Toeplitz operators in Bergman spaces. Some new trends

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$\mathcal{H}$ a Hilbert space of functions (on some domain Ω in $\mathbb{R}^d$ or $\mathbb{C}^d$ or on some smooth manifold $\mathbf{M}$), usually, L^2. $\mathcal{L}$ a closed subspace in $\mathcal{H}$. $\mathbf{P} : \mathcal{H} \to \mathcal{L}$ orthogonal projection. F a function (symbol).

$$\mathbf{T} = \mathbf{T}_F = \mathbf{T}_{F,\mathcal{L}} = \mathbf{P}F,$$

$$\mathbf{T}u = \mathbf{P}Fu, u \in \mathcal{L}.$$

Another point of view

A acting in $\mathcal{H}$, $\mathcal{K} = \mathcal{H} \ominus \mathcal{L}$, the Toeplitz operator $\mathbf{T}_A$ associated with A and acting in $\mathcal{L}$ ($\equiv$ the compression of A onto $\mathcal{L}$, or the angle of the operator A) is defined as

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In such a general setting, different operators A' and A'', and even different enveloping spaces $\mathcal{H}$ can generate the same Toeplitz operator

$$A' = \begin{pmatrix} A'_{1,1} & A'_{1,2} \\ A'_{2,1} & A'_{2,2} \end{pmatrix}, \; A'' = \begin{pmatrix} A''_{1,1} & A''_{1,2} \\ A''_{2,1} & A''_{2,2} \end{pmatrix}$$

, Then $T_{A'} = T_{A''}$ if and only if $A'_{1,1} = A''_{1,1}$.

Examples

1. $\mathbf{M} = \mathbb{T} = S^1$, the unit circle. $\mathcal{H} = L^2(S^1)$, $\mathcal{L} = H^2$ - the Hardy space, boundary values of analytic functions in the disk $\mathbb{D}$.
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2c. Generalization. Solutions of some elliptic equation or system. Little is known in general. Important particular case- solutions of the Helmholtz equation. Will study later.

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For individual operators: Definition, boundedness and compactness conditions, closedness and closability when boundedness is not granted, uniqueness of the symbol, finite rank property, extension to wider classes of symbols, spectral properties, Fredholm property.

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The picture with enveloping space is not always feasible.

Consider $\mathcal{B}^\alpha$ and H^2 both as spaces of analytic functions. Taylor series:
$u(z) = \sum_0^\infty u_k z^k$. For $\mathcal{B}^\alpha$:

$$\int |u(z)|^2 (1-|z|^2)^\alpha dA(z) = 2\pi \sum |u_k|^2 \int_0^1 (1-|r|^2)^\alpha r^{2k+1} dr =$$
$$2\pi \sum |u_k|^2 \frac{k!\Gamma(2+\alpha)}{\Gamma(k+2+\alpha)}$$

So, by the Stirling's formula

$$\|u\|^2_{\mathcal{B}^\alpha} \asymp \sum |u_k|^2 k^{-(\alpha+1)}, \ \|u\|^2_{H^2} \asymp \sum |u_k|^2,$$

Therefore, formally, one can consider the Hardy space as a limit case of the weighted Bergman space as $\alpha = -1$. This analogy is sometimes useful. On the other hand, H^2 can be identified with the space of analytic functions in $\mathbb{D}$ with boundary values in L^2, with the above norm. But there is no reasonable candidate for the enveloping space $\mathcal{H}$. Redefining $\mathcal{B}^\alpha$ the space of analytic fictions in $\mathbb{D}$, having boundary values in the negative Sobolev space $\mathbf{H}^{-(\alpha+1)/2}(\mathbf{S})$. Here H^2 becomes $\mathcal{B}^{-1}$ in a natural way. Again, no candidate for the enveloping space.

Bergman type spaces vs Hardy type spaces.

The reproducing kernel, $\mathbf{P} : \mathcal{H} \to \mathcal{L}$. The orthogonal projection $\mathbf{P}^2 = \mathbf{P}$. For Bergman space $\mathcal{B}^\alpha$, $\mathbf{P}$ is an integral operator with smooth kernel. $\mathbf{P}u(z) = \int_\Omega K(z, w)u(w)d\mu(w)$.

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For weighted Bergman,
$K(z,w) = (\alpha+1)(1-|w|^2)^\alpha(1-z\bar{w})^{-(2+\alpha)}$,
For Fock: $K(z,w) = \frac{1}{\pi}e^{z\bar{w}}$.

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For Hardy type spaces: Not necessarily.
$\mathbf{P}u(z) = (2\pi^{-1})(V.P.)\int_{\mathbb{T}}(1-z\bar{w})^{-1}u(w)dA(w)$ (principal value integral).
Since $\mathbf{P}$ is an orthogonal projection, $K(z,w) = \overline{K(w,z)}$,
$\int K(z,w)K(w,\zeta)d\mu(w) = K(z,\zeta)$.

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$\langle u, v\rangle = \int u(z)\overline{v(z)}d\mu(z) = \int\int u(w)\overline{\mathbf{k}_z(w)v(z)}d\mu(z)d\mu(w)$.
On the other hand,
$\langle u, v\rangle = \int u(w)\overline{v(w)}d\mu(w) = \int\int u(w)\mathbf{k}_w(z)\overline{v(z)}d\mu(z)d\mu(w)$.
Comparing: $\mathbf{k}_w(z) = \overline{\mathbf{k}_z(w)}$. $\int \mathbf{k}_z(w)\mathbf{k}_w(\zeta)dw = \mathbf{k}_z(\zeta)$. Thus, the operator with kernel $\mathbf{k}_w(z)$ is an orthogonal projection from $\mathcal{H}$ onto $\mathcal{L}$.

$$K(z, w) = \mathbf{k}_w(z).$$

$K(z, w)$ belongs to $\mathcal{L}$ in z variable for w fixed, and belongs to the complex adjoint of $\mathcal{L}$ in w variable, for z fixed.

