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Pushing the envelope of the test functions in the Szegö and Avram–Parter theorems

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Abstract

The Szegö and Avram–Parter theorems give the limit of the arithmetic mean of the values of certain test functions at the eigenvalues of Hermitian Toeplitz matrices and the singular values of arbitrary Toeplitz matrices, respectively, as the matrix dimension goes to infinity. The question on whether these theorems are true whenever they make sense is essentially the question on whether they are valid for all continuous, nonnegative, and monotonously increasing test functions. We show that, surprisingly, the answer to this question is negative. On the other hand, we prove the two theorems in a general form which includes all versions known so far.

Keywords: Toeplitz matrix; Eigenvalue; Singular value; Test function; Asymptotic distribution

1. Introduction

Let a be a complex-valued function in $L^1 := L^1(0, 2\pi)$. We denote by a_n the nth Fourier coefficient of a

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} a(\theta) e^{-in\theta} \, d\theta \quad (n \in \mathbf{Z})$$

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and by $T_n(a)$ the $n \times n$ Toeplitz matrix $(a_{j-k})_{j,k=1}^n$. The function a is usually referred to as the symbol of the sequence $T_1(a), T_2(a), \ldots$ All the matrices $T_n(a)$ are Hermitian if (and only if) a is real-valued. Theorems of the Szegö type say that, under certain conditions on a and F, including that a be real-valued

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(T_n(a))) = \frac{1}{2\pi} \int_0^{2\pi} F(a(\theta))\mathrm{d}\theta, \tag{1}$$

where $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ are the eigenvalues of a Hermitian $n \times n$ matrix A, while theorems of the Avram–Parter type state that, again under appropriate assumptions on a and F,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \frac{1}{2\pi} \int_0^{2\pi} F(|a(\theta)|)\mathrm{d}\theta, \tag{2}$$

where $s_1(A) \leqslant \cdots \leqslant s_n(A)$ are the singular values of an $n \times n$ matrix A. The function F in (1) and (2) is called a test function. Throughout this paper we assume that F is real-valued and that F is continuous on $\mathbf{R}$, $F \in C(\mathbf{R})$, when considering (1) and continuous on $[0, \infty)$, $F \in C[0, \infty)$, when dealing with (2).

Formula (1) goes back to Szegö [13], who proved it for real-valued functions a in $L^\infty := L^\infty(0, 2\pi)$ and compactly supported continuous functions F on $\mathbf{R}$ (see also [6]). Formula (2) was first established by Parter [8] for all $F \in C[0, \infty)$ under the assumptions that a is in L^∞ and that a is locally selfadjoint, which means that $a = bc$ with a continuous 2π-periodic function c and a real-valued function b. Avram [1] subsequently proved (2) for all $F \in C[0, \infty)$ and all $a \in L^\infty$. Then Tyrtyshnikov [16,17] showed that (1) and (2) hold for all continuous functions F with compact support if a is merely required to be in $L^2 := L^2(0, 2\pi)$ and to be real-valued when dealing with (1). Zamarashkin and Tyrtyshnikov [18,19] were finally able to prove that (1) and (2) are true whenever F is continuous and compactly supported and a is in L^1, again requiring that a be real-valued when considering (1). A very simple proof of the Zamarashkin–Tyrtyshnikov result was given by Tilli [15], who also extended (1) and (2) to all uniformly continuous functions F and all $a \in L^1$, assuming that a is real-valued in the case of (1). Eventually Serra Capizzano [10] derived (2) under the assumption that $a \in L^p := L^p(0, 2\pi)$ $(1 \leqslant p < \infty)$ and $F \in C[0, \infty)$ satisfies $F(s) = O(s^p)$ as $s \to \infty$. Serra Capizzano's result implies in particular that (2) is valid for all $a \in L^1$ under the sole assumption that $F(s) = \mathrm{O}(s)$, which includes all the results concerning (2) listed before.

In [3], we raised the question whether (1) and (2) are true whenever they make sense. To be more precise and to exclude "∞–∞" cases, the question is whether (1) and (2) hold for all symbols $a \in L^1$ (being real-valued in (1)) and all nonnegative and continuous test functions. Here we make the following convention: we denote the functions under the integrals in (1) and (2), that is, the functions $\theta \mapsto F(a(\theta))$ and $\theta \mapsto F(|a(\theta)|)$, by $F(a)$ and $F(|a|)$, respectively, and if these functions are not in L^1, we define the right-hand sides of (1) and (2) to be ∞ and interpret (1) and (2) as the statement that the limit on the left-hand side is ∞.

It turns out that the answer to the question cited in the preceding paragraph is negative: in [3], we constructed a positive $a \in L^1$ and a continuous $F : \mathbf{R} \to [0, \infty)$ such that (1) and (2) are false. The test function F in that counterexample is not monotonous. This leaves us with the question whether (1) is always true if $a \in L^1$ is real-valued, $F \in C(\mathbf{R})$, and $F(\lambda)$ increases monotonously to infinity as $\lambda \to \infty$ and as $\lambda \to -\infty$ and with the problem whether (2) is always valid if $a \in L^1$ and $F : [0, \infty) \to [0, \infty)$ increases monotonously to infinity. (We use "increasing" as a synonym for "nondecreasing", that is, by a monotonously increasing function F we understand a function satisfying $F(x) \leqslant F(y)$ for $x \leqslant y$.) Our first main result shows that the answer to this question is negative.

To state things in other terms, let ST denote the set of all continuous functions $F : \mathbf{R} \to [0, \infty)$ for which (1) is true for all real-valued $a \in L^1$ and let APT be the set of all continuous $F : [0, \infty) \to [0, \infty)$ for which (2) is true for all $a \in L^1$. We know that ST and APT are proper subsets of the sets of nonnegative functions in $C(\mathbf{R})$ and $C[0, \infty)$, respectively, that ST contains all nonnegative uniformly continuous functions on $\mathbf{R}$, and that APT contains all nonnegative functions $F \in C[0, \infty)$ satisfying $F(s) = O(s)$ as $s \to \infty$. The result mentioned at the end of the previous paragraph tells us that APT does not contain all nonnegative monotonously increasing functions in $C[0, \infty)$.

In [3], we showed that if $a \in L^1$ and $F : [0, \infty) \to [0, \infty)$ is any continuous function, then

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) \geqslant \frac{1}{2\pi} \int_0^{2\pi} F(|a(\theta)|)\mathrm{d}\theta. \tag{3}$$

This implies in particular that (2) is always true if $F(|a|) \notin L^1$. Hence, in connection with (2) it remains to consider only the case where $F(|a|) \in L^1$.

We write $F(s) \simeq G(s)$ as $s \to \infty$ if there are positive constants C_1 and C_2 such that $C_1 G(s) \leqslant F(s) \leqslant C_2 G(s)$ for all sufficiently large $s > 0$. Combining (3) with the result by Serra Capizzano [10], we arrive at the conclusion that if a is in L^1 (but not necessarily in L^p) and $F : [0, \infty) \to [0, \infty)$ satisfies $F(s) \simeq s^p$ $(1 \leqslant p < \infty)$, then (2) holds. Thus, APT contains all nonnegative $F \in C[0, \infty)$ with $F(s) \simeq s^p$ $(1 \leqslant p < \infty)$. Other classes of convex functions F in APT were introduced in [3]. For example, we there showed that $F \in APT$ if

$$F(s) = \sum_{p=0}^{\infty} F_p s^p \quad \text{with } F_p \geqslant 0 \text{ for all } p.$$

This includes such functions as $F(s) = \mathrm{e}^s$, but the convex function $F(s) = s \log(s + 1)$ does not have such a representation. Another main result of the present paper is that APT contains all convex functions $F : [0, \infty) \to [0, \infty)$. Moreover, we can even weaken convexity to essential convexity, which means that $F(s) \simeq \Psi(s)$ with some convex function Ψ as $s \to \infty$.

The paper is organized as follows. In Section 2 we construct a nonnegative function $a \in L^1$ and a nonnegative and monotonously increasing function $F \in C[0, \infty)$ such that $F(a) \in L^1$ but (2) fails. Clearly, for these a and F, formula (1) is false as well. The remaining part of the paper is devoted to results in the positive direction. In Section 3 we introduce our main technical tool, a variational characterization of the sums $\sum \Phi(s_j(A))$ which mimics the variational characterization of unitarily invariant norms due to Serra Capizzano and Tilli. Section 4 contains a proof of the original Avram–Parter theorem and in Section 5 we cite Tilli's proof of the Zamarashkin–Tyrtyshnikov version of the Avram–Parter theorem. We present these proofs for the reader's convenience only. Those who are interested in the proofs of the main results may entirely skip these two sections. In Section 6 we derive a key inequality (Proposition 6.1) and show that the Avram–Parter theorem for monotonously increasing and convex test functions is equivalent to that theorem for compactly supported test functions. As the Avram–Parter theorem is available in the latter case, we therefore get it for the former. In Section 7 we employ Proposition 6.1 to prove our second main result (Corollary 7.3), which states that all nonnegative and essentially convex test functions belong to APT. Section 8 contains our third main result (Corollary 8.4). This result says that all nonnegative and essentially convex functions on $\mathbf{R}$ are in the class ST.

