value of the difference between $b^{(32,128)}_s$ calculated with Matlab and (2.2), without $Q_1$. Note that the second term is important only for the very last part of the eigenvector. The value of $t_{63,128}$ was obtained by means of a Newton–Raphson routine as the exact solution of the equation $a(t_{63,128}) = \lambda_{32,128}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
We are left with expanding $t_{s,j,n}$ and $\lambda_{j,n}$ in (2.3). Formula (1.1) gives us that
\[
\frac{1}{t_{s,j,n}^3} = \frac{1}{n^{(\alpha+1)s/n}} \left( 1 + \log D_1(\omega_n^j) \frac{1}{n} + S_1(j,n) \right)^{-s} \\
= \frac{1 + Q_2(j,n,s)}{n^{(\alpha+1)s/n} \omega_n^j D_1(\omega_n^j)^{s/n}}
\]
with $Q_2(j,n,s) = O(s/n^{\alpha+1}) + O(s \log n/n^2)$ as $s, n \to \infty$, uniformly with respect to $j \in J_n$. Similarly, formula (1.2) shows that $1/\lambda_{j,n}^2$ is
\[
\frac{1}{a^2(\omega_n^j)} \left( 1 + \frac{(\alpha + 1)\omega_n^j a'(\omega_n^j) \log n}{a(\omega_n^j)} + \frac{\omega_n^j a'(\omega_n^j) \log D_1(\omega_n^j)}{a(\omega_n^j)} + S_2(j,n) \right)^{-2}
\]
which equals
\[
\frac{1}{a^2(\omega_n^j)} - \frac{2(\alpha + 1)\omega_n^j a'(\omega_n^j) \log n}{a^3(\omega_n^j) n} - \frac{2\omega_n^j a'(\omega_n^j) \log D_1(\omega_n^j)}{a^3(\omega_n^j) n} + Q_4(j,n)
\]
\[
= D_3(\omega_n^j)(1 + Q_5(j,n)),
\]
where $Q_4(j,n) = O(1/n^{\alpha_0+1}) + O((\log n)^2/n^2)$ and $Q_5(j,n) = O(\log n/n)$ as $n \to \infty$, uniformly with respect to $j \in J_n$. Inserting the last expression and (2.4) in (2.3) we obtain
\[
b_{(j,n)}^s = \frac{D_2(\omega_n^j)}{n^{(\alpha+1)s/n} \omega_n^j D_1(\omega_n^j)^{s/n}} \left( 1 + Q_2(j,n) \right) \left( 1 + Q_3(j,n,s) \right)
\]
\[
+ \frac{D_3(\omega_n^j)}{s^{\alpha+1}} \left( 1 + Q_5(j,n) + \frac{s^{\alpha+1}}{D_3(\omega_n^j)} Q_1(j,n,s) \right)
\]
Clearly
\[
R_1(j,n,s) := Q_2(j,n) + Q_3(j,n,s) + Q_2(j,n) Q_3(j,n,s)
\]
\[
= O(s/n^{\alpha_0+1}) + O(\log n/n).
\]
Furthermore,
\[
R_2(j,n,s) := Q_5(j,n) + \frac{s^{\alpha+1}}{D_3(\omega_n^j)} Q_1(j,n,s)
\]
\[
= O(\log n/n) + s^{\alpha+1} O(1/s^{\alpha+\alpha_0+1}).
\]
This completes the proof. □

Lemma 2.1 of [1] shows that
\[
D_n(a - \lambda) = (-1)^n h_0^{n+1} b_n^{(\lambda)},
\]
where $b_n^{(\lambda)}$ is the $n$th Fourier coefficient of the function $b^{(\lambda)}(t) = 1/(h(t) - \lambda t)$. As $\lambda_{j,n}$ is an eigenvalue of $T_n(a)$, we must have $b_n^{(\lambda_{j,n})} = b_n^{(j,n)} = 0$. Note that the main terms of (1.4) annihilate each other when $s = n$. The following result shows that the very last component of the eigenvector $v_{j,n}$ is nonzero whenever $n$ is large enough.