For the Hardy space,
$\mathbf{P}u(z) = (2\pi^{-1})(V.P.) \int_{\mathbb{T}}(1 - z\bar{w})^{-1} u(w) dl(w)$, not an integral operator, the functional $\phi_z(u) = u(z)$ is not bounded. In Lecture 3 we will see what can be done about it.

Which spaces admit a reproducing kernel?

Until recently: spaces of solutions of elliptic equations or systems. If $\mathcal{A}$ is an elliptic operator in Ω, $x_0 \in \Omega$, then, from the elliptic regularity:

$$|u(x_0)| \leq C_G(\|u\|_{L^2(G)} + \|\mathcal{A}u\|_{L^2(G)})$$

where G is an open set containing x_0.

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where G is an open set containing x_0.
The Hardy space on $\mathbf{S}$ is not a space of solutions of an elliptic equation or system. No reproducing kernel. The projection $\mathbf{P} : L^2(\mathbb{T}) \to H^2$, singular integral operator

$$\mathbf{P}u(z) = (2\pi^{-1})(V.P.)\int_{\mathbb{T}}(1 - z\bar{w})^{-1}u(w)dl(w).$$

The kernel DOES NOT belong to H^2 for fixed w. (!)

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A possible way out. $\mathbf{P} : \mathcal{H} \to \mathcal{L}$. If $\mathcal{L}$ consists of smooth functions, $\mathbf{P}$, actually, acts as $\mathbf{P}^0 : \mathcal{E}(\Omega)' \to \mathcal{E}(\Omega)$, $\mathcal{E}(\Omega) \equiv C^{\infty}(\Omega)$ and, probably, even to $\mathbf{P}^0 : \mathcal{E}(\Omega)' \to \mathcal{L}$. Therefore for $F \in \mathcal{E}'$, the operator $\mathbf{T}^? = (\mathbf{P}^0)' F : \mathcal{L} \to \mathcal{L}$ makes sense. Serious questions: when $\mathbf{T}^?$ acts into $\mathcal{L}$, when is it bounded. Some attempts have been made (Zorboska, Taskinen-Virtanen...). Not quite satisfactory.

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Problem. **To define in a reasonable Toeplitz operators with bad symbols.**

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$$(\mathbf{J}e_k)(z) = (-1)^k e_k(z).$$

So, the eigenvalue sequence $\boldsymbol{\gamma}$ is $1, -1, 1, -1 \ldots$. (Sic! this is not simply the spectrum but the specially ordered sequence).

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So, the eigenvalue sequence $\boldsymbol{\gamma}$ is $1, -1, 1, -1....$ (Sic! this is not simply the spectrum but the specially ordered sequence).But this is impossible for a Toeplitz operator with bounded symbol since for such operators the spectral sequence must satisfy (Grudsky, Maximenko, Vasilevskii 2013) $\lim_{n/m \to 1} |\gamma_n - \gamma_m| \to 0$ (slowly oscillating), which is not the fact.

The same question in $\mathcal{F}$. Even harder (Berger/Coburn, 1994) - not only **J** is not a Toeplitz operator with bounded symbol, not only it is not a Toeplitz operator with (possibly) unbounded symbol such that $F(z) = O(e^{|z^2|-A|z|})$ (for all A), but it cannot be norm approximated by such Toeplitz operators, and the 0 operator is the best approximation.

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Quite nice an operator is not Toeplitz.

Finite rank operators

Can a finite rank operator be a Toeplitz one? No. Both in $\mathcal{B}$ and in $\mathcal{F}$ a finite rank Toeplitz operator (with function-symbol) must be zero (Luecking 2007, Borichev-GR, 2015). In particular a rank 1 operator, $P_{k,l} : e_k \mapsto e_l$, and zero elsewhere ($P_{k,l}u = \langle u, e_k \rangle e_l$) is not Toeplitz.

So, the direct definition is not quite satisfactory. Need for the enveloping space (which does not necessarily exist), complications with singular symbols, distributions, non-Toeplitz for some important operators. Another approach is needed.

Sesquilinear forms

Definition $\Phi(u, v)$, linear in u, antilinear in v, $|\Phi(u, v)| \leq C\|u\|_{\mathcal{L}}\|v\|_{\mathcal{L}}$. By Riesz theorem, such form defined a bounded operator in $\mathcal{L}$, $\mathbf{T}_\Phi$, $\langle \mathbf{T}_\Phi u, v\rangle = \Phi(u, v)$.

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$$(\mathbf{T}_\Phi u)(z) = \langle \mathbf{T}_\Phi u, \mathbf{k}_z\rangle = \Phi(u(.), \mathbf{k}_z(.)).$$

Explicit action of the Toeplitz operator.

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4. For a measure ν on $\mathbb{D}$ or $\mathbb{C}$: $\mathbf{F}_\nu(u,v) = \int u(z)\overline{v(z)}d\nu(z)$

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$\langle \mathbf{T}_F u, v\rangle = \langle \mathbf{P}Fu, v\rangle = \langle Fu, \mathbf{P}v\rangle = \langle Fu, v\rangle$.
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5. (generalization of 3,4) For a distribution $F \in \mathcal{E}'(\Omega)$:
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A general structure

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X-valued sesquilinear form $\mathbf{G}$ on $\mathcal{L}$: a continuous mapping

$$\mathbf{G}(\cdot,\cdot) \; : \; \mathcal{L} \oplus \mathcal{L} \longrightarrow \; X,$$

For a continuous X-valued sesquilinear form $\mathbf{G}$ and $\Psi \in X'$, *the sesquilinear form* $\mathbf{F}_{\mathbf{G},\Phi}$ on $\mathcal{L}$:

$$\mathbf{F}_{\mathbf{G},\Psi}(u,v) = \Psi(\mathbf{G}(u,v)) = (\Psi, \mathbf{G}(u,v)).$$

$$\mathbf{T}_{\mathbf{G},\Psi} u(z) := (\mathbf{T}_{\mathbf{F}_{Gb,\Phi}} u)(z) = (\Phi, \mathbf{G}(u, \mathbf{k}_z)).$$

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The choice of forms - to be a natural generalization of usual ones.