2. The counterexample

We denote by $\|A\|$ the spectral norm (=largest singular value) of a matrix A and use the norms

$$\|f\|_p = \left(\int_0^{2\pi} |f(\theta)|^p \frac{d\theta}{2\pi}\right)^{1/p}, \quad \|f\|_\infty = \operatorname{ess\,sup} |f(\theta)|$$

in L^p $(1 \leqslant p < \infty)$ and L^∞, respectively.

In this section we prove the following theorem.

Theorem 2.1. *There exist nonnegative functions $a \in L^1$ and nonnegative and monotonously increasing functions $F \in C[0,\infty)$ such that $F(a) \in L^1$ but*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \infty. \tag{4}$$

We explicitly construct such a and F. We put $b_0 = 0$, define b_k, β_k, δ_k for $k \geqslant 1$ by

$$b_k = 2^{2^{k^2}}, \quad \beta_k = 1/b_k, \quad \delta_k = 2^{-k^2}$$

and let

$$a(\theta) = \begin{cases} b_k & \text{for } \theta \in [(1-\delta_k)\beta_k, \beta_k] =: I_k \quad (k \geqslant 1), \\ 0 & \text{for } \theta \in [0, 2\pi) \setminus \cup_{k=1}^{\infty} I_k, \end{cases}$$

$$F(s) = \begin{cases} b_k & \text{for } s \in [b_{k-1}+1, b_k] \quad (k \geqslant 1), \\ (s-b_k)(b_{k+1}-b_k)+b_k & \text{for } s \in [b_k, b_k+1] \quad (k \geqslant 0). \end{cases}$$

It is obvious that $F : [0,\infty) \to [0,\infty)$ is continuous and monotonously increasing. Clearly, $F(b_k) = b_k$ and $F(b_k+1) = b_{k+1}$. We have

$$\|a\|_1 = \sum_{k=1}^{\infty} b_k \delta_k \beta_k = \sum_{k=1}^{\infty} \delta_k = \sum_{k=1}^{\infty} 2^{-k^2} < \infty,$$

and since $F(a(\theta)) = F(b_k) = b_k$ for $\theta \in I_k$ and $F(a(\theta)) = F(0) = 0$ for $\theta \notin \cup_{k=1}^{\infty} I_k$, we get

$$\|F(a)\|_1 = \sum_{k=1}^{\infty} b_k \delta_k \beta_k = \sum_{k=1}^{\infty} \delta_k = \sum_{k=1}^{\infty} 2^{-k^2} < \infty.$$

We are therefore left with the verification of (4).

Let

$$D_n(\theta) := \sum_{|j|\leqslant n-1} e^{ij\theta} = \frac{\sin\left(n-\frac{1}{2}\right)\theta}{\sin(\theta/2)}$$

be the Dirichlet kernel. Since $D_n'(\theta) = \sum_{|j|\leqslant n-1} ij e^{ij\theta}$, we see that

$$\|D_n'\|_\infty \leqslant \sum_{|j|\leqslant n-1} |j| = (n-1)n < n^2. \tag{5}$$

Parseval's equality gives

$$\|D_n\|_2 = \left(\sum_{|j| \leqslant n-1} 1^2 \right)^{1/2} = \sqrt{2n-1}. \tag{6}$$

For $c, d \in [0, 2\pi)$ and $n \in \mathbf{N}$, put

$$E_{c,d}^{(n)}(\theta) = \frac{1}{2\pi} \int_c^d D_n(\theta - \varphi) \mathrm{d}\varphi.$$

Without the concrete bound 3, the following result is Lemma 8.2 of Chapter II of [20]. We include a full proof for the reader's convenience.

Lemma 2.2. *We have* $\|E_{c,d}^{(n)}\|_\infty \leqslant 3$ *for all* $c, d \in [0, 2\pi)$ *and all* $n \in \mathbf{N}$.

Proof. Clearly, it suffices to show that

$$\frac{1}{2\pi} \left| \int_c^d D_n(x) \mathrm{d}x \right| \leqslant 3 \tag{7}$$

for $c, d \in [-\pi, \pi]$ and $n \in \mathbf{N}$. Let $f(y) = y - \sin y - y \sin y$. We have $f(0) = 0$ and

$$f'(y) = 1 - \cos y - \sin y - y \cos y = 1 - (1 + 2 \sin y \cos y)^{1/2} - y \cos y \leqslant 0$$

for all $y \in [0, \pi/2]$, which implies that $f(y) \leqslant 0$ for all $y \in [0, \pi/2]$. Therefore it follows that $0 < 1/\sin y - 1/y \leqslant 1$ for $y \in (0, \pi/2]$ or equivalently

$$\left| \frac{1}{\sin(x/2)} - \frac{1}{x/2} \right| \leqslant 1 \quad \text{for } x \in [-\pi, \pi] \backslash \{0\}. \tag{8}$$

We write

$$\int_c^d D_n(x) \mathrm{d}x = \int_c^d \frac{\sin\left(n - \frac{1}{2}\right) x}{x/2} \mathrm{d}x + \int_c^d \sin\left(n - \frac{1}{2}\right) x \left[\frac{1}{\sin(x/2)} - \frac{1}{x/2} \right] \mathrm{d}x.$$

By (8), the absolute value of the second integral on the right does not exceed

$$\int_c^d \left| \sin\left(n - \frac{1}{2}\right) \right| \mathrm{d}x \leqslant \int_c^d \mathrm{d}x \leqslant 2\pi. \tag{9}$$

The change of variables $(n - 1/2)x = t$ in the first integral on the right shows that its absolute value is

$$\left| \int_{(n-1/2)c}^{(n-1/2)d} \frac{\sin t}{t/2} \mathrm{d}t \right| \leqslant 2 \left| \int_0^{(n-1/2)|c|} \frac{\sin t}{t} \mathrm{d}t \right| + 2 \left| \int_0^{(n-1/2)|d|} \frac{\sin t}{t} \mathrm{d}t \right|. \tag{10}$$

The integral sine $\mathrm{Si}(v) := \int_0^v \frac{\sin t}{t} \mathrm{d}t$ is positive on $(0, \infty)$, attains its maximum at $v = \pi$, and

$$\mathrm{Si}(\pi) = \int_0^\pi \frac{\sin t}{t} \mathrm{d}t < \int_0^\pi \mathrm{d}t = \pi.$$

Consequently, (10) is at most 4π. This in conjunction with (9) yields (7). □

We now consider the partial sum

$$(P_n a)(\theta) := \sum_{|j| \leqslant n-1} a_j \mathrm{e}^{\mathrm{i}j\theta} = \frac{1}{2\pi} \int_0^{2\pi} a(\varphi) D_n(\theta - \varphi) \mathrm{d}\varphi.$$

For $k \in \mathbf{N}$, put

$$n_k := \sqrt{b_{k+1}} = 2^{2^{k^2+2k}}. \tag{11}$$

Lemma 2.3. *We have*

$$\|P_{n_k} a\|_2 \geqslant \frac{1}{5\pi} \delta_{k+1}\, n_k^{1/2}$$

for all sufficiently large k.

Proof. Obviously,

$$(P_{n_k} a)(\theta) = \sum_{j=1}^{\infty} \frac{b_j}{2\pi} \int_{I_j} D_{n_k}(\theta - \varphi)\mathrm{d}\varphi = \sum_{j=1}^{\infty} b_j E^{(n_k)}_{(1-\delta_j)\beta_j,\beta_j}(\theta). \tag{12}$$

Our aim is to show that the L^2 norm of the term with $j = k + 1$ is greater than a constant times $\delta_{k+1} n_k^{1/2}$ while the sum of the L^2 norms of the remaining terms is at most $o(1)$ times $\delta_{k+1} n_k^{1/2}$ as $k \to \infty$.