**Corollary 2.1.** We have $b_{n-1}^{(j,n)} \neq 0$ for all sufficiently large $n$ and all $j \in J_n$. 


Proof. Replacing $s$ by $n - 1$ in (1.4) we get

$$b_{n-1}^{(j,n)} = \frac{D_2(\omega_n^j)\omega_n^j(1-n)}{(D_1(\omega_n^j)\omega_n^j)^{(n-1)/n}} (1 + Q_6(j, n)) + \frac{D_3(\omega_n^j)}{(n-1)^{\alpha+1}} (1 + Q_7(j, n))$$

$$= \frac{D_2(\omega_n^j)\omega_n^j}{D_1(\omega_n^j)\omega_n^j} (D_1(\omega_n^j)\omega_n^j)^{(n-1)/n} (1 + Q_6(j, n)) + \frac{D_2(\omega_n^j)}{n^{\alpha+1}} (1 - 1/n)^{-(\alpha+1)} (1 + Q_7(j, n)),$$

where $Q_6$ and $Q_7$ are $O(1/n^{\alpha_0}) + O(\log n/n)$ as $n \to \infty$ uniformly in $j \in J_n$. Using that

$$(D_1(\omega_n^j)\omega_n^j)^{(n-1)/n} = 1 + Q_8(j, n), \quad (1 - 1/n)^{-(\alpha+1)} = 1 + Q_9(n),$$

and $D_2/D_1 = -D_3$, where $Q_8(j, n) = O(\log n/n)$, $Q_9(n) = O(1/n)$ as $n \to \infty$ uniformly in $j \in J_n$, we obtain

$$b_{n-1}^{(j,n)} = \frac{D_3(\omega_n^j)(1 - \omega_n^j)}{n^{\alpha+1}} + Q_{10}(j, n),$$

where $Q_{10}(j, n) = O(1/n^{\alpha+\alpha_0+1}) + O(\log n/n^{\alpha+2})$ as $n \to \infty$ uniformly in $j \in J_n$. Thus, $b_{n-1}^{(j,n)} \neq 0$ for all sufficiently large $n$. □

3. Norm of the eigenvector

Using that

$$\|f\|^2 = \frac{1}{2\pi} \int_\mathbb{T} |f(t)|^2 \frac{dt}{t}$$

for every $f \in L^2(\mathbb{T})$, we calculate the norm of the vector (1.3) by determining first $\|b^{(j,n)}\|^2 = \sum_{s=0}^{\infty} |b_s^{(j,n)}|^2$ and then estimating the tail $\sum_{s=n}^{\infty} |b_s^{(j,n)}|^2$.

Lemma 3.1. We have

$$\sum_{s=0}^{\infty} |b_s^{(j,n)}|^2 = \frac{n}{2(\alpha + 1)|a'(t_{j,n})|^2 \log n} \left( 1 - \text{Re} \frac{D_1(\omega_n^j)}{(\alpha + 1) \log n} + Q_{11}(j, n) \right),$$

where $Q_{11}(j, n) = O(1/(\log n)^2)$ as $n \to \infty$, uniformly in $j \in J_n$.

Proof. By Lemma 3.1 of [1], the point $t_{j,n}$ is a simple pole of $b^{(j,n)}$. Thus, we can split $b^{(j,n)}$ as

$$b^{(j,n)}(t) = \frac{1}{t_{j,n} a'(t_{j,n})(t - t_{j,n})} + \varphi^{(j,n)}(t),$$

where the function $\varphi^{(j,n)}$ is analytic and bounded in some open neighborhood of $\mathbb{T}$. It follows that

$$\|b^{(j,n)}(t)\|^2 = \frac{1}{2\pi i} \int_\mathbb{T} \left( \frac{1}{t_{j,n} a'(t_{j,n})(t - t_{j,n})} + \varphi^{(j,n)}(t) \right) \left( \frac{1}{t_{j,n} a'(t_{j,n})(\bar{t} - t_{j,n})} + \overline{\varphi^{(j,n)}(t)} \right) \frac{dt}{t},$$
and multiplying term by term, we get \( \|b^{(j,n)}(t)\|_2^2 = I_1 + I_2 + I_3 + I_4 \), where