Hardy space as Bergman space

In the model: $H^2 = \{u \mathrm{analytic\,in} \mathbb{D}, u|_{\mathbb{T}} \in L^2(\mathbb{T})\}$. No reasonable enveloping space. But: for a measure ν on $\mathbb{D}$, sesquilinear form $\mathbf{F}(u, v) = \int_{\mathbb{D}} Fu\bar{v}dA(z)$. Moreover, $F \in \mathcal{E}'(\mathbb{D})$, $\mathbf{F}(u, v) = (F, u\bar{v})$, which makes sense since $u\bar{v}$ is a smooth function on $\mathbb{D}$.

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In this space we can consider sesquilinear forms, for example $\int_{\mathbb{D}} u(z)\bar{v}(z)d\mu(z)$, where μ is a measure on $\mathbb{D}$. If $\operatorname{supp}\mu \subset \partial\mathbb{D} = \mathbb{T}$, we arrive to the usual Riesz-Toeplitz operators.

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We meet other important examples in Lecture 3.

Symbols with compact support

We start 'hard' analysis part of our presentation by considering a wide class of symbols F, for which, in any Bergman type space, the sesquilinear form $\mathbf{F}_F(u, v)$ is bounded, moreover, compact (i.e., defines a compact operator).

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Proof. The structure theorem for distributions with compact support: If $F \in \mathcal{E}'(\Omega), \Omega \subseteq \mathbb{R}^d$, there exit $N \geq 0$ and continuous functions $g_\alpha(x), |\alpha| \leq N$ such that $F = \sum_{|\alpha| \leq N} D^\alpha g_\alpha$, moreover, the support $\mathbb{K}'$ of g_α can be chosen compact and arbitrarily close to $\operatorname{supp} F$.

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$$(F, u\bar{v}) = \sum_{|\alpha| \leq N} (D^\alpha g_\alpha, u\bar{v}) = \sum_{|\alpha| \leq N} (g_\alpha D^\alpha (u\bar{v})) \leq$$

$$\sum_{|\alpha| \leq N} \int_{\mathbb{K}'} |g_{\alpha,\beta}||D^\beta u||D^{\alpha-\beta}\bar{v}|dA \leq C \sum_{|\alpha|+|\beta| \leq N} \int |\partial^\alpha u||\partial^\beta v|dA.$$

As it follows from elliptic regularity, $\int_{\mathbb{K}'} D^\beta |u|^2 dA \leq C \int_\Omega |u|^2$, and this gives boundedness. The embedding theorem gives compactness.

Similar reasoning proves a more general result. Let $F \in \mathcal{E}'(\Omega \times \Omega)$. Consider the sesquilinear form

$$\mathbf{F}_F(u, v) = (F, u \otimes \bar{v}), u, v \in \mathcal{L}.$$

defines a compact Toeplitz operator, $\mathbf{T}_{\mathbf{F}} u(z) = (F, u \otimes \mathbf{k}_z)$.

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More. $\mathcal{L} = H^2$, Hardy space in our representation as $\mathcal{B}^{-1}$. Again, for F with compact support in $\mathbb{D}$ (or $\mathbb{D} \times \mathbb{D}$), the sesquilinear form $(F, u\bar{v})$, (or $(F, u \otimes \bar{v})$) defines a compact operator.

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Examples.1. $F = \delta(z - z_0)$.
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(**Theorem.**) Any finite rank Toeplitz operator is a linear combination of these ones.
3. An important special case. $z_0 = 0 \in \mathbb{D}$ or in $\mathbb{C}$. $\mathbf{T} e_k = C_{k,l} e_l$.
$\mathbf{T} e_j = 0, j \neq k$.

Remarks.1. Since the operators are compact, the important topic is the study of the quality of this compactness, the rate of decay of eigenvalues or singular numbers. It is known not that much. For the Bergman space, the s-numbers decay exponentially, and sometimes the exact exponential order is found. For the Fock space, the s-numbers decay superexponentially, as n^{-cn}. A lot of questions open.

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2.The above results show that a substantial study in the topic should involve symbols whose support touches the boundary (for Bergman type spaces) or spreads to infinity (for Fock type spaces). This will be done further on.

3. Bergman spaces under consideration consist of analytic functions. Therefore, the functions $u(z)\overline{v(z)}$ and $u \otimes v(z,w) = u(z)\overline{v(w)}$ are not only smooth but *real-analytic*. Therefore it is possible to consider symbols F even more singular than distributions, hyperfunctions, that involve derivatives of all orders.