From Lemma 2.2 we infer that

$$\begin{aligned}
\delta_{k+1}^{-1} n_k^{-1/2} \sum_{j=1}^{k} \|b_j E^{(n_k)}_{(1-\delta_j)\beta_j,\beta_j}\|_2 &\leqslant \delta_{k+1}^{-1} n_k^{-1/2} \sum_{j=1}^{k} 3b_j \\
&= 2^{(k+1)^2}\, 2^{-2^{k^2+2k-1}}\, 3 \sum_{j=1}^{k} 2^{2^{j^2}} \\
&\leqslant 2^{(k+1)^2}\, 2^{-2^{k^2+2k-1}}\, 3\, k\, 2^{2^{k^2}} \\
&= 3k\, 2^{(k+1)^2}\, 2^{2^{k^2}(1-2^{2k-1})} = \mathrm{o}(1)
\end{aligned} \tag{13}$$

as $k \to \infty$. To tackle the terms with $j \geqslant k + 1$ on the right of (12) we write

$$b_j E^{(n_k)}_{(1-\delta_j)\beta_j,\beta_j}(\theta) = \frac{b_j}{2\pi} D_{n_k}(\theta - \beta_j)\delta_j \beta_j + R_j(\theta) \tag{14}$$

with

$$R_j(\theta) := \frac{b_j}{2\pi} \int_{I_j} \left[D_{n_k}(\theta - \varphi) - D_{n_k}(\theta - \beta_j)\right] \mathrm{d}\varphi.$$

The mean value theorem and (5) give

$$|D_{n_k}(\theta - \varphi) - D_{n_k}(\theta - \beta_j)| = |D'_{n_k}(\xi)|(\beta_j - \varphi) \leqslant n_k^2(\beta_j - \varphi),$$

whence

$$|R_j(\theta)| \leqslant \frac{b_j n_k^2}{2\pi} \int_{(1-\delta_j)\beta_j}^{\beta_j} (\beta_j - \varphi)\mathrm{d}\varphi = \frac{b_j n_k^2}{2\pi} \frac{\beta_j^2 \delta_j^2}{2} = \frac{n_k^2 \beta_j \delta_j^2}{4\pi}. \tag{15}$$

By virtue of (6), the L^2 norm of the function $D_{n_k}(\theta - \beta_j)$ is at most $\sqrt{2n_k}$. Thus, from (14) we obtain that

$$\|b_j E^{(n_k)}_{(1-\delta_j)\beta_j,\beta_j}\|_2 \leqslant \frac{\delta_j}{2\pi} \sqrt{2n_k} + \frac{n_k^2 \beta_j \delta_j^2}{4\pi} = \frac{1}{4\pi}\, \delta_j n_k^{1/2} \left[2\sqrt{2} + n_k^{3/2} \beta_j \delta_j\right]. \tag{16}$$

If $j = k + m$ with $m \geqslant 1$, then

$$\begin{aligned} n_k^{3/2}\beta_j\delta_j &= 2^{3\cdot 2^{k^2+2k-1}}\, 2^{-2^{(k+m)^2}}\, 2^{-(k+m)^2} \\ &= 2^{2^{k^2}(3\cdot 2^{2k-1}-2^{2mk+m^2})}\, 2^{-(k+m)^2} \\ &< 2^{2^{k^2}(3\cdot 2^{2k-1}-2^{2k+1})} = 2^{-2^{k^2+2k-1}} < 1. \end{aligned} \tag{17}$$

Consequently, (16) implies that

$$\|b_j E^{(n_k)}_{(1-\delta_j)\beta_j,\beta_j}\|_2 < \frac{1+2\sqrt{2}}{4\pi}\,\delta_j n_k^{1/2} < \delta_j n_k^{1/2} \tag{18}$$

for $j \geqslant k + 1$. It follows that

$$\begin{aligned} \delta_{k+1}^{-1}\, n_k^{-1/2} \sum_{j=k+2}^{\infty} \|b_j E^{(n_k)}_{(1-\delta_j)\beta_j,\beta_j}\|_2 &\leqslant \delta_{k+1}^{-1}\, n_k^{-1/2} \sum_{j=k+2}^{\infty} \delta_j n_k^{1/2} \\ &= 2^{(k+1)^2} \sum_{j=k+2}^{\infty} 2^{-j^2} = 2^{(k+1)^2} \sum_{\ell=1}^{\infty} 2^{-(k+1+\ell)^2} \\ &= \sum_{\ell=1}^{\infty} 2^{-2(k+1)\ell-\ell^2} < 2^{-2(k+1)} \sum_{\ell=1}^{\infty} 2^{-\ell^2} = \mathrm{o}(1) \end{aligned} \tag{19}$$

as $k \to \infty$. We finally consider the term with $j = k + 1$, which may be written in the form (14). Due to (6), the L^2 norm of the function

$$\frac{b_{k+1}}{2\pi}\, D_{n_k}(\theta - \beta_{k+1})\delta_{k+1}\beta_{k+1} = \frac{\delta_{k+1}}{2\pi}\, D_{n_k}(\theta - \beta_{k+1})$$

is $\delta_{k+1}(2n_k - 1)^{1/2}/(2\pi)$. From (15) we know that $\|R_{k+1}\|_2 \leqslant n_k^2\beta_{k+1}\delta_{k+1}^2/(4\pi)$. Hence (14) gives

$$\begin{aligned} \|b_{k+1} E^{(n_k)}_{(1-\delta_{k+1})\beta_{k+1},\beta_{k+1}}\|_2 &\geqslant \frac{\delta_{k+1}}{2\pi}(2n_k - 1)^{1/2} - \frac{n_k^2\beta_{k+1}\delta_{k+1}^2}{4\pi} \\ &= \frac{\delta_{k+1} n_k^{1/2}}{4\pi}\left[2\left(2 - \frac{1}{n_k}\right)^2 - n_k^{3/2}\beta_{k+1}\delta_{k+1}\right]. \end{aligned}$$

From (17) we see that $n_k^{3/2}\beta_{k+1}\delta_{k+1} < 1$. Consequently,

$$\delta_{k+1}^{-1}\, n_k^{-1/2}\, \|b_{k+1} E^{(n_k)}_{(1-\delta_{k+1})\beta_{k+1},\beta_{k+1}}\|_2 \geqslant \frac{1}{4\pi}\,[2 - 1] = \frac{1}{4\pi} \tag{20}$$

for all sufficiently large k. Inserting (13), (19), (20) in (12) we arrive at the estimate

$$\delta_{k+1}^{-1}\, n_k^{-1/2}\, \|P_{n_k} a\|_2 \geqslant \frac{1}{4\pi} - \mathrm{o}(1) - \mathrm{o}(1) \geqslant \frac{1}{5\pi}$$

for all k large enough. □

Proof of Theorem 2.1. As already said, it remains to prove (4). With n_k given by (11),

$$\frac{1}{n_k}\sum_{j=1}^{n_k} F(s_j(T_{n_k}(a))) \geqslant \frac{1}{n_k} F(s_{n_k}(T_{n_k}(a))) = \frac{1}{n_k} F(\|T_{n_k}(a)\|). \tag{21}$$

For $|j| \leqslant n_k - 1$, the jth Fourier coefficients of a and $P_{n_k}a$ coincide. Consequently, $T_{n_k}(a) = T_{n_k}(P_{n_k}a)$. As the norm of a matrix is at least the ℓ^2 norm of its first column, we obtain that

$$\|T_{n_k}(a)\|^2 = \|T_{n_k}(P_{n_k}a)\|^2 \geqslant \sum_{j=0}^{n_k-1} |(P_{n_k}a)_j|^2 \geqslant \frac{1}{2} \sum_{|j|\leqslant n_k-1} |(P_{n_k}a)_j|^2$$

and hence, by Parseval's equality, $\|T_{n_k}(a)\| \geqslant \|P_{n_k}a\|_2/\sqrt{2}$. Lemma 2.3 therefore implies that (21) is at least

$$\frac{1}{n_k} F\left(\frac{\delta_{k+1}\, n_k^{1/2}}{5\sqrt{2}\,\pi}\right). \tag{22}$$

If k is large enough, then

$$2^{2^{k^2}} + 1 < \frac{1}{5\sqrt{2}\,\pi}\, 2^{-(k+1)^2}\, 2^{2^{k^2+2k-1}} < 2^{2^{(k+1)^2}}$$

or equivalently,

$$b_k + 1 < \frac{\delta_{k+1}\, n_k^{1/2}}{5\sqrt{2}\,\pi} < b_{k+1}.$$

Thus, if k is sufficiently large, then (22) equals $b_{k+1}/n_k = \sqrt{b_{k+1}}$, and since $b_{k+1} \to \infty$ as $k \to \infty$, it follows that the left-hand side of (21) goes to infinity as $k \to \infty$. $\square$

Remark 2.4. If $G \in C[0,\infty)$ is any test function such that $G(s) \geqslant F(s)$ for all $s \in [0,\infty)$, then, obviously, (4) holds with F replaced by G. By changing F only slightly, we can clearly produce a $G \geqslant F$ such that $G(a) \in L^1$ and such that G is C^∞ and strictly monotonously increasing.

