Analysis on Bounded Domains

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Overview

• In our talks we will discuss the application of representation theory to analysis on bounded domains and representations theory.

The talk is (approximately) organized in the following way:

- Short overview over symmetric spaces and bounded domain from the point of view of Lie Theory.
- The Harish-Chandra realization of bounded domains.
- Basic representation theory.
- The Bergman spaces
  - Starting with the unit ball
  - Then general bounded domains
- The work with M. Dawson and R. Quiroga-Barranco
- Example of commuting families.
- The restriction principle and Segal-Bargmann transform on bounded domains.
- Representation theory, Coorbit spaces and sampling in Bergman spaces.

PART I

LIE GROUPS, HOMOGENEOUS SPACES

AND

BOUNDED DOMAINS

Notation

• In the following G will stand for a (finite dimensional) simple Lie group. Up to local isomorphism (and we will mostly assume that) those are closed subgroup of some $\mathrm{GL}(n, \mathbb{C})$ such that

- $G^* = \{a^* \mid a \in G\} = G \subseteq \mathrm{GL}(n, \mathbb{C})$ for some n, closed,
- G does not contain any normal subgroup of positive dimension.

• Thus $\theta(a) = [a^{-1}]^*$ defines an involution on G with fixed point group

$$K = G^\theta = \{a \in G \mid a^* = a^{-1}\} = \mathrm{U}(n) \cap G \text{ maximal compact in } G$$

• In general, an involution θ such that $K = G^\theta$ is maximal compact is called Cartan involution.

• $\mathbf{X} := G/K$ and $x_o = eK$. Homogeneous space with (up to constant unique) G-invariant measure $\mu = \mu_{\mathbf{X}}$. $L^p = L^p(\mathbf{X})$ always with respect to this measure. Symmetric with symmetry $s_{x_o}(g \cdot x_o) = \theta(g) \cdot x_o$.

The Lie algebra

The Lie algebra $\mathfrak{g}$ of G is the sub-algebra of $\mathrm{GL}(n,\mathbb{C})$ such that for all $t \in \mathbb{R}$, $\exp tX \in G$. The involution θ defines a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}$. We set

- $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\} = \{X \in \mathfrak{g} \mid X^* = -X\} = \mathfrak{su}(n) \cap \mathfrak{g}$. This is also the Lie algebra of K.

- $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\} = \{X \in \mathfrak{g} \mid X^* = X\} = \mathrm{sym}_n(\mathbb{C}) \cap \mathfrak{g}$.

- Note that $\mathfrak{p}$ is not a Lie algebra, but it is isomorphic to the tangent space

$$\mathfrak{p} \simeq T_{x_o}(\mathbf{X}), \quad X \mapsto D_X, \; D_X f(x_o) = \left.\frac{d}{dt}\right|_{t=0} f(\exp tX \cdot x_o).$$

Examples

Our standard examples includes

- $\mathrm{SL}(2,\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \det A = 1 \right\}$ the group
- $\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| |\alpha|^2 - |\beta|^2 = 1 \right\} \simeq \mathrm{SL}(2,\mathbb{R})$
- Let $\beta_{p,q}(z,w) = -\sum_{j=1}^{p} z_j \bar{w}_j + \sum_{j=p+1}^{p+q} z_j \bar{w}_j = -(z^1, w^1) + (z^2, w^2)$ where $p+q=n$, and $z = (z^1, z^2) \in \mathbb{C}^n$. Then

$$\mathrm{U}(p,q) = \{a \in \mathrm{GL}(n,\mathbb{C}) \mid (\forall z, w \in \mathbb{C}^n)\ \beta(a(z), a(w)) = \beta(z,w) \text{ (not simple)}$$

$$\mathrm{SU}(p,q) = \{a \in \mathrm{U}(p,q) \mid \det a = 1\} \text{ simple}\,.$$

- Here $K = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{matrix} A \in \mathrm{U}(p), B \in \mathrm{U}(q) \\ \det A \det B = 1 \end{matrix} \right\}$.