Removing compact support condition

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$$(F, u\bar{v}) = \sum_{|\alpha|\leq N} (D^\alpha g_\alpha, u\bar{v}) = \sum C_{\alpha,\beta} \int_{|\alpha|+|\beta|\leq N} |\partial^\alpha u||\partial^\beta v| dA.$$

The left-hand side may not defined if the compact support condition is dropped, but the right-hand side can (probably!) be extended. New notion: *Carleson measures for derivatives.*

Carleson measures

L.Carleson, 1962. A measure μ on the disk $\mathbb{D}$, such that for any $u \in H^2 = \mathcal{B}^{-1}$,

$$\int_{\mathbb{D}} |u|^2 d\mu \leq C\|u\|_{H^2}^2.$$

In our language: the sesquilinear form $\mathbf{F}(u, v) = \int_{\mathbb{D}} u\bar{v} d\mu$ is bounded on $\mathcal{B}^{-1}$. Carleson: condition:
$S(r_0, \theta_0) = \{re^{i\theta}\}, r \in (r_0, 1), \theta \in (\theta_0 - (1 - r_0), \theta_0 + (1 - r_0))$.
$|\mu(S(r_0, \theta_0))| \leq C(1 - r_0)$, all θ_0, r_0.

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$|\mu(S(r_0,\theta_0))| \leq C(1-r_0)$, all θ_0, r_0. After this, results for many other spaces.

Carleson measures for Fock space

$$\int_{\mathbb{C}} |u|^2 e^{-|z|^2} d\mu \leq C\|u\|_{\mathcal{F}}^2 \equiv C \int |u(z)|^2 e^{-|z|^2} dA(z)$$

Theorem (Zhu): A positive measure μ on $\mathbb{C}^1$ is F-C measure if and only if for a fixed ρ_0, $\mu(B(z_0, \rho_0)) \leq C$ for all $z_0 \in \mathbb{C}^1$, with constant C not depending on z_0.

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Definition. Measure μ is called F-C-measure for derivative of order k (k-F-C measure) if

$$\int_{\mathbb{C}} |u^{(k)}|^2 e^{-|z|^2} d\mu \leq \varpi_k(\mu)\|u\|_{\mathcal{F}}^2 \equiv \varpi_k(\mu) \int |u(z)|^2 e^{-|z|^2} dA(z).$$

Questions: to find a condition for μ to be a k-FC measure. Dependence of $\varpi_k(\mu)$ on k is important.

Theorem. (GR, NV, 2014) A measure μ is a k-FC measure if and, for $\mu \geq 0$, only if, for some (and, therefore, for any) $r > 0$, the quantity

$$C_k(\mu, r) = (k!)^2 \sup_{z \in \mathbb{C}} \left\{ |\mu|(B(z, r))(1 + |z|^2)^k \right\} \qquad (0.1)$$

is finite. For a fixed r, the constant $\varpi_k(\mu)$ in can be taken as $\varpi_k(\mu) = C(r)C_k(\mu, r)$, with some coefficient $C(r)$ depending only on r.

In other words, Theorem states that μ is a k-FC measure if and only if $(1+|z|^2)^k\mu$ is a FC measure. If $k=0$, then any 0-FC measure is just a FC measure. The quantity $\varpi_k(\mu)$ in will be called the k-FC norm of the measure μ.

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It is convenient to extend the notion of k-FC measures to half-integer values of k, defining these measures as those for which $(k!)^2 \sup_{z\in\mathbb{C}} \left\{|\mu|(B(z,r))(1+|z|^2)^k\right\} = \varpi_k(\mu) < \infty$.

$|\mu|$ is the variation of measure μ.

In other words, Theorem states that μ is a k-FC measure if and only if $(1+|z|^2)^k\mu$ is a FC measure. If $k=0$, then any 0-FC measure is just a FC measure. The quantity $\varpi_k(\mu)$ in will be called the k-FC norm of the measure μ.

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Relation of measures for different k:

For any integers $p\in\mathbb{Z}_+$ and integer or half-integer k a measure μ is a k-FC measure if and only if the measure $\mu_p=(1+|z|^2)^{(k-p)}\mu$ is a p-FC measure, moreover, for integer k, $C_p(\mu,r)\asymp C_k(\mu_p,r)$.

Estimates for distributional sesquilinear forms

Let μ be a k-FC measure, with integer or half-integer k. With μ we associate the sesquilinear form

$$\mathbf{F}(u,v)=\int_{\mathbb{C}} u^{(\alpha)}(z)\overline{v^{(\beta)}(z)}e^{-|z|^2}d\mu(z),\ \ u,v\in\mathcal{F}(\mathbb{C}), \tag{0.2}$$

for some α,β with $\alpha+\beta=2k$. This form is bounded in $\mathcal{F}(\mathbb{C})$, moreover

$$|\mathbf{F}(f,g)|\le C(\mathbf{F})\|u\|_{\mathcal{F}}\|v\|_{\mathcal{F}},\ \text{with}\ C(F)\le(\varpi_\alpha(\mu)\varpi_\beta(\mu))^{\frac{1}{2}}. \tag{0.3}$$

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$$\varpi_k(\mu)(k!)^2\sup_{z\in\mathbb{C}}\left\{|\mu|(B(z,r))(1+|z|^2)^k\right\}.$$

The higher is derivative, the faster 'decay' of the measure at ∞.

If μ is k-Fc measure then the distribution

$$(e^{-|z|^2}\mu, u^{(\alpha)}\overline{v^{(\beta)}}) = (-1)^{\alpha+\beta}(\partial^\alpha\overline{\partial}^\beta(e^{-|z|^2}\mu), u\bar{v})$$

extends from $\mathcal{D}(\mathbb{C})$ to a continuous functional on functions $u, v \in \mathcal{F}$, so defines a bounded Toeplitz operator with distributional symbol.

Compact forms

As usual, for any norm estimate for the operator defined by a symbol, the boundedness result is accompanied by a compactness result.

Definition. A measure μ is called *vanishing k-FC measure* if

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Corollary: Let μ be a vanishing k-FC measure, with integer or half-integer k. Then the operator in $\mathcal{F}(\mathbb{C})$ defined by the form with $\alpha+\beta=2k$ is compact.