3. A modification of a result by Serra Capizzano and Tilli

We equip $\mathbf{C}^n$ with the inner product $(z,w) = \sum_{j=1}^n z_j \overline{w}_j$, denote by $M_n(\mathbf{C})$ the algebra of all complex $n \times n$ matrices, and think of matrices in $M_n(\mathbf{C})$ as linear operators on $\mathbf{C}^n$ in the natural fashion. Given a function $\Phi : [0,\infty) \to [0,\infty)$, we put

$$S_\Phi(A) = \sum_{j=1}^n \Phi(s_j(A)).$$

In [11], Serra Capizzano and Tilli derived a beautiful variational characterization of unitarily invariant norms on $M_n(\mathbf{C})$. The following theorem is a modification of their result; paper [11] contains the implication (i) $\Rightarrow$ (ii) of the theorem for $\Phi(s) = s^p$ $(1 \leqslant p < \infty)$.

Theorem 3.1. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a continuous function and let $n \geqslant 2$. Then the following are equivalent*:

(i) *Φ is monotonously increasing and convex*;
(ii) *for every $A \in M_n(\mathbf{C})$ we have*

$$S_\Phi(A) = \max \sum_{k=1}^n \Phi(|(Au_k, v_k)|),$$

the maximum over all pairs $\{u_1,\dots,u_n\}$ and $\{v_1,\dots,v_n\}$ of orthonormal bases of $\mathbf{C}^n$.

Proof. (i) $\Rightarrow$ (ii). Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be orthonormal bases of $\mathbf{C}^n$ and let $A = V^*DU$ with $D = \operatorname{diag}(s_1, \ldots, s_n)$ be the singular value decomposition of A. We put $u'_k = Uu_k$ and $v'_k = Vv_k$. Clearly,

$$|(Au_k, v_k)| = |(Du'_k, v'_k)| = \left|\sum_{j=1}^n s_j (u'_k)_j \overline{(v'_k)_j}\right| \leqslant \sum_{j=1}^n s_j \left|(u'_k)_j\right| \left|(v'_k)_j\right|$$

$$\leqslant \frac{1}{2}\sum_{j=1}^n s_j |(u'_k)_j|^2 + \frac{1}{2}\sum_{j=1}^n s_j |(v'_k)_j|^2 = \frac{1}{2}(Du'_k, u'_k) + \frac{1}{2}(Dv'_k, v'_k).$$

Since Φ is monotonously increasing and convex, we therefore obtain that

$$\Phi(|(Au_k, v_k)|) \leqslant \frac{1}{2}\Phi((Du'_k, u'_k)) + \frac{1}{2}\Phi((Dv'_k, v'_k)). \tag{23}$$

But

$$\sum_{k=1}^n \Phi((Du'_k, u'_k)) = \sum_{k=1}^n \Phi\left(\sum_{j=1}^n s_j |(u'_k)_j|^2\right) \tag{24}$$

and taking into account that Φ is convex and

$$\sum_{j=1}^n |(u'_k)_j|^2 = \sum_{k=1}^n |(u'_k)_j|^2 = 1,$$

we see that (24) is at most

$$\sum_{k=1}^n \sum_{j=1}^n \Phi(s_j)|(u'_k)_j|^2 = \sum_{j=1}^n \Phi(s_j) \sum_{k=1}^n |(u'_k)_j|^2 = \sum_{j=1}^n \Phi(s_j) = S_\Phi(A).$$

Analogously we get that

$$\sum_{k=1}^n \Phi((Dv'_k, v'_k)) \leqslant S_\Phi(A).$$

Thus, summing up (23) we arrive at the inequality

$$\sum_{k=1}^n \Phi(|(Au_k, v_k)|) \leqslant \frac{1}{2}S_\Phi(A) + \frac{1}{2}S_\Phi(A) = S_\Phi(A).$$

It remains to show that there exist two orthonormal bases $\{\tilde{u}_1, \ldots \tilde{u}_n\}$ and $\{\tilde{v}_1, \ldots \tilde{v}_n\}$ such that $\sum \Phi(|(A\tilde{u}_k, \tilde{v}_k)|)$ equals $S_\Phi(A)$. Let $\tilde{u}_k$ and $\tilde{v}_k$ be the kth column of U^* and V^*, respectively. Since $AU^* = V^*D$, we get $A\tilde{u}_k = s_k\tilde{v}_k$ and hence $(A\tilde{u}_k, \tilde{v}_k) = s_k$. It follows that

$$\sum_{k=1}^n \Phi(|(A\tilde{u}_k, \tilde{v}_k)|) = \sum_{k=1}^n \Phi(s_k) = S_\Phi(A),$$

as desired.

(ii) $\Rightarrow$ (i). We denote by $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbf{C}^n$. By assumption,

$$S_\Phi(A) \geqslant \sum_{k=1}^n \Phi(|(Ae_k, e_k)|) = \sum_{k=1}^n \Phi(|A_{kk}|) \tag{25}$$

for every $A \in M_n(\mathbf{C})$. Let $0 \leqslant \alpha \leqslant \beta < \infty$ and let A be the $n \times n$ matrix whose upper-left 2×2 block is

$$B = \begin{pmatrix} \sin\gamma & \cos\gamma \\ -\cos\gamma & \sin\gamma \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$$

and the remaining entries of which are zero. The singular values of B are α and β, while the diagonal entries of B are

$$B_{11} = B_{22} = \frac{\alpha+\beta}{2}\sin 2\gamma.$$

Thus, (25) gives

$$\Phi(\alpha) + \Phi(\beta) + (n-2)\Phi(0) \geqslant 2\,\Phi\left(\frac{\alpha+\beta}{2}|\sin 2\gamma|\right) + (n-2)\Phi(0).$$

Taking γ so that $\sin 2\gamma = 2\alpha/(\alpha+\beta)$ we get $\Phi(\alpha) + \Phi(\beta) \geqslant 2\Phi(\alpha)$, that is, $\Phi(\alpha) \leqslant \Phi(\beta)$, and taking $\gamma = \pi/4$ we obtain the inequality $\Phi(\alpha) + \Phi(\beta) \geqslant 2\,\Phi((\alpha+\beta)/2)$. This proves that Φ is monotonously increasing and convex. □

We remark that inequality (25) for monotonously increasing and convex functions Φ can already be found in [7,5, p. 72] (see also [12,14]).

Given $x = (x_1, \dots, x_n) \in \mathbf{C}^n$, we let $x(\theta)$ be the trigonometric polynomial

$$x(\theta) = x_1 + x_2 e^{i\theta} + \cdots + x_n e^{i(n-1)\theta}.$$

It is well known and easily seen that

$$(T_n(a)z, w) = \frac{1}{2\pi}\int_0^{2\pi} a(\theta)z(\theta)\overline{w(\theta)}\,d\theta. \tag{26}$$

In what follows we frequently use the abbreviation

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta =: \int f.$$

For $\Phi(s) = s^p$ $(1 \leqslant p < \infty)$, the following corollary is already in the work of Avram [1], Fasino and Tilli [4], and Serra Capizzano and Tilli [11].

Corollary 3.2. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a monotonously increasing and convex function. If $a, b \in L^1$ and $|a| \leqslant b$ almost everywhere, then*

$$S_\Phi(T_n(a)) \leqslant S_\Phi(T_n(b))$$

for all $n \geqslant 1$.

Proof. By Theorem 3.1, there exist two orthonormal bases $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ such that

$$S_\Phi(T_n(a)) = \sum_{k=1}^{n} \Phi(|(T_n(a)u_k, v_k)|). \tag{27}$$

From (26) we infer that

$$\begin{aligned} |(T_n(a)u_k, v_k)| &= \left|\int a u_k \overline{v_k}\right| \leqslant \int |a|\,|u_k|\,|v_k| \leqslant \int b\,|u_k|\,|v_k| \\ &\leqslant \frac{1}{2}\int b\,|u_k|^2 + \frac{1}{2}\int b\,|v_k|^2 = \frac{1}{2}(T_n(b)u_k, u_k) + \frac{1}{2}(T_n(b)v_k, v_k). \end{aligned}$$

Thus, using that Φ is monotonously increasing and convex we obtain that (27) does not exceed

$$\frac{1}{2}\sum_{k=1}^{n}\Phi((T_n(b)u_k,u_k))+\frac{1}{2}\sum_{k=1}^{n}\Phi((T_n(b)v_k,v_k)),$$

and again by Theorem 3.1, this is at most

$$\frac{1}{2}S_\Phi(T_n(b))+\frac{1}{2}S_\Phi(T_n(b))=S_\Phi(T_n(b)). \qquad \square$$

4. Bounded symbols

In this section we prove the Avram–Parter theorem in the version of Avram [1], that is, we show that

$$\lim_{n\to\infty}\frac{1}{n}S_F(T_n(a))=\int F(|a|) \tag{28}$$

for $a\in L^\infty$ and $F\in C_0[0,\infty)$, where $C_0[0,\infty)$ stands for functions in $C[0,\infty)$ which are eventually identically zero.

First of all we remark that in order to prove (28) for some $a\in L^1$ and some test function F, it suffices to prove (28) for the same a and some sequence $F_1,F_2,\ldots$ of test functions which converge uniformly to F on $[0,\infty)$. This follows from an easy $\varepsilon/3$-argument.