• We make $\mathbf{X}$ into a Riemannian manifold by fixing a K-invariant inner product $(\cdot\,,\,\cdot)$ on $\mathfrak{p}$ and then translating to other points

$$(u,v)_{a\cdot x_o} = (d\ell_{a^{-1}}u, d\ell_{a^{-1}}u)$$

where $\ell_a(x) = a \cdot x$. Well defined because of the K-invariance of $(\cdot\,,\,\cdot)$.

If G is linear then we can simply take $(X,Y) = \mathrm{Tr} XY^*$.

• $\mathbf{X}$ is said to be a hermitian symmetric space or a bounded symmetric domain if if is diffeomorphic to a bounded complex domain $\mathbf{D} \subset \mathbb{C}^d$ invariant under multiplication by $|z| \leq 1$. We can then assume that $x_o \longleftrightarrow 0 \in \mathbf{D}$.

Theorem

If $G/K = \mathbf{D}$ is a bounded domain, then the connected component of the group of holomorphic automorphism of $\mathbf{D}$ is locally isomorphic to G.

Example 1: $\mathbf{D} = \{z \in \mathbb{C} \mid |z| < 1\}$

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ and $z \in \mathbf{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ define

$$A \cdot z = \frac{az + b}{cz + d}\,.$$

Then $\Gamma = \{\gamma \in \mathrm{SL}(2, \mathbb{C}) \mid \gamma \cdot \mathbf{D} \subseteq \mathbf{D}\}$ is a semigroup and $\Gamma \cap \Gamma^{-1} = \mathrm{SU}(1, 1)$. The stabilizer of 0 in $\mathrm{SU}(1, 1)$ is

$$K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) = \left\{ k_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{-it} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Thus

$$\mathbf{D} = \mathrm{SU}(1, 1)/K \simeq \mathbb{R} + i\mathbb{R}^+ \simeq \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)\,.$$

The geodesic reflection around 0 is simply $z \mapsto -z$.

Important Submanifolds/orbits

- Circles: The orbits of K are 0 and circles centered at the origin:

$$C_r = \{e^{2i\theta} r \mid \theta \in \mathbb{R}\} \quad r > 0$$

- Geodesics through: The orbits of K-conjugates of $A = \left\{ a_t = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \;\middle|\; t \in \right\}$ are given by

$$\gamma_\theta = e^{2i\theta}\{\gamma(t) = a_t \cdot 0 = e^{2i\theta}\tanh(t) \mid t \in \mathbb{R}\} \to e^{2i\theta}, \quad t \to \pm\infty$$

are geodesics through zero meeting the boundary at the points $\pm e^{2i\theta}$. Note in particular $\gamma_0 = (-1, 1) = \mathbf{D}^\tau$, $\tau(z) = \bar{z}$.

Example-2

- Horocycles: Let $N = \left\{ n_x = \begin{pmatrix} 1+ix & -ix \\ -ix & 1-ix \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$. The orbits through $r = \tanh(t)$

$$\xi(t,0) = \left\{ n_x \cdot r = \frac{r+(r-1)ix}{1+(r-1)ix} \;\middle|\; x \in \mathbb{R} \right\}$$

are the horocycles meeting the boundary tangentially at the point 1.

- The other horocicles are obtained by rotating those by $e^{2i\theta}$:

$$\xi(t,\theta) = k_\theta \xi(t,0)$$

$e^{2i\theta}$ is sometimes called the normal and t the signed distance from the origin.

Totally real sub-manifolds

Definition

Let $\mathcal{M}$ be a complex manifold and $\mathcal{N}$ a real sub-manifold. Then $\mathcal{N}$ is a totally real sub-manifold if locally we can find coordinates such that the embedding $\mathcal{N} \hookrightarrow \mathcal{M}$ looks like $\mathbb{R}^d \hookrightarrow \mathbb{C}^d$.