Symbols and operators of weak almost-infinite type

Definition. Let $\mathbf{F}(u,v)$ be a sesquilinear form on the Fock space $\mathcal{F}(\mathbb{C})$. This form is a symbol of *weak almost-finite type* if there exist a collection $\boldsymbol{\mu} = \{\mu_{\alpha,\beta}\}_{\alpha,\beta=0,1,2,\ldots}$ of $(\alpha+\beta)/2$-FC measures such that, for each $u, v \in \mathcal{F}^2$, the series

$$\sum_j \sum_{\max(\alpha,\beta)=j} \mathbf{F}_{\alpha,\beta}(u,v) \equiv \sum_j \sum_{\max(\alpha,\beta)=j} (\partial^\alpha \overline{\partial}^\beta \mu_{\alpha,\beta}, u\bar{v}),$$

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Any bounded operator in $\mathcal{F}$ is a Toeplitz operator with symbol of weak almost finite type.(??)

Recall symbols $h_\alpha = C_\alpha \partial^\alpha \overline{\partial^\alpha} \delta(z)$. They define rank one projections $\mathbf{P}_\alpha$ onto e_α ($e_\alpha = c_\alpha z^\alpha$ is the standard monomial basis). For a given bounded $\mathbf{T}$, $\mathbf{T}_{\alpha,\beta} = \mathbf{P}_\alpha \mathbf{T} \mathbf{P}_\beta$ is a rank one operator with symbol constant times $\partial^\alpha \overline{\partial^\beta} \delta(z)$, a Toeplitz one with distributional symbol. And the sum $\sum \mathbf{T}_{\alpha,\beta}$ converges weakly to $\mathbf{T}$.

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To prove for particular examples, it is important that we know the dependence of constants on the order of derivatives.

Example

Let $h_{\alpha,\beta}(z)$, $\alpha, \beta = 0, 1, \ldots,$ be bounded functions in $\mathbb{C}$ such that

$$\sup_{z\in\mathbb{C}}\{|h_{\alpha,\beta}(z)|(1+|z|)^{\alpha+\beta}\} \leq (\alpha!\beta!)^{-1}c_{\alpha,\beta}$$

with $\sum c_{\alpha,\beta} < \infty$. With each function $h_{\alpha,\beta}(z)$ we associate the measure $\mu_{\alpha,\beta}$ that has the density $h_{\alpha,\beta}(z)$ with respect to the Lebesgue measure, consider the symbol

$$\boldsymbol{\mu} \simeq \sum_{\alpha,\beta} \partial^{\alpha}\overline{\partial}^{\beta}\mu_{\alpha,\beta},$$

the convergence condition is satisfied, and therefore, the formal series $\boldsymbol{\mu}$ can serve as a symbol of a bounded Toeplitz operator $T_{\boldsymbol{\mu}}$ with differentiation of all orders.

example

Let $\delta_{\mathbf{n}}$, $\mathbf{n} = n_1 + in_2$, be the delta-measure placed at the point $\mathbf{n}$. With each point $\mathbf{n}$ we associate the distribution

$$\theta_{\mathbf{n}} = \partial^{\alpha_{\mathbf{n}}} \overline{\partial}^{\beta_{\mathbf{n}}} \delta_{z-\mathbf{n}},$$

with some $\alpha_{\mathbf{n}}$, $\beta_{\mathbf{n}}$. Each term defines a compact operator. Norm estimate: $p_{\mathbf{n}} \leq C\alpha_{\mathbf{n}}!\beta_{\mathbf{n}}!(1+|\mathbf{n}|)^{\alpha_{\mathbf{n}}+\beta_{\mathbf{n}}}$ So, if we take the sum of these distributions with sufficiently small coefficients, we have norm convergence.

Toeplitz operators in the Bergman space

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Carleson measures for $\mathcal{B}$. μ on $\mathbb{D}$ is a C measure if $\int_{\mathbb{D}} |u|^2 d\mu \leq C\|u\|^2 = C\int_{\mathbb{D}} |u|^2 dA$ for all $u \in \mathcal{B}$.

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Theorem (Hastings, 1973, Luecking 1983, Zhu 1988). Fix $p \in (0,1)$. For a measure μ to be a Carleson measure for $\mathcal{B}$ it is sufficient, and for a positive measure necessary, that

$$|\mu|(B(z, p(1-|z|))) \leq C(1-|z|)^2.$$

And this constant C determines the norm of the Toeplitz operator with symbol μ.

Similar to Fock space, we define sesquilinear forms

$$\mathbf{F} = \mathbf{F}_{\alpha,\beta,\mu}(u, v) = \int_{\mathbb{D}} u^{(\alpha)}(z)\overline{v}^{(\beta)}(z) d\mu.$$

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$$|\mu|(B(z, p(1-|z|))) \leq \varpi_k(\mu, p)(1-|z|)^{2(1+k)}.$$

and this $\varpi_k(\mu, p)$ determines the constant $C(k,\mu) = Cp^{-2k}(k!)^2\varpi_k(|\mu|, p)$.

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and this $\varpi_k(\mu, p)$ determines the constant $C(k,\mu) = Cp^{-2k}(k!)^2\varpi_k(|\mu|, p)$. Further the theory goes on similar to Fock. Weak and norm almost finite type. Representation of any bounded operator as Toeplitz. Examples of operators with unbounded order of derivatives.