To start somewhere, we take the following observation for granted: if $a_1,\ldots,a_m$ are functions in L^∞, then

$$T_n(a_1)\cdots T_n(a_m)=T_n(a_1\cdots a_m)+M_n\quad\text{with }\frac{\|M_n\|_1}{n}\to 0\text{ as }n\to\infty.$$

Here $\|\cdot\|_1$ is the trace norm. In particular, if $a\in L^\infty$ and p is a natural number, then

$$T_n(\bar a)T_n(a)=T_n(|a|^2)+K_n\quad\text{and}\quad T_n^p(|a|^2)=T_n(|a|^{2p})+L_n \tag{29}$$

where $\|K_n\|_1/n\to 0$ and $\|L_n\|_1/n\to 0$ as $n\to\infty$. As to our knowledge, the first to mention this result explicitly was SeLegue [9]. A simple proof can be found in [2, Lemma 5.16], for example.

Take $a\in L^\infty$. We denote the eigenvalues of a positive semi-definite $n\times n$ matrix A by $\lambda_1(A)\leqslant\cdots\leqslant\lambda_n(A)$. Thus,

$$\frac{1}{n}\sum s_j^{2p}(T_n(a))=\frac{1}{n}\sum\lambda_j^p(T_n(\bar a)T_n(a)),$$

and since $0\leqslant\lambda_j(T_n(\bar a)T_n(a))\leqslant\|a\|_\infty^2$ and $0\leqslant\lambda_j(T_n(|a|^2))\leqslant\|a\|_\infty^2$, we obtain from (29) and the inequality $\sum|\lambda_j(A)-\lambda_j(B)|\leqslant\|A-B\|_1$ that

$$\begin{aligned}&\sum\left|\lambda_j^p(T_n(\bar a)T_n(a))-\lambda_j^p(T_n(|a|^2)\right|\\&\quad\leqslant p\|a\|_\infty^{2(p-1)}\sum\left|\lambda_j(T_n(\bar a)T_n(a))-\lambda_j(T_n(|a|^2))\right|\\&\quad\leqslant p\|a\|_\infty^{2(p-1)}\|T_n(\bar a)T_n(a)-T_n(|a|^2)\|_1=p\|a\|_\infty^{2(p-1)}\|K_n\|_1.\end{aligned}$$

Consequently,

$$\frac{1}{n}\sum s_j^{2p}(T_n(a))=\frac{1}{n}\sum\lambda_j^p(T_n(|a|^2))+\mathrm{o}(1). \tag{30}$$

The matrix $T_n(|a|^2)$ is positive semi-definite. Hence, denoting by $\operatorname{tr}A$ the trace of A, we get from (29) that

$$\begin{aligned}\frac{1}{n}\sum\lambda_j^p(T_n(|a|^2)) &= \frac{1}{n}\mathrm{tr}\,T_n^p(|a|^2) = \frac{1}{n}\mathrm{tr}\,(T_n(|a|^{2p})+L_n)\\ &= \frac{1}{n}\mathrm{tr}\,T_n(|a|^{2p})+\mathrm{o}(1) = \int|a|^{2p}+\mathrm{o}(1).\end{aligned} \tag{31}$$

Combining (30) and (31) we arrive at the conclusion that (28) is true for $F(s)=s^{2p}$. It follows that (28) is valid whenever $F(s)=P(s^2)$ with a polynomial P and thus whenever $F(s)=G(s^2)$ with $G\in C_0[0,\infty)$. As every $F\in C_0[0,\infty)$ may be written in the form $F(s)=G(s^2)$ with $G\in C_0[0,\infty)$, we get (28) for all $F\in C_0[0,\infty)$.

5. Uniformly continuous test functions

Zamarashkin and Tyrtyshnikov [18,19] proved that

$$\lim_{n\to\infty}\frac{1}{n}S_F(T_n(a)) = \int F(|a|) \tag{32}$$

if $a\in L^1$ and $F\in C_0[0,\infty)$. An extremely lucid and short proof was given by Tilli [15]. This proof even yields (32) for all uniformly continuous (and not necessarily bounded) test functions F. It is as follows. Let first F be Lipschitz continuous, $|F(s)-F(t)|\leqslant K|s-t|$. For $M>0$, define $a_M\in L^\infty$ by $a_M(\theta)=a(\theta)$ if $|a(\theta)|\leqslant M$ and $a_M(\theta)=0$ if $|a(\theta)|>M$. Then $\int|a-a_M|\to 0$ as $M\to\infty$. Fix $\varepsilon>0$. Taking into account the inequality $\sum|s_j(A)-s_j(B)|\leqslant\|A-B\|_1$ and using Corollary 3.2 for $\Phi(s)=s$ we obtain that

$$\begin{aligned}&\frac{1}{n}\left|\sum F(s_j(T_n(a)))-\sum F(s_j(T_n(a_M)))\right|\\ &\quad\leqslant\frac{K}{n}\sum|s_j(T_n(a))-s_j(T_n(a_M))|\\ &\quad\leqslant\frac{K}{n}\|T_n(a-a_M)\|_1=\frac{K}{n}\sum s_j(T_n(a-a_M))\leqslant\frac{K}{n}\sum s_j(T_n(|a-a_M|))\\ &\quad=\frac{K}{n}\sum\lambda_j(T_n(|a-a_M|))=\frac{K}{n}\mathrm{tr}\,T_n(|a-a_M|)=K\int|a-a_M|<\frac{\varepsilon}{3}\end{aligned}$$

for all $n\geqslant 1$ if $M\geqslant M_0$. We also have

$$\int|F(|a|)-F(|a_M|)|\leqslant K\int||a|-|a_M||\leqslant K\int|a-a_M|<\frac{\varepsilon}{3}$$

for $M\geqslant M_0$. For each $M\geqslant M_0$, formula (28) gives

$$\left|\frac{1}{n}\sum F(s_j(T_n(a_M)))-\int F(|a_M|)\right|<\frac{\varepsilon}{3}$$

if $n\geqslant n_0(M)$. Thus,

$$\left|\frac{1}{n}\sum F(s_j(T_n(a)))-\int F(|a|)\right|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

for all sufficiently large n, which completes the proof for Lipschitz continuous functions. Every uniformly continuous function on $[0,\infty)$ is the uniform limit of Lipschitz continuous functions. (Indeed, fix $\varepsilon>0$. There is a $\delta>0$ such that $|F(s)-F(t)|\leqslant\varepsilon$ whenever $|s-t|<\delta$. Let F_ε be the continuous and piecewise linear function that satisfies $F_\varepsilon(k\delta)=F(k\delta)$ for $k=0,1,2,\ldots$ and is linear on $[k\delta,(k+1)\delta]$ for all k. It is easily seen that F_ε is Lipschitz continuous,

$|F_\varepsilon(s) - F_\varepsilon(t)| \leqslant (\varepsilon/\delta)|s-t|$, and that $\|F - F_\varepsilon\|_\infty \leqslant 2\varepsilon$ on $[0,\infty)$.) We therefore arrive at the conclusion that (32) is true for all uniformly continuous functions on $[0,\infty)$.

6. Convex test functions

For $\Phi(s) = s^p$ $(1 \leqslant p < \infty)$, the following Proposition 6.1 and Corollary 6.4 are again already in [10] and [11].

Proposition 6.1. *If $a \in L^1$ and $\Phi : [0,\infty) \to [0,\infty)$ is monotonously increasing and convex, then*

$$\frac{1}{n} S_\Phi(T_n(a)) \leqslant \int \Phi(|a|)$$

for all $n \geqslant 1$.

Proof. By Corollary 3.2, $S_\Phi(T_n(a) \leqslant S_\Phi(T_n(|a|))$. The matrix $T_n(|a|)$ is positive semi-definite. Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis of eigenvectors and $T_n(|a|)w_k = s_k w_k$. Then

$$\Phi(s_k) = \Phi((T_n(|a|)w_k, w_k)) = \Phi\left(\int |a|\,|w_k|^2\right).$$

Taking into account that $\int |w_k|^2 = 1$ we can use Jensen's inequality to get

$$\Phi\left(\int |a|\,|w_k|^2\right) \leqslant \int \Phi(|a|)|w_k|^2 = (T_n(\Phi(|a|))w_k, w_k).$$

Consequently,

$$S_\Phi(T_n(|a|)) \leqslant \sum_{k=1}^{n} ((T_n(\Phi(|a|))w_k, w_k)) = \operatorname{tr} T_n(\Phi(|a|)) = n \int \Phi(|a|). \qquad \square$$

If $a(\theta) = \mathrm{e}^{\mathrm{i}\theta}$, then $s_1(T_n(a)) = 0$ and $s_2(T_n(a)) = \cdots = s_n(T_n(a)) = 1$. The inequality of Proposition 6.1 so amounts to the inequality $\Phi(0) + (n-1)\Phi(1) \leqslant n\Phi(1)$, that is, $\Phi(0) \leqslant \Phi(1)$. This reveals that the convex functions for which Proposition 6.1 is true must necessarily be monotonously increasing on $[1,\infty)$. The proof of Proposition 6.1 also shows that if $a \geqslant 0$ almost everywhere and $\Phi : [0,\infty) \to [0,\infty)$ is a concave function, then $(1/n)S_\Phi(T_n(a)) \geqslant \int \Phi(|a|)$ for all $n \geqslant 1$.