- Denote by $\mathcal{O}(\mathcal{M})$ the space of holomorphic functions on $\mathcal{M}$. The importance of totally real sub-manifolds is that $\mathcal{O}(\mathcal{M}) \to C^\infty(\mathcal{N})$, $F \mapsto F|_{\mathcal{N}}$ is injective..

Theorem

The above orbits in $\mathbf{D}$ *except the trivial* K*-orbit* 0 *are totally real submanifolds.*

The unit ball

- $\mathbf{D} = \mathbf{B}_1(0) = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$, the unit ball in $\mathbb{C}^n$. Then $\mathbf{D} = \mathrm{SU}(n,1)/K$ where

$$K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1)) = \left\{ k_A = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{T}\,,\, A \in \mathrm{U}(n) \\ \det k_A = a \det A = 1 \end{array} \right\}.$$

- To determine the action of $\mathrm{SU}(n,1)$ on $\mathbf{D}$ one embeds the ball into the projective space $\mathbb{P}[\mathbb{C}^{n+1}]$ by $\mathbf{z} \overset{\Phi}{\mapsto} [\mathbf{z}, 1]$. The image is exactly the open sub manifold

$$\mathbb{P}[\mathbb{C}^{n+1}]_+ = \{[\mathbf{w}] \mid \beta_{n,1}(\mathbf{w}, \mathbf{w}) > 0\}.$$

The action of $\mathrm{SU}(n,1)$

- $\mathrm{SU}(n,1)$ acts linearly on $\mathbb{P}_+$ and we set

$$A \cdot \mathbf{z} = \Phi^{-1}([A(z,1)^T]) .$$

Thus, with $g = g(A, v, w) = \begin{pmatrix} A & v \\ w^t & a \end{pmatrix}$ and $a \cdot b = \sum a_j b_j$:

$$g \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} A & Az + v \\ w^t & w \cdot z + a \end{pmatrix} = (v \cdot z + a) \begin{pmatrix} 1 \\ (Az + w)(v \cdot z + a)^{-1} \end{pmatrix} .$$

Thus the action is given by fractional linear transformations:

$$\boxed{g \cdot z = (Az + w)(v \cdot z + a)^{-1} .}$$

A totally real sub-manifold

- Let $\mathbf{B}_{\mathbb{R}} = \{z \in \mathbb{R}^n \mid \|z\| < 1\} = \{z \in \mathbf{B} \mid \bar{z} = z\}$ be the real ball of radius one.

- On the group level we define the involution

$$\tau : \mathrm{SU}(n,1) \to \mathrm{SU}(n,1)\,, \quad \tau(A) = \overline{A}$$

Then $\mathrm{SO}(n,1) = \mathrm{SU}(n,1)^{\tau}$ and

$$\mathbf{B}_{\mathbb{R}} = \mathrm{SO}_o(n,1) \cdot 0 = \mathrm{SO}_o(n,1)/\mathrm{SO}(n)$$

is a totally real symmetric sub-manifold.

Symmetric matrices

- Let $\mathbf{D} = \{Z \in \mathrm{M}_n(\mathbb{C}) \mid \mathrm{I} - Z^*Z > 0\}$. The group $\mathrm{SU}(n,n)$ acts transitively on $\mathbf{D}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}$$

and the stabilizer is

$$K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n)) \simeq \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \,\middle|\, A.B \in \mathrm{U}(n),\ \det A \det B = 1 \right\}.$$

More about this later.

Structure of bounded domains

Theorem

$\mathbf{X}$ is a bounded symmetric domain if and only if the center $\mathfrak{z}$ of $\mathfrak{k}$ is one dimensional. In that case, there exists an element $Z_o \in \mathfrak{k}$ such that $\mathrm{ad}Z_o : \mathfrak{p} \to \mathfrak{p}$, $X \mapsto [Z_o, X] (= Z_o X - X Z_o)$ defines a G-invariant complex structure on $\mathbf{X}$.