\[
I_1 := \frac{1}{2\pi i |t\alpha_{j,n} \gamma'_{j,n}|} \int_T \frac{1}{(1 - t\alpha_{j,n})^{n-1}} dt, \quad I_2 := \frac{1}{2\pi i} \int_T |\phi^{(j,n)}(t)|^2 dt,
\]

\[
I_3 := \frac{1}{2\pi i} \int_T \frac{\phi^{(j,n)}(t) dt}{t - t\alpha_{j,n}}.
\]

Now we calculate the preceding integrals. By residue theory

\[
I_1 = \frac{1}{|t\alpha_{j,n} \gamma'_{j,n}|^2} \lim_{t \to 1/t\alpha_{j,n}} \frac{1}{1/t\alpha_{j,n} - t} = \frac{1}{|t\alpha_{j,n} \gamma'_{j,n}|^2},
\]

From (1.1) we know that

\[
t\alpha_{j,n} = \omega_n^{\frac{j}{2}} \left[ 1 + (\alpha + 1) \frac{\log n}{n} + \frac{\log D_1(\omega_n^{j})}{n} + Q_{12}(j, n) \right],
\]

where \( Q_{12}(j, n) = O(1/n^\alpha) + O((\log n)^2/n^2) \) as \( n \to \infty \), uniformly with respect to \( j \in \mathcal{J}_n \). Multiplying \( t\alpha_{j,n} \) by \( 1/t\alpha_{j,n} \) we therefore obtain

\[
|t\alpha_{j,n}|^2 = 1 + 2(\alpha + 1) \frac{\log n}{n} + \frac{2 \Re \log D_1(\omega_n^{j})}{n} + Q_{13}(j, n),
\]

Inserting (3.2) in (3.1) we see that \( I_1 \) equals

\[
\frac{n |a'(t\alpha_{j,n})|^{-2}}{2(\alpha + 1) \log n (1 + O(\log n/n))} \left( 1 + \frac{\Re \log D_1(\omega_n^{j})}{(\alpha + 1) \log n} + Q_{14}(j, n) \right)^{-1}
\]

with \( Q_{14}(j, n) = O(1/(n^\alpha \log n)) + O(\log n/n) \) as \( n \to \infty \), uniformly in \( j \in \mathcal{J}_n \). Hence

\[
I_1 = \frac{n}{2(\alpha + 1) |a'(t\alpha_{j,n})|^2 \log n} \left( 1 - \frac{\Re \log D_1(\omega_n^{j})}{(\alpha + 1) \log n} + Q_{15}(j, n) \right).
\]

where \( Q_{15}(j, n) = O(1/(\log n)^2) \) as \( n \to \infty \), uniformly in \( j \in \mathcal{J}_n \).

The result will follow once we have shown that \( I_2, I_3, I_4 \) are \( O(1) \) uniformly in \( n \) and \( j \in \mathcal{J}_n \). Making in \( I_2 \) the change of variables \( u = 1/t \) and using residue theory again, we obtain

\[
I_2 = \frac{1}{2\pi i t^2 \alpha_{j,n} \gamma'_{j,n}} \int_T \frac{\varphi^{(j,n)}(\frac{1}{u})}{u} du = \frac{\varphi^{(j,n)}(t\alpha_{j,n})}{t^2 \alpha_{j,n} \gamma'_{j,n}} = O(1)
\]

as desired. Similarly,

\[
I_3 = -\frac{\varphi^{(j,n)}(1/t\alpha_{j,n})}{\omega_n^{\frac{j}{2}} \gamma'_{j,n}} = O(1).
\]

Finally, as the \( L^\infty \) norm of the function \( \varphi^{(j,n)} \) is uniformly bounded, we conclude that \( I_4 = \|\varphi^{(j,n)}\|_2^2 = O(1) \). \( \square \)
Lemma 3.2. We have $\sum_{s=n}^{\infty} |b_s^{(j,n)}|^2 = O(1)$, uniformly in $n$ and $j \in J_n$.