Operators in the Herglotz space

(GR, NV, 2016). Solutions of the Helmholtz eqn. $\Delta u + u = 0$ in $\mathbb{R}^d$. What Hilbert space? NB! no nontrivial solutions in $L^2(\mathbb{R}^d)$. Important for applications: a little bit worse than L^2:

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$$\int_{|x|=R} |\partial_r u - iu|^2 dS = o(R^{d-1}).$$

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($dx = dA(x)$) The norm is nonlocal. Cannot serve as an enveloping space.Pérez-Esteva, Barcelo: candidates for enveloping space.Spaces with differential norm, do not admit multiplication by a nondifferential function.

A different approach

Herglotz representation

Fourier transform. $(1 - |\xi|^2)\hat{u}(\xi) = 0$. $\hat{u}(\xi)$ a distribution supported on $\mathbf{S} = S^{d-1}$. Suppose: $\phi(\xi)$ is a function on $\mathbf{S}$.

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$$u(x)=(\mathbf{I}\phi)(x)=c_d\int_{\mathbf{S}}\phi(\xi)e^{ix\xi}\,dS(\xi),\quad \text{with}\quad c_d=\frac{\sqrt{\pi}}{(2\pi)^{d/2}},$$

Conversely, ϕ can be recovered from u as 'far field pattern'.
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Unitary equivalence of $\mathcal{H}^2$ and $L^2(\mathbf{S})$
Reproducing kernel space. The reproducing kernel can be found explicitly

Let $u(x) = \mathbf{I}\phi \in \mathcal{H}^2$, for some $\phi \in L^2(\mathbf{S})$. Thus, for the reproducing kernel $K(x, y) = \mathbf{k}_x(y)$, we have

$$u(x) = \langle u(\cdot), \mathbf{k}_x(\cdot) \rangle.$$

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$\mathbf{k}_x(\cdot)$ belongs to $\mathcal{H}^2$ for any $x \in \mathbb{R}^d$, it can be represented as

$$\mathbf{k}_x(y) = (\mathbf{I}\Psi_x)(y) = c_d \int_{\mathbf{S}} \Psi_x(\xi) e^{i\xi y} dS(\xi),$$

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And, so, $K(x,y) = \mathbf{k}_x(y) = c_d^2 \int_{\mathbf{S}} e^{-i(x-y)\xi} dS(\xi)$. Is a difference kernel, Let $u(x) = \mathbf{I}\phi \in \mathcal{H}^2$, for some $\phi \in L^2(\mathbf{S})$. Thus, for the reproducing kernel $K(x,y) = \mathbf{k}_x(y)$, we have

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$= \frac{\pi}{(2\pi)^{d/2}} |x-y|^{-(d-2)/2} J_{(d-2)/2}(|x-y|)$.

Sesquilinear forms

Following the general pattern: $\mathbf{F}$ sesquilinear form. $\mathbf{T}_{\mathbf{F}} u(x) = \mathbf{F}(u(.), \mathbf{k}_z(.))$. Which forms to study?

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Following the general pattern: **F** sesquilinear form. $\mathbf{T}_{\mathbf{F}} u(x) = \mathbf{F}(u(.), \mathbf{k}_z(.))$. Which forms to study?
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$\mathbf{T}_{\mathbf{F}} u(x) = \mathbf{F}(u(.), \mathbf{k}_z(.)) = \langle F u(.), k_x(.) \rangle$. Non-convenient since the scalar product is nonlocal. But: important results Barcelo, Pérez-Esteva and their groups. Based upon fine estimates for Bessel functions.

2. $\mathbf{F}(u,v) = \mathbf{F}_F(u,v) = \int_{\mathbb{R}^d} F(x)u(x)\bar{v}(x)dx$. Important. The Born approximation for the scattering matrix in the classical and quantum scattering theory.

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Set $\widehat{F}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\xi} F(x)\, dx$, then

$$\mathbf{F}_F(u,v) = \frac{\pi}{(2\pi)^{d/2}} \int_{\mathbf{S}} \int_{\mathbf{S}} \widehat{a}(\eta - \xi)\phi(\xi)\psi(\eta) dS(\xi) dS(\eta) = \lfloor \check{\mathbf{T}}_F \phi, \psi \rfloor,$$

where $\check{\mathbf{T}}_F := \check{\mathbf{T}}_{\mathbf{F}_F}$ is the operator in $L_2(\mathbf{S})$, defined by

$$(\check{\mathbf{T}}_a \phi)(\eta) = \frac{\pi}{(2\pi)^{d/2}} \int_{\mathbf{S}} \widehat{F}(\eta - \xi) \phi(\xi) dS(\xi).$$

The Toeplitz operator generated by $\mathbf{F}_F(u, v)$ in the Herglotz space is unitarily equivalent to the integral operator in $L^2(\mathbf{S})$

$$(\check{\mathbf{T}}_a\phi)(\eta) = \frac{\pi}{(2\pi)^{d/2}} \int_{\mathbf{S}} \widehat{F}(\eta - \xi)\phi(\xi)dS(\xi).$$

Integral operator of convolution type – but not this exactly. $\check{\mathbf{T}}_F$ acts on functions defined on the sphere $\mathbf{S}$, while the values of the kernel $\widehat{F}(\eta - \xi)$, that are involved in the action of the operator, are calculated at all points $\eta - \xi \in \overline{\mathbf{B}(0,2)}$, where $\overline{\mathbf{B}(0,2)}$ is the closed ball in $\mathbb{R}^d$ with radius 2 and centered at the origin.