- We fix Z_o as above. Then $\mathrm{ad}Z_o : \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}_{\mathbb{C}}$ and has eigenvalues $\pm i$. We let

$$\mathfrak{p}^{\pm} = \{Z \in \mathfrak{p}_{\mathbb{C}} \mid \mathrm{ad}Z_o(Z) = \pm iZ\} = \mathfrak{p}_{\mathbb{C}}(\mathrm{ad}Z_o, \pm i)\,.$$

Clearly $\mathfrak{p}^{\pm}$ is an abelian Lie algebra and in fact isomorphic to

$$P^{\pm} = \exp \mathfrak{p}^{\pm} \subset G_{\mathbb{C}}\,.$$

The Harish-Chandra realiziation

Theorem

$P^+ \times K_{\mathbb{C}} \times P^- \to G_{\mathbb{C}}$, $(p^+, k, p^-) \mapsto p^+ k p^-$ is a holomorphic diffeomorphism onto an open dense subset and $G \subset P^+ K_{\mathbb{C}} P^-$.

- We denote the inverse by $z \mapsto (p^+(z), k_{\mathbb{C}}(z), p^-(z))$.

Theorem (Harish-Chandra)

The map $gK \mapsto \log p^+(g)$ induces a diffeomorphism $G/K \simeq \mathbf{D} \subset \mathfrak{p}^+$ where $\mathbf{D}$ is a bounded symmetric domain in $\mathfrak{p}^+$. The action of G on $\mathbf{D}$ is given by

$$g \cdot Z = \log(p^+(g \exp Z))\,.$$

Examples continued

For $G = \mathrm{SU}(1,1)$, then we can take

$$\mathfrak{p}^+ = \left\{ p_z^+ = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \middle| z \in \mathbb{C} \right\} \text{ and } \mathfrak{p}^- = \left\{ p_w^- = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \middle| w \in \mathbb{C} \right\}.$$

Then

$$\begin{aligned} P^+ K_{\mathbb{C}} P^- &= \left\{ p_z^+ k_\gamma p_w^- = \begin{pmatrix} * & \gamma^{-1} z \\ \gamma^{-1} w & \gamma^{-1} \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| d \neq 0, ad - bc = 1 \right\}. \end{aligned}$$

Example continued

- Hence with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $d \neq 0$:

$$p^{+}(A) = b/d, \quad k_{\mathbb{C}}(A) = 1/d \quad \text{and} \quad p^{-}(A) = c/d$$

and the action is given by

$$(A, z) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & az+b \\ c & cz+d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$$

which is the same action as before.

The unit ball

- For $\mathrm{SU}(n,1)$ one takes

$$\mathfrak{p}^+ = \left\{ p_z^+ = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \,\middle|\, z \in \mathbb{C}^n \right\} \text{ and } \mathfrak{p}^- = (\mathfrak{p}^+)^t\,.$$

Then, with $p_w^- = (p_w^+)^t$ we have for

$$g(B,z,y) = \begin{pmatrix} B & z \\ y^t & b \end{pmatrix} = p_v^+ k_A p_w^- = \begin{pmatrix} A + av\cdot w & av \\ aw^t & a \end{pmatrix}$$

if and only if

$$a = (\det A)^{-1} = b,\ \ v/a = z,\ \text{ and } v = y/a\,.$$

$\mathrm{SU}(n,1)$ continued

- In particular we have

$$\begin{pmatrix} A & v \\ w^t & a \end{pmatrix} \begin{pmatrix} I_n & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & Az+v \\ w^t & a+w\cdot z \end{pmatrix} \mapsto (Az+v)(a+w\cdot z)^{-1}$$

and we recover the previous action.