The following proposition is just (3) and was established in [3].

Proposition 6.2. *Let $a \in L^1$ and let $F : [0,\infty) \to [0,\infty)$ be a continuous function. If*

$$C := \liminf_{n\to\infty} \frac{1}{n} S_F(T_n(a)) < \infty, \tag{33}$$

then $F(|a|) \in L^1$ and $\int F(|a|) \leqslant C$.

Proof. Fix $\varepsilon > 0$ and choose $n_1 < n_2 < \cdots$ so that $(1/n_k)S_F(T_{n_k}(a)) < C + \varepsilon$. For a natural number M, define the function $F_M : [0,\infty) \to [0,\infty)$ by $F_M(s) = F(s)$ for $s \in [0, M]$, $F_M(s) = (M+1-s)F(s)$ for $s \in [M, M+1]$, and $F_M(s) = 0$ for $s \in [M+1,\infty)$. Since $F_M \in C_0[0,\infty)$, we deduce from (32) that

$$\int F_M(|a|) = \lim_{k\to\infty} \frac{S_{F_M}(T_{n_k}(a))}{n_k} \leqslant \limsup_{k\to\infty} \frac{S_F(T_{n_k}(a))}{n_k} \leqslant C + \varepsilon,$$

which implies that $\int F(|a|) \leqslant C + \varepsilon$. $\square$

Proposition 6.3. *Let $a \in L^1$. Then the following are equivalent*:

(i) $(1/n)S_F(T_n(a)) \to \int F(|a|)$ *for every* $F \in C_0[0,\infty)$;
(ii) $(1/n)S_\Phi(T_n(a)) \to \int \Phi(|a|)$ *for every monotonously increasing and convex function* $\Phi : [0,\infty) \to [0,\infty)$.

In other words, $C_0[0,\infty)$ *is a subset of* APT *if and only if all nonnegative*, *monotonously increasing, and convex functions are in* APT.

Proof. (i) $\Rightarrow$ (ii). Assumption (i) was used in the proof of Proposition 6.2. But this proposition and Proposition 6.1 imply (ii).

(ii) $\Rightarrow$ (i). It is sufficient to prove that $(1/n)S_F(T_n(a)) \to \int F(|a|)$ for every twice continuously differentiable $F \in C_0[0,\infty)$. We then have $F''(s) = \phi(s) - \psi(s)$ with nonnegative continuous functions ϕ, ψ which vanish identically for $s > s_0$. Put

$$\Phi(s) = F(0) + \gamma s + \int_0^s \int_0^t \phi(\sigma)\mathrm{d}\sigma\,\mathrm{d}t, \quad \Psi(s) = \delta s + \int_0^s \int_0^t \psi(\sigma)\mathrm{d}\sigma\,\mathrm{d}t,$$

where $\gamma = F'(0)$, $\delta = 0$ if $F'(0) \geqslant 0$ and $\gamma = 0$, $\delta = -F'(0)$ if $F'(0) \leqslant 0$. Clearly, $F(s) = \Phi(s) - \Psi(s)$. Considering the first and second derivatives, we see that Φ and Ψ are monotonously increasing and convex functions. Since $\Phi''(s) = \Psi''(s) = 0$ for $s > s_0$, there are constants α and β such that $\Phi(s) = \Psi(s) = \alpha + \beta s$ for $s > s_0$, which implies that $\Phi(|a|)$ and $\Psi(|a|)$ are in L^1 together with a. From (ii) we therefore deduce that

$$\frac{S_F(T_n(a))}{n} = \frac{S_\Phi(T_n(a)) - S_\Psi(T_n(a))}{n} \to \int \Phi(|a|) - \int \Psi(|a|) = \int F(|a|). \quad \square$$

Corollary 6.4. *If* $a \in L^1$ *and* $\Phi : [0,\infty) \to [0,\infty)$ *is monotonously increasing and convex*, *then*

$$\lim_{n\to\infty} \frac{1}{n} S_\Phi(T_n(a)) = \int \Phi(|a|).$$

Thus, *all monotonously increasing and convex functions* $F : [0,\infty) \to [0,\infty)$ *are in* APT.

Proof. As (i) of Proposition 6.3 is guaranteed by (32), the assertion follows from the implication (i) $\Rightarrow$ (ii) of Proposition 6.3. $\square$

7. Essentially convex test functions

Here are our main results concerning the Avram–Parter theorem. For $\Phi(s) = s^p$ and $\Psi(s) = s^p$, these results were previously established by Serra Capizzano [10]. The proof of the following lemma makes also use of ideas of [10].

Lemma 7.1. *Let* $a \in L^1$, *let* $\Phi : [0,\infty) \to [0,\infty)$ *be a monotonously increasing and convex function*, *and suppose* $\Phi(|a|) \in L^1$. *Then for every* $\varepsilon > 0$ *there exist* $M \in (0,\infty)$ *and* $n_0 \in \mathbf{N}$ *such that*

$$\frac{1}{n} \sum_{\{j : s_j(T_n(a)) > M\}} \Phi(s_j(T_n(a))) < \varepsilon \tag{34}$$

for all $n \geqslant n_0$.

Proof. Since $\Phi(|a|) \in L^1$ and Φ is monotonously increasing, there is an M such that

$$\frac{1}{2\pi}\int_{\{\theta:|a(\theta)|>M/2\}} \Phi(|a(\theta)|)\mathrm{d}\theta < \frac{\varepsilon}{2}. \tag{35}$$

We define a continuous function $H : [0, \infty) \to [0, \infty)$ by $H(s) = \Phi(s)$ for $0 \leqslant s \leqslant M/2$, $0 \leqslant H(s) \leqslant \Phi(s)$ for $M/2 \leqslant s \leqslant M$, and $H(s) = 0$ for $s \geqslant M$. Then $\int \Phi(|a|) - \int H(|a|)$ does not exceed (35) and hence

$$\int H(|a|) > \int \Phi(|a|) - \frac{\varepsilon}{2}. \tag{36}$$

Since H has finite support, (32) yields an $n_0 \in \mathbf{N}$ such that

$$-\frac{\varepsilon}{2} < \frac{1}{n}S_H(T_n(a)) - \int H(|a|) < \frac{\varepsilon}{2}$$

for all $n \geqslant n_0$. Thus, for $n \geqslant n_0$ we have

$$\begin{aligned}&\frac{1}{n}\sum_{\{j:s_j(T_n(a))\leqslant M\}} \Phi(s_j(T_n(a)))\\ &\quad\geqslant \frac{1}{n}S_H(T_n(a)) > \int H(|a|) - \frac{\varepsilon}{2} > \int \Phi(|a|) - \varepsilon.\end{aligned} \tag{37}$$

On the other hand, Proposition 6.1 tells us that

$$\frac{1}{n}\sum_{j=1}^{n} \Phi(s_j(T_n(a))) \leqslant \int \Phi(|a|) \tag{38}$$

for all $n \geqslant 1$. Clearly, (37) and (38) imply (34). □

Theorem 7.2. *Let $a \in L^1$, let $\Phi : [0, \infty) \to [0, \infty)$ be a monotonously increasing and convex function, and suppose $\Phi(|a|) \in L^1$. Let $F : [0, \infty) \to [0, \infty)$ be a continuous function such that $F(s) \leqslant \Phi(s)$ for all $s > s_0$. Then*

$$\lim_{n\to\infty} \frac{1}{n}S_F(T_n(a)) = \int F(|a|).$$

Proof. Fix $\varepsilon > 0$. We have to show that

$$\left|\frac{1}{n}S_F(T_n(a)) - \int F(|a|)\right| < \varepsilon \tag{39}$$

for all sufficiently large n. Taking into account that $\Phi(|a|) \in L^1$ and using Lemma 7.1 we get $M \in (0, \infty)$ and $n_1 \in \mathbf{N}$ such that

$$\frac{1}{2\pi}\int_{\{\theta:\Phi(|a(\theta)|)>M\}} \Phi(|a(\theta)|)\mathrm{d}\theta < \frac{\varepsilon}{3} \tag{40}$$

and

$$\frac{1}{n}\sum_{\{j:s_j(T_n(a))>M\}} \Phi(s_j(T_n(a))) < \frac{\varepsilon}{3} \tag{41}$$

for $n \geqslant n_1$. Let $G : [0, \infty) \to [0, \infty)$ be a continuous function satisfying $G(s) = F(s)$ for $0 \leqslant s \leqslant M$, $0 \leqslant G(s) \leqslant F(s)$ for $m \leqslant s \leqslant 2M$, and $G(s) = 0$ for $s \geqslant 2M$. By (40),

$$\left|\int F(|a|) - \int G(|a|)\right| \leqslant \frac{1}{2\pi} \int_{\{\theta:\Phi(|a(\theta)|)>M\}} \Phi(|a(\theta)|)\mathrm{d}\theta < \frac{\varepsilon}{3},$$

from (32) we infer that

$$\left|\frac{1}{n} S_G(T_n(a)) - \int G(|a|)\right| < \frac{\varepsilon}{3}$$

for all $n \geqslant n_2$, and due to (41),

$$\left|\frac{1}{n} S_F(T_n(a)) - \frac{1}{n} S_G(T_n(a))\right| \leqslant \frac{1}{n} \sum_{\{j:s_j(T_n(a))>M\}} \Phi(s_j(T_n(a))) < \frac{\varepsilon}{3}$$

for all $n \geqslant n_1$. Adding the last three inequalities we obtain inequality (39) for $n \geqslant \max(n_1, n_2)$. □