Interesting properties

1. (Non) degeneracy. Is it possible that $\mathbf{T}_F = 0$ but $F \neq 0$? If F has compact support ($F \in \mathcal{E}'$), $\bar{F}$ must be zero in the ball $\overline{\mathbf{B}(0,2)}$, but for an entire function $\hat{F}$ this is impossible! The condition of compact support can be relaxed to he reasoning extends to any class of symbols with quasi-analytic Fourier transform. This, in particular, holds for those symbols a that satisfy the condition $|F(x)| = O(\gamma(|x|))$ for $|x| \to \infty$ with a monotone function $\gamma(t) \searrow 0$ satisfying

$$\int_0^\infty \frac{|\log(\gamma(t))|}{1+t^2} dt < \infty.$$

Degeneracy

On the other hand, quite a lot of 'degenerate' symbols. Since the values of $\widehat{F}$ that are involved in the expression for the action of $\check{\mathbf{T}}_F$ are only the ones that are attained at the points of $\overline{\mathbf{B}(0,2)}$, the operator $\check{\mathbf{T}}_F$ is zero and, therefore,

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For example, let $F(x) \in \mathcal{S}'$, say, belong to L^p_{loc}. Cut-off function $\omega(\xi) \in C^\infty$, $\omega(\xi) = 1, |\xi| \leq 2$, and $\omega(\xi) = 0, |\xi| \geq 3$. The inverse Fourier transform $\check{\omega}(x)$ of ω is a smooth function on $\mathbb{R}^d$ with decay, faster than any negative power of $|x|$. Therefore, the convolution $F_\omega = F * \check{\omega}$ makes sense.

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(Compare with previous spaces. For $\mathcal{B}$ unknown. For $\mathcal{F}$ there are examples of F that produce zero operator but they grow VERRRRY fast, as $e^{|z|^2 - |z|^\alpha}$, $\alpha < 1$. And this is rather sharp. For $\alpha > 1$ such examples are impossible.!!)

Finite rank property

Let F has compact support. Is it possible that $\mathbf{T}_F$ has finite rank?

Finite rank property

Let F has compact support. Is it possible that $\mathbf{T}_F$ has finite rank? Yes! but only if F is a finite sum of δ distributions and their derivatives. Similar to $\mathcal{F}$. Proof (GR,NV, 2016): a long way. We take the result for $\mathcal{F}$ then derive a similar result for the harmonic Fock space, and then pass to $\mathcal{H}^2$. By the way, at some moment we loose one dimension. So, the result is not proved in dimension 2. We hope so much but still no approaches to dim 2 (:().

Symbol F with compact support

. $F \in \mathcal{E}'(\mathbb{R}^d)$. Similar to the Fock space: the form $\mathbf{F}(u, v) = (F, u\bar{v})$ is bounded in $\mathcal{H}^2$ since $F = \sum_{|\alpha| \leq N} D^\alpha g_\alpha$. The operator is bounded, moreover, compact.

Radial symbols

A special, important and most transparent, case when the function F is *radial*: $F = F(|x|) = F(r)$.
The orthonormal basis in $\mathcal{H}^2$

$$\mathbf{e}_{n,j}(x) = \sqrt{\pi}\, i^n \frac{J_{n+(d-2)/2}(r)}{r^{(d-2)/2}} Y_{n,j}(\xi),$$

Given a locally integrable radial function $F = F(|x|) = F(r)$, we introduce the *spectral sequence* $\boldsymbol{\gamma}_F = \{\gamma_F(n)\}_{n\in\mathbb{Z}_+}$, where

$$\gamma_F(n) = \pi \int_{\mathbb{R}_+} F(r)[J_{n+(d-2)/2}(r)]^2\, r\, dr.$$

the eigenvalues, repeated with multiplicity
$N_{n,d} = \frac{(2n+d-2)(n+d-3)!}{n!(d-2)!}$, if $n \geq 1$, 1 for $n = 0$. If $\gamma_F(n)$ is bounded, the operator can be defined as bounded. A simple boundedness condition: $F \in L^1(\mathbb{R}_+)$. This condition is almost sharp. $r^{-\sigma}, \sigma \lesseqgtr 1$.

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Question: which sequences of numbers $\boldsymbol{\gamma}$ may serve as the spectral sequence of an operator $\mathbf{T}_a$, with certain symbol $F(r) \in L_1(\mathbb{R}_+)$.
(For Bergman space on the disk, found recently by

General symbols

$$(\check{\mathbf{T}}_F\phi)(\eta) = \frac{\pi}{(2\pi)^{d/2}} \int_{\mathbf{S}} \widehat{F}(\eta-\xi)\phi(\xi)dS(\xi).$$

Theorem. Let $F(x) \in L^1_{\text{loc}} \cap \mathcal{S}'(\mathbb{R}^d)$. For $\xi \in \mathbf{S}$, $|\xi| = 1$, denote by S_ξ the unit sphere $\{\eta : |\eta-\xi| = 1\}$, centered at ξ. Suppose that for any $\xi \in \mathbf{S}$, the restriction of $\widehat{F}$ to S_ξ belongs to $L_1(S_\xi)$ and $\|\widehat{F}\|_{L_1(S_\xi)} \leq C$.(A kind of restriction condition.) Then $\check{\mathbf{T}}_F$ is bounded in $L_2(\mathbf{S})$, and, therefore, the operator $\mathbf{T}_F$, is bounded in $\mathcal{H}$. Moreover $\|\mathbf{T}_F\| \leq C$.

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Local singularities do not play role in the boundedness problem. Therefore, F can be a distribution, with conditions set on the behavior at infinity, say, for $F * \omega$.

Spectral properties

$F(x)$ almost homogeneous,
$F(x) = \theta(x/|x|)t^{-\sigma} + o(t^{-\sigma})$, $t \to +\infty$, $\sigma > 1$ (important for scattering). The operator $\check{\mathbf{T}}_F$ is a pseudodifferential operator on $\mathbf{S}$.

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Asymptotics of scattering phases. If F has compact support, the eigenvalues decay very fast. $|\lambda_n^{\pm}| \preceq n^{-n}$.