- It is also good to notice that the important quantity

$$\begin{pmatrix} I_n & 0 \\ \bar{w}^t & 1 \end{pmatrix} \begin{pmatrix} I_n & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & z \\ w^t & 1-(z,w) \end{pmatrix} \mapsto k(z,w) := 1-(z,w)$$

is also recovered by the triangular decomposition $P^+ K_{\mathbb{C}} P^-$.

The morale of the story

The point is, that functions that shows up in analysis on the unit ball, including the reproducing kernel and the factor in the invariant measure have a simple explanation in terms of the triangular decomposition

$$G \subset P^+ K_{\mathbb{C}} P^-$$

and hence can be generalized to arbitrary symmetric bounded domains.

PART II

BASIC REPRESENTATION THEORY

Representation Theory

• $\mathcal{S}$ will always denote a Fréchet space. Denote by $\mathrm{GL}(\mathcal{S})$ the space of continuous linear maps $\mathcal{S} \to \mathcal{S}$ with continuous inverse. A representation π of G is a map $\pi : G \to \mathrm{GL}(\mathcal{S})$ such that for all $u \in \mathcal{S}$ the orbit map

$$G \to \mathcal{S}\,, \quad g \mapsto \hat{u}(g) := \pi(g)u \quad \text{is continuous}$$

• Let $\mathcal{S}^*$ denote the conjugate linear dual of $\mathcal{S}$. Define a continuous representation of G on $\mathcal{S}^*$ by

$$\langle \pi^*(g)\varphi, u\rangle := \langle \varphi, \pi(g^{-1})u\rangle\,.$$

We assume that $\mathcal{S} \hookrightarrow \mathcal{S}^*$ (continuously) with a dense image and such that $\pi^*(g)|_{\mathcal{S}} = \pi(g)$.

The Voice transform

- For a fixed non-zero u the linear map $V_u : \mathcal{S}^* \to C(G)$

$$V_u(\varphi)(g) := \langle \varphi, \pi(g)u \rangle$$

is a G-intertwining operator,

$$V_u(\pi^*(g)\varphi)(x) = L_g[V_u(\varphi)](x) = V_u(\varphi)(g^{-1}x) \, .$$

It is sometimes called the Voice transform(lots of other names).

Definition

1) The representation $(\pi, \mathcal{S})$ is (topologically) irreducible if there are no closed invariant subspaces except the trivial one $\{0\}$ and $\mathcal{S}$.

2) $u \in \mathcal{S}$ is cyclic if

$$\mathcal{S} = \text{closure of}\{\sum_{\text{finite}} c_j \pi(g_j)u \mid g_j \in G, \ c_j \in \mathbb{C}\} = \overline{< \pi(G)u >}$$

3) $u \in \mathcal{S}$ is weakly cyclic if $\langle \varphi, \pi(g)u \rangle = 0$ for all $g \in G$ implies $\varphi = 0$.

Example: Windowed Fourier Transform

- A typical example is the action of the (reduced) Heisenberg group on the space of rapidly decreasing functions $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}^*$ the space of (anti)-linear tempered distributions.

$$\pi(x, y, t)f(z) = te^{iy \cdot z} f(z - x) .$$

Here the voice transform is given

$$V_u(f)(x, y, 1) = \int_{\mathbb{R}^n} f(z)\overline{u(x - z)}e^{-iy \cdot z}\, dz$$

is the windowed Fourier transform also called short time Fourier transform or Gabor transform.

Unitary representations

• One of the most important example are the unitary representations. In this the space on which π acts is a Hilbert space $\mathcal{H}$ and $\pi(a)$ is unitary for all $a \in G$. We have $\mathcal{H} = \mathcal{H}^*$ via the $u \mapsto \varphi_u$; $v \mapsto (u, v) = \varphi_u(v)$. Furthermore $\pi^* = \pi$.