Proof. Using expression (2.2) and multiplying $b_s^{(j,n)}$ by $\overline{b_s^{(j,n)}}$ we obtain

$$|b_s^{(j,n)}|^2 = \left( \frac{E_{j,n}}{t_{\lambda,j,n}} + \frac{F_{j,n}}{s^{\alpha+1}} + Q_1(j, n, s) \right) \left( \frac{E_{j,n}}{t_{\lambda,j,n}} + \frac{F_{j,n}}{s^{\alpha+1}} + Q_1(j, n, s) \right)$$

$$= \frac{|E_{j,n}|^2}{|t_{\lambda,j,n}|^{2s}} + \frac{|F_{j,n}|^2}{s^{2\alpha+2}} + 2 \Re \left( \frac{E_{j,n}F_{j,n}}{t_{\lambda,j,n}^2 s^{\alpha+1}} \right) + Q_{16}(j, s),$$

where

$$E_{j,n} := \frac{-1}{t_{\lambda,j,n}^2} a'(\lambda,j,n) = O(1), \quad F_{j,n} := \frac{1}{c_\alpha \lambda_{j,n}^2} = O(1),$$

and $Q_{16}(j, s) = O(1/s^{\alpha+\alpha_0+1})$ uniformly in $n$ and $j \in J_n$. Since $b_n^{(j,n)} = 0$ (see Eq. (2.1) of [1]), our sum may be written as

$$\sum_{s=n+1}^{\infty} |b_s^{(j,n)}|^2 = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 := \frac{|E_{j,n}|^2}{|t_{\lambda,j,n}|^{2s}(|t_{\lambda,j,n}|^2 - 1)}, \quad S_2 := \frac{|F_{j,n}|^2}{s^{2\alpha+2}},$$

$$S_3 := 2 \sum_{s=n+1}^{\infty} \Re \left( \frac{E_{j,n}F_{j,n}}{t_{\lambda,j,n}^2 s^{\alpha+1}} \right), \quad S_4 := \sum_{s=n+1}^{\infty} Q_{16}(j, n, s).$$

Summing up the geometric series for $S_1$ we obtain

$$S_1 = \frac{|E_{j,n}|^2}{|t_{\lambda,j,n}|^{2n}(|t_{\lambda,j,n}|^2 - 1)}.$$

From (3.2) we see that $1/(|t_{\lambda,j,n}|^2 - 1) = O(n/ \log n)$ and

$$\frac{1}{|t_{\lambda,j,n}|^{2n}} = \exp \left[ -n \left( \log \left( 1 + 2(\alpha + 1) \frac{\log n}{n} + O(1/n) \right) \right) \right]$$

$$= \exp \left[ -2(\alpha + 1) \log n + O(1) \right] = O(1/n^{2\alpha+2}),$$

uniformly in $j \in J_n$. Consequently, $S_1 = O(1/n^{2\alpha+1} \log n) = O(1)$. The remaining three sums are trivial. Indeed for $S_2$ we have

$$S_2 = O \left( \sum_{s=n+1}^{\infty} \frac{1}{s^{2\alpha+2}} \right) = O(1)$$

the sum $S_3$ admits the estimate

$$S_3 \leq 2|E_{j,n}| |F_{j,n}| \sum_{s=n+1}^{\infty} \frac{1}{|t_{\lambda,j,n}|^{s\alpha+1}} \leq 2|E_{j,n}| |F_{j,n}| \sum_{s=n+1}^{\infty} \frac{1}{s^{\alpha+1}} = O(1),$$
and finally,

\[
S_4 = O \left( \sum_{s=n+1}^{\infty} \frac{1}{s^{\alpha+\alpha_0+1}} \right) = O(1).
\]

This completes the proof. \(\square\)

We remark that the previous proof actually shows that

\[
\sum_{s=n}^{\infty} |b_s^{(j,n)}|^2 = O(1/n^{\alpha}).
\]

**Proof of Theorem 1.3.** Combine Lemmas 3.1 and 3.2. \(\square\)

**References**