One more type of sesquilinear forms.

Recall $\mathbf{I}: L^2(\mathbf{S}) \to \mathcal{H}^2$, $(\mathbf{I}\phi)(x) = c_d \int_{\mathbf{S}} e^{-ix\xi}\phi(\xi)dS(\xi)$, isometry. Consider $\mathbf{I}^*: \mathcal{H}^2 \to L^2(\mathbf{S})$ and the sesquilinear form

$$\mathbf{F}_a(u,v) = \int_{\mathbf{S}} a(\xi)(\mathbf{I}^*u)(\xi)\overline{(\mathbf{I}^*v)(\xi)}\, dS(\xi),$$

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for a function $a \in L^\infty(\mathbf{S})$. Unitary equivalent to the multiplication operator by a in $L^2(\mathbf{S})$. Action:

$$\begin{aligned}
(\mathbf{T}_{\mathbf{F}_a}u)(x) &= \int_{\mathbf{S}} a(\xi)(\mathbf{I}^*u)(\xi)\overline{(\mathbf{I}^*k_x)(\xi)}\, dS(\xi) \\
&= c_d \int_{\mathbf{S}} a(\xi)(\mathbf{I}^*u)(\xi)e^{ix\xi}\, dS(\xi) = (\mathbf{I}a\mathbf{I}^*u)(x).
\end{aligned}$$

Such operators are bounded, moreover they form a commutative C^* algebra, isometrically isomorphic to $L^\infty(\mathbf{S}.)$

Topics for further studies

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– The results concerning sesquilinear forms in Bergman and Fock spaces are obtained for the complex dimension 1. To carry over to higher dimensions (some complications in obtaining criteria for Carleson measures for derivatives)

Topics for further studies. High dimensions

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Analysis in poly-Fock spaces, the images of Fock spaces under the 'creation' operator $Q^* = \partial_z + \bar{z}$. Some new effects to expect.

Topics for further studies

Unbounded forms. If the form $\mathbf{F}(u, v)$ is unbounded, it still may define a reasonable operator in the Hilbert space. For this, it is necessary that $\mathbf{F}$ is defined on a dense set $\mathcal{D}(\mathbf{F})$, is sectorial $|\operatorname{Im}(\mathbf{F}(u, u))| \leq C \operatorname{Re}(\mathbf{F}(u, u))$ and the form is closable, this means that $u_n \in \mathcal{D}(\mathbf{F}), u_n \to 0$ and $\mathbf{F}(u_n - u_m, u_n - u_m) \to 0$ imply $\mathbf{F}(u_n, u_n) \to 0$. Alternatively, the closability of the operator $\mathbf{T}$: $u_n \in \mathcal{D}(\mathbf{T}), u_n \to 0, ||\mathbf{T}(u_n - u_m)|| \to 0$ imply $\|\mathbf{T} u_n\| \to 0$. Closed operators possess many nice properties of bounded ones. Very little is known. There are some very recent results by Yafaev for the Hardy space.

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$$H^2_\varphi = H^2 \ominus \varphi H^2.$$

If φ is differentiable on $\mathbb{T}$, H^2_φ is a reproducing kernel space,

$$K_z(w) = \frac{1 - \overline{\varphi z}}{\varphi w(1 - \bar{z}w)}.$$

Space is invariant wrt Conjugation: $\mathbf{C}u = \bar{z}\varphi\bar{u}$.
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Unlike the whole Hardy space, w can be on $\mathbb{T}$.

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$\mathbf{T}_F^{\varphi} u = \mathbf{T}_F^{\varphi} u = \mathbf{P}_{\varphi} F u$, $u \in H^2_{\varphi}$ a closed densely defined operator with $\mathcal{D}(\mathbf{T}_F) = \{u \in H^2_{\varphi}, \mathbf{P}_{\varphi} u \in H^2_{\varphi}\}$.

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In many aspects similar to Bergman-Toeplitz. For bounded F automatically bdd, but also for many unbounded as well.
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Topics for further studies. General weights

Let $\psi(z)$ be an almost everywhere positive function. $\mathcal{B}^{\psi} = \{u \mathrm{analytic} \int \psi z |u|^2 dAz < \infty\}$. The same questions: boundedness, compactness, positivity, invertibility Schatten classes...

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Similar problem for weighted Fock space. Was started by Luecking, 30 years ago. Some serious results recently, Olivia Constantin, A. Aleman, K. Guo, D.Zheng....In many cases the description of Carleson measures. Very recent: Z.Wang, X.Zhao. Harmonic weighted Bergman spaces.

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Let $\psi(z)$ be an almost everywhere positive function. $\mathcal{B}^{\psi} = \{u \text{analytic} \int \psi z |u|^2 dAz < \infty\}$. The same questions: boundedness, compactness, positivity, invertibility Schatten classes...
Difference with classical weights - no explicit formula for the reproducing kernel, moreover, why this kernel exists? Some progress for Mackenhaupt weights:

$$\int_{B(a,r)} \psi dA \int_{B(a,r)} \psi^{-1} \leq C|B(a,r)|^2$$

Similar problem for weighted Fock space. Was started by Luecking, 30 years ago. Some serious results recently, Olivia Constantin, A. Aleman, K. Guo, D.Zheng....In many cases the description of Carleson measures. Very recent: Z.Wang, X.Zhao. Harmonic weighted Bergman spaces.

Topics for further studies. Vector-valued Bergman spaces and matrix (operator valued) symbols.

Here even the reproducing kernel property is unclear. But Aleman-Constantin: Boundedness of Bergman projection under Mackenhaupt condition. More recent results.

Topics for further studies. More general Bergman spaces

Solutions of an elliptic equation with variable coefficients.

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Thank you!!!!!!!