• This situation gives rise to several examples, in particular, if $\mathcal{S} \subset \mathcal{H}$ is a $\pi(G)$-invariant and dense Fréchet space, with continuous inclusion, and such that $g \mapsto \pi(g)|_{\mathcal{S}}$ is strongly continuous, then we have a Gelfand tripple

$$\mathcal{S} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{S}^*$$

which explains why we use conjugate linear dual. The representation π^* is an extension of the initial representation π.

Smooth Vectors

- A way to get a Fréchet space from an unitary representation is to use $\mathcal{S} = \mathcal{H}^\infty$ the space of smooth vectors: Let $(\pi, \mathcal{H})$ be a representation. A vector $u \in \mathcal{H}$ is smooth if the function

$$\widetilde{u} : \mathfrak{g} \to \mathcal{H}, \quad \widetilde{u}(X) = \pi(\exp X)u = \hat{u}(\exp X) \text{ is smooth.}$$

The space of smooth vectors is denoted by $\mathcal{H}^\infty$ (topology in a moment). Its conjugate linear dual is denoted by $\mathcal{H}^{-\infty}$. The elements of $\mathcal{H}^{-\infty}$ are called distribution vectors. One can also replace "smooth" by "analytic" to get the spaces $\mathcal{H}^\omega \subseteq \mathcal{H}^{-\omega}$ of analytic/hyperfunction vectors.

- Representation π^∞ of $\mathfrak{g}$ on $\mathcal{H}^\infty$ by

$$\pi^\infty(X)u = \lim_{t \to 0} \frac{\pi(\exp(tX))u - u}{t} = D_X \hat{u}(0).$$

The Topology

- Fix a basis $X_1, \ldots, X_n \in \mathfrak{g}$. For $\alpha \in \mathbb{N}_0^n$:

$$D^\alpha = \pi(X^\alpha) := D_{X_1}^{\alpha_1} \cdots D_{X_n}^{\alpha_n} .$$

Assume for simplicity that $\mathcal{H}$ is a Hilbert space and π is unitary. Then we define a family of semi norms by $p_\alpha(u) = \|D^\alpha u\|$. Then $\mathcal{S} = \mathcal{H}^\infty$ is a Fréchet space invariant under G and $\mathfrak{g}$ and the G action is continuous (in fact smooth).

Example

If $\mathcal{H} = L^2(\mathbf{X})$ then $\mathcal{H}^\infty$ is the space of smooth functions such that $D^\alpha f \in L^2(\mathbf{X})$ for all α. If $\mathcal{H} = L^2(\mathbb{R}^n)$ and G is the Heisenberg group. Then $L^2(\mathbb{R}^n)^\infty$ is $\mathcal{S}(\mathbb{R}^n)$.

p-integrable Representations, $1 \leq p < \infty$

Definition

An (irreducible) unitary representation $(\pi, \mathcal{H})$ is p-integrable if there exists a non-zero $u \in \mathcal{H}$ such that $V_u(u) \in L^p(G)$.

- If π is irreducible and u is L^p, then

$$\sum_{\text{finite}} c_j V_u(\pi(g_j)u) = V_u\left(\sum_{\text{finite}} c_j \pi(g_j)u\right) \in L^p(G)$$

so the space $\{v \in \mathcal{H} \mid V_u(v) \in L^p(G)\}$ is G-invariant and hence dense.

- If $\|u\|, \|v\| = 1$, $1 \leq p \leq 2$ then $|(v, \pi(g)u)|^2 \leq |(v, \pi(g)u)|^p \leq 1$. Hence L^p $(1 \leq p \leq 2)$ implies L^2 and the closure in the norm $\|v\|_p = \|V_u\|_{L^p}$ is in L^2. Hence the space of p-integrable vectors is a Banach subspace of $\mathcal{H}$ if $(\pi, \mathcal{H})$ is square integrable.

For square integrable representations we can take as $\mathcal{S}$ the space of p-integrable vectors or the intersection over all $1 < p \leq 2$. Assuming that those spaces are non-zero.