Recall that we write $F(s) \simeq \Phi(s)$ as $s \to \infty$ if there exist positive constants C_1 and C_2 such that $C_1\Phi(s) \leqslant F(s) \leqslant C_2\Phi(s)$ for all sufficiently large s.

Corollary 7.3. *Let a be a function in* $\in L^1$, *let* $\Psi : [0, \infty) \to [0, \infty)$ *be a convex function, and let* $F : [0, \infty) \to [0, \infty)$ *be a continuous function such that* $F(s) \simeq \Psi(s)$ *as* $s \to \infty$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} S_F(T_n(a)) = \int F(|a|). \tag{42}$$

In other terms, APT *contains all nonnegative and essentially convex functions*.

Proof. From Proposition 6.2 it follows that both sides of (42) are infinite if $F(|a|) \notin L^1$. So suppose $F(|a|) \in L^1$. We have $C_1\Psi(s) \leqslant F(s) \leqslant C_2\Psi(s)$ for all $s > s_0$. Let first Ψ be a bounded function, $\Psi(s) \leqslant M$ for all $s \in [0, \infty)$. The constant function Φ given by $\Phi(s) = C_2M$ is monotonously increasing and convex, we have $F(s) \leqslant \Phi(s)$ for $s > s_0$, and $\Phi(|a|) \in L^1$. Theorem 7.2 therefore implies (42). Now suppose Ψ is unbounded. Then Ψ is monotonously increasing on some half-line $[s_0, \infty)$. The function $\Phi(s) := C_2\Psi(s)$ is monotonously increasing and convex on $[s_0, \infty)$ together with $\Psi(s)$, the inequality $F(s) \leqslant \Phi(s)$ is satisfied for all $s > s_0$, and since $C_1\Psi(s) \leqslant F(s)$, we conclude that $\Phi(|a|) \in L^1$. Thus, Theorem 7.2 yields (42). □

8. The Szegö theorem

We finally turn to Szegö's theorem. Using the abbreviation

$$\Lambda_F(T_n(a)) = \sum_{j=1}^{n} F(\lambda_j(T_n(a))),$$

we can write this theorem as

$$\lim_{n\to\infty} \frac{1}{n} \Lambda_F(T_n(a)) = \int F(a). \tag{43}$$

For real-valued $a \in L^\infty$ and compactly supported F in $C(\mathbf{R})$, (43) can be easily derived from (28). Indeed, we can write $a = m + b$ with $m \in \mathbf{R}$ and an L^∞ function $b \geqslant 0$, we then have

$$\lambda_j(T_n(a)) = m + \lambda_j(T_n(b)) = m + s_j(T_n(b)),$$

and (28) with $F(s)$ replaced by $G(s) = F(m+s)$ therefore yields

$$\lim_{n\to\infty}\frac{1}{n}\Lambda_F(T_n(a)) = \int G(|b|) = \int F(m+|b|) = \int F(m+b) = \int F(a).$$

Tilli [15] gave a very simple proof of (43) for real-valued $a \in L^1$ and uniformly continuous $F \in C(\mathbf{R})$. This proof is nearly identical with the proof given in Section 5, the only difference being that now the inequality $\sum |\lambda_j(A) - \lambda_j(B)| \leqslant \|A - B\|_1$ has to be used, which holds for Hermitian matrices A and B. The purpose of this section is to establish the Szegö type versions of the results of Section 7.

For a real-valued function $a \in L^1$, we define $a_+ = \max(a, 0)$ and $a_- = \max(-a, 0)$. Then $a_\pm \in L^1$, $a_\pm \geqslant 0$, and $a = a_+ - a_-$. It is well known that $\lambda_j(T_n(b)) \leqslant \lambda_j(T_n(c))$ whenever $b, c \in L^1$ are real-valued and $b \leqslant c$. In particular, $\lambda_j(T_n(a_\pm)) \geqslant 0$ for all j.

Lemma 8.1. *Let $a \in L^1$ be real-valued, let $\Phi : [0,\infty) \to [0,\infty)$ be a monotonously increasing and convex function, and suppose $\Phi(a_+) \in L^1$. Then for every $\varepsilon > 0$ there exist $M \in (0,\infty)$ and $n_0 \in \mathbf{N}$ such that*

$$\frac{1}{n}\sum_{\{j:\lambda_j(T_n(a))>M\}} \Phi(\lambda_j(T_n(a))) < \varepsilon \tag{44}$$

for all $n \geqslant n_0$.

Proof. There is an M such that

$$\frac{1}{2\pi}\int_{\{\theta:a_+(\theta)>M/2\}} \Phi(a_+(\theta))\mathrm{d}\theta < \frac{\varepsilon}{2}.$$

Continue Φ to a function $\Phi : \mathbf{R} \to [0,\infty)$ by putting $\Phi(\lambda) = \Phi(0)$ for $\lambda \leqslant 0$ and let $H : \mathbf{R} \to [0,\infty)$ be any continuous function such that $H(\lambda) = \Phi(\lambda)$ for $\lambda \leqslant M/2$, $0 \leqslant H(\lambda) \leqslant \Phi(\lambda)$ for $M/2 \leqslant \lambda \leqslant M$, and $H(\lambda) = 0$ for $\lambda \geqslant M$. Then

$$\int H(a) = \int H(a_+) > \int \Phi(a_+) - \frac{\varepsilon}{2}. \tag{45}$$

The function H is uniformly continuous and hence we can use (43) with F replaced by H to see that

$$\left|\frac{1}{n}\Lambda_H(T_n(a)) - \int H(a)\right| < \frac{\varepsilon}{2} \tag{46}$$

for $n \geqslant n_0$. Thus

$$\begin{aligned}\frac{1}{n}\sum_{\{j:\lambda_j(T_n(a))\leqslant M\}} \Phi(\lambda_j(T_n(a))) &\geqslant \frac{1}{n}\sum_{j=1}^{n} H(\lambda_j(T_n(a)))\\ &= \frac{1}{n}\Lambda_H(T_n(a)) > \int H(a) - \frac{\varepsilon}{2} > \int \Phi(a_+) - \varepsilon \end{aligned}\tag{47}$$

for $n \geqslant n_0$. Since $\lambda_j(T_n(a)) \leqslant \lambda_j(T_n(a_+))$ for all j and Φ is monotonously increasing, we deduce from Proposition 6.1 that

$$\frac{1}{n}\sum_{j=1}^{n}\Phi(\lambda_j(T_n(a))) \leqslant \frac{1}{n}\sum_{j=1}^{n}\Phi(\lambda_j(T_n(a_+)))$$
$$= \frac{1}{n}\sum_{j=1}^{n}\Phi(s_j(T_n(a_+))) = \frac{1}{n}S_\Phi(T_n(a_+)) \leqslant \int \Phi(a_+) \tag{48}$$

for all $n \geqslant 1$. Combining (47) and (48) we arrive at (44). □

Theorem 8.2. *Let $a \in L^1$ be real-valued, let $\Phi_\pm : [0,\infty) \to [0,\infty)$ be monotonously increasing and convex functions such that $\Phi_-(0) = \Phi_+(0)$, and suppose $\Phi_+(a_+)$ and $\Phi_-(a_-)$ are in L^1. Let $F : \mathbf{R} \to [0,\infty)$ be a continuous function such that $F(\lambda) \leqslant \Phi_+(\lambda)$ and $F(-\lambda) \leqslant \Phi_-(\lambda)$ whenever $\lambda > \lambda_0$. Then*