Compact Groups

- If G is compact, then every irreducible representation is square integrable and we also have for two such representation

Orthogonality Relations

$$\int_G (v, \pi(g)u)\overline{(z, \tau(g)w)}\, dg = \frac{\delta_{[\pi][\tau]}}{\dim V_\pi}(v, z)(w, u)\,.$$

or

$$(V_u^\pi(v), V_w^\tau(z))_{L^2} = \frac{\delta_{[\pi][\tau]}}{\dim V_\pi}(v, z)(w, u)\,.$$

Square Integrable Representations

This gives the reproducing formula: Assume $\|u\| = \sqrt{\dim V_\pi}$:

$$\begin{aligned} V_u(v) * V_u(u)(a) &= \int_G (v, \pi(b)u)(u, \pi(b^{-1}a)u)\, db \\ &= \int_G (v, \pi(b)u)\overline{(\pi(a)u, \pi(b)u)}\, db \\ &= \frac{(u,u)}{\dim V_\pi}\, (v, \pi(a)u) \\ &= V_u(v)(a)\,. \end{aligned}$$

- This leads to the theorem:

Theorem

$f * V_u(u) = \begin{cases} f & \text{if} \quad f \in V_u(\mathcal{H}_\pi) \\ 0 & \text{if} \quad f \in V_u(\mathcal{H}_\pi)^\perp \end{cases}$ Thus the convolution operator $f \mapsto f * V_u(u)$ is the orthogonal projection onto $\mathrm{Im}\, V_u \subset L^2(G)$.

Square integrable representations

Those formulas are still correct for irreducible square integrable representations, where now one has to replace the dimension by the formal dimension. In particular, one can find $u \in \mathcal{H}_\pi$ such that

$$V_u(u) * V_u(u) = V_u(u)$$

and

$$f \mapsto f * V_u(u)$$

is a orthogonal projection onto the space $\mathrm{Im} V_u \simeq_G \mathcal{H}_\pi$.

PART III

THE BERGMAN SPACES
AND
CONNECTION TO REPRESENTATION THEORY
HIGHEST WEIGHT REPRESENTATION

The invariant measure on $\mathbf{X}$

- We start by discussing the case of the unit ball in $\mathbb{C}^n$ and then discuss the general case.
- Let $\mathbf{D} = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1)) = \mathbb{B}^n$.
- Let $d\mu(\mathbf{z})$ be the measure $c_n(1-\|\mathbf{z}\|^2)^{-(n+1)}d\mathbf{z}$.

The measure μ is G-invariant: For all $f \in L^2(\mathbf{X},\mu)$ and $a \in G$ we have

$$\int f(a\cdot\mathbf{z})\,d\mu(\mathbf{z}) = \int f d\mu\,.$$

We simply write $L^2(\mathbf{X})$ for $L^2(\mathbf{X},\mu)$

Bergman spaces

For $\alpha > -1$ let

$$d\mu_\alpha(z) = c_\alpha(1 - \|\mathbf{z}\|^2)^\alpha d\mathbf{z} = c_\alpha(1 - \|\mathbf{z}|^2)^{\alpha+n+1}\, d\mu(\mathbf{z})\,.$$

were c_α is so that $\int d\mu_\alpha = 1$.

$$\text{Define } \|f\|_{p,\alpha} := \left(\int_{\mathbf{X}} |f(\mathbf{z})|^p\, d\mu_\alpha(\mathbf{z})\right)^{1/p} \text{ and } L^p_\alpha(\mathbf{X}) = L^p(\mathbf{X}, \mu_\alpha)$$

the weighted L^p-spaces. Then define

$$\mathbb{A}^p_\alpha(\mathbf{X}) = \mathcal{O}(\mathbf{X}) \cap L^p_\alpha(\mathbf{X})\,.$$

As the measure μ_α is finite it follows that for $1 \leq p < q$: $\mathbb{A}^q_\alpha \subseteq \mathbb{A}^p_\alpha \subseteq \mathbb{A}^\alpha_1$ and the embeddings are continuous.

Reproducing Formula

Definition

A Banach space **B** of function on a topological space **M** is a reproducing kernel Banach space if for each compact set $L \subset \mathbf{M}$ there exists a $C_L > 0$ such that for all $F \in \mathbf{B}$ we have

$$|f(x)| \leq C_L \|f\|_{\mathbf{B}} \quad \text{for all } x \in L\,.$$

- **Remark:** Sometimes it is only required that the point evaluation map is continuous.

The reproducing kernel

Theorem

$|F(z)| \leq \frac{\|F\|_{p,\alpha}}{(1-\|\mathbf{z}\|^2)^{(n+1+\alpha)/p}}$ *for all* $F \in \mathbb{A}^p_\alpha$ *and* $\mathbf{z} \in \mathbf{X}$.

Theorem

Let $K_{\alpha,\mathbf{w}}(\mathbf{z}) = K_\alpha(\mathbf{z},\mathbf{w}) = \frac{1}{(1-(z,w))^{n+1+\alpha}}$. *Then for all* $F \in \mathbb{A}^p_\alpha$

$$F(\mathbf{z}) = \int_{\mathbf{X}} f(\mathbf{w}) K_\alpha(\mathbf{z},\mathbf{w})\, d\mu_\alpha(\mathbf{w})$$

Theorem

The Bergman projection is continuous if and only if $1 < p < \infty$.

Special cases $p = 2$

• $p = 2$ reproducing kernel Hilbert space. It is the irreducible holomorphic discrete series representation or part of the highest weight representations with the action

$$\pi_\alpha(g)f(z) = (a + w \cdot z)^{-\alpha-n-1} f(g^{-1} \cdot \mathbf{z}) \text{ if } g^{-1} = \begin{pmatrix} A & v \\ w^t & a \end{pmatrix} .$$

• Note that this is well defined for all $F \in L^2_\alpha(\mathbf{X})$ and leaves the spade of holomorphic functions invariant.

• Note also that strictly speaking this is not a representation of G because we need to take the α-rooth of the factor $a + w \cdot z$. If α is rational then we only need a finite covering of G. If α is irrational one needs to go to the universal covering. We will ignore this in the following.

Irreducible

Theorem

The representation π_α is unitary and irreducible. Furthermore, it is square integrable for $\alpha > -1$ and integrable for $\alpha > n - 1$.

- The special case $d\mu_\alpha = d\mathbf{z}$ is the Bargmann space of holomorphic functions on $\mathbf{X}$ such that $\int |F(\mathbf{z})|^2 \, d\mathbf{z} < \infty$.

The general case

- I will not spend much time on the general case, just to make clear one can define Bergman space for all bonded symmetric space. In short it goes the following way:

- The group $K_{\mathbb{C}}$ acts on $\mathfrak{p}^+$ by conjugation. Define a homomorphism $\chi_{2\rho_n}(k) = \det \mathrm{Ad}(k)|_{\mathfrak{p}^+}$. It is of the form

$$\chi_{2\rho}(\exp tZ_0) = \exp(-2i\rho_n)t), t \in \mathbb{R}$$

where $2\rho_n$ is some number. Any other character is of $K_{\mathbb{C}}$ is of the form

$$\chi_\nu(k) = \chi(k)^{\nu/2\rho_n}$$

- Fix a G-invariant measure μ on $\mathbf{X}$. Define $J : G \times \mathbf{D} \to K_{\mathbb{C}}$ by

$$J(g, z) = k_{\mathbb{C}}(g \exp z)$$
$$j_\nu(g, z) = \chi_\nu(J(g, z)) \longleftrightarrow (a+ < w, z >)^{\text{some power}}$$

$J(g, z)$ is called the universal automorphic factor. It satisfies the