$$\lim_{n\to\infty}\frac{1}{n}\Lambda_F(T_n(a)) = \int F(a). \tag{49}$$

Proof. Assume first that $F(0) = 0$. Fix $\varepsilon > 0$. Since $\Phi_+(a_+) \in L^1$, Lemma 8.1 delivers $M > \lambda_0$ and $n_1 \in \mathbf{N}$ such that

$$\frac{1}{2\pi}\int_{\{\theta:a_+(\theta)>M\}}\Phi_+(a_+(\theta))\mathrm{d}\theta < \frac{\varepsilon}{3}$$

and

$$\frac{1}{n}\sum_{\{j:\lambda_j(T_n(a))>M\}}\Phi_+(\lambda_j(T_n(a))) < \frac{\varepsilon}{3}$$

for $n \geqslant n_1$. Put $F(\lambda) = 0$ for $\lambda \leqslant 0$ and let $G : \mathbf{R} \to [0,\infty)$ be any continuous function satisfying $G(\lambda) = F(\lambda)$ for $\lambda \leqslant M$, $0 \leqslant G(\lambda) \leqslant F(\lambda)$ for $M \leqslant \lambda \leqslant 2M$, and $G(\lambda) = 0$ for $\lambda \geqslant 2M$. We have

$$\left|\int F(a_+) - \int G(a)\right| = \left|\int F(a_+) - \int G(a_+)\right|$$
$$\leqslant \frac{1}{2\pi}\int_{\{\theta:a_+(\theta)>M\}}\Phi_+(a_+(\theta))d\theta < \frac{\varepsilon}{3}$$

and

$$\frac{1}{n}\left|\sum_{\{j:\lambda_j(T_n(a))\geqslant 0\}}F(\lambda_j(T_n(a))) - \sum_{\{j:\lambda_j(T_n(a))\geqslant 0\}}G(\lambda_j(T_n(a)))\right|$$
$$\leqslant \frac{1}{n}\sum_{\{j:\lambda_j(T_n(a))>M\}}\Phi_+(\lambda_j(T_n(a))) < \frac{\varepsilon}{3}.$$

Using (43) with the compactly supported and continuous function G, we get

$$\left|\frac{1}{n}\sum_{\{j:\lambda_j(T_n(a))\geqslant 0\}}G(\lambda_j(T_n(a))) - \int G(a)\right| < \frac{\varepsilon}{3}$$

for $n \geqslant n_2$. The last three inequalities give

$$\left| \frac{1}{n} \sum_{\{j:\lambda_j(T_n(a))\geqslant 0\}} F(\lambda_j(T_n(a))) - \int F(a_+) \right| < \varepsilon$$

for $n \geqslant \max(n_1, n_2)$. Analogously one can show that

$$\left| \frac{1}{n} \sum_{\{j:\lambda_j(T_n(a))<0\}} F(\lambda_j(T_n(a))) - \int F(-a_-) \right| < \frac{\varepsilon}{3}$$

for all sufficiently large n. (Notice that $F(0) = 0$, so that it does not matter whether we take $\lambda_j(T_n(a))) < 0$ or $\lambda_j(T_n(a)) \leqslant 0$.) Thus,

$$\lim_{n\to\infty} \frac{1}{n} \Lambda_F(T_n(a)) = \int F(a_+) + \int F(-a_-) = \int F(a).$$

In case $F(0) > 0$, we choose a compactly supported and continuous function $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(0) = -F(0)$, $\varphi(\lambda) \geqslant -F(\lambda)$ for $|\lambda| \leqslant \lambda_0$, and $\varphi(\lambda) = 0$ for $|\lambda| \geqslant \lambda_0$. From what was already proved we know that

$$\frac{1}{n}\Lambda_F(T_n(a)) + \frac{1}{n}\Lambda_\varphi(T_n(a)) = \frac{1}{n}\Lambda_{F+\varphi}(T_n(a)) \to \int F(a) + \int \varphi(a),$$

and since $(1/n)\Lambda_\varphi(T_n(a)) \to \int \varphi(a)$ by (43), it follows that $(1/n)\Lambda_F(T_n(a))$ converges to $\int F(a)$. $\square$

Proposition 8.3. *Let $a \in L^1$ be real-valued and let $F : \mathbf{R} \to [0, \infty)$ be a continuous function. If*

$$C := \liminf_{n\to\infty} \frac{1}{n} \Lambda_F(T_n(a)) < \infty,$$

then $F(a) \in L^1$ and $\int F(a) \leqslant C$.

Proof. We proceed as in the proof of Proposition 6.2. Fix $\varepsilon > 0$ and choose $n_1 < n_2 < \cdots$ so that $(1/n_k)\lambda_F(T_{n_k}(a)) < C + \varepsilon$. Define $F_M : \mathbf{R} \to [0, \infty)$ by $F_M(\lambda) = F(\lambda)$ for $|\lambda| \leqslant M$, $F_M(\lambda) = (M + 1 - |\lambda|)F(\lambda)$ for $M \leqslant |\lambda| \leqslant M + 1$, and $F_M(\lambda) = 0$ for $|\lambda| \geqslant M + 1$. Since F_M has compact support, formula (43) implies that

$$\int F_M(a) = \lim_{k\to\infty} \frac{\Lambda_{F_M}(T_{n_k}(a))}{n_k} \leqslant \limsup_{k\to\infty} \frac{\Lambda_F(T_{n_k}(a))}{n_k} \leqslant C + \varepsilon.$$

Letting $M \to \infty$ we see that $F(a) \in L^1$ and $\int F(a) \leqslant C + \varepsilon$. $\square$

Corollary 8.4. *Let $\Psi_\pm : [0, \infty) \to [0, \infty)$ be two convex functions and let $F : \mathbf{R} \to [0, \infty)$ be a continuous function such that $F(\lambda) \simeq \Psi_+(\lambda)$ as $\lambda \to \infty$ and $F(\lambda) \simeq \Psi_-(-\lambda)$ as $\lambda \to -\infty$. Then $F \in ST$, that is,*

$$\lim_{n\to\infty} \frac{1}{n} \Lambda_F(T_n(a)) = \int F(a) \tag{50}$$

for every real-valued function $a \in L^1$.

Proof. If $F(a) \notin L^1$, then both sides of (50) are infinite by Proposition 8.3. Thus, let $F(a) \in L^1$. Then $F(a_+) \in L^1$ and $F(-a_-) \in L^1$. There are finite and positive constants C_1 and C_2 such that

$$C_1\Psi_+(\lambda) \leqslant F(\lambda) \leqslant C_2\Psi_+(\lambda), \quad C_1\Psi_-(\lambda) \leqslant F(-\lambda) \leqslant C_2\Psi_-(\lambda)$$

for all $\lambda > \nu$. It follows that $\Psi_+(a_+) \in L^1$ and $\Psi_-(a_-) \in L^1$. If both Ψ_+ and Ψ_- are bounded, we define Φ_+ and Φ_- as the functions on $[0, \infty)$ that take the constant value $C_2 \max(\Psi_-(0), \Psi_+(0))$. If Ψ_- is bounded and Ψ_+ is unbounded, there is a $\mu \in (0, \infty)$ such that Ψ_+ is monotonously increasing on (μ, ∞) and $\Psi_+(\mu) \geqslant \Psi_-(0)$. In that case we put $\Phi_-(\lambda) = C_2\Psi_-(\mu)$ for all $\lambda \geqslant 0$, $\Phi_+(\lambda) = C_2\Psi_+(\mu)$ for $0 \leqslant \lambda \leqslant \mu$, and $\Phi_+(\lambda) = C_2\Psi_+(\lambda)$ for $\lambda \geqslant \mu$. A similar construction is made if Ψ_+ is bounded and Ψ_- is unbounded. Finally, if Ψ_+ and Ψ_- are both unbounded, there exist $\mu_\pm \in (0, \infty)$ such that $\Psi_\pm$ is monotonously increasing on $(\mu_\pm, \infty)$ and $\Psi_-(\mu_-) = \Psi_+(\mu_+) \geqslant \max(\Psi_-(0), \Psi_+(0))$. We then put $\Phi_\pm(\lambda) = C_2\Psi_\pm(\mu_\pm)$ for $0 \leqslant \lambda \leqslant \mu_\pm$ and $\Phi_\pm(\lambda) = C_2\Psi_\pm(\lambda)$ for $\lambda \geqslant \mu_\pm$. The functions $\Phi_\pm$ obtained in this way satisfy all hypotheses of Theorem 8.2, and (50) therefore results from (49). □

Corollary 8.5. *The set ST contains all nonnegative and convex functions, that is, if $a \in L^1$ is real-valued and $F : \mathbf{R} \to [0, \infty)$ is convex, then*

$$\lim_{n\to\infty} \frac{1}{n} \Lambda_F(T_n(a)) = \int F(a).$$

Proof. Use Corollary 8.4 with $\Psi_\pm(\lambda) = F(\pm\lambda)$ for $\lambda \geqslant 0$. □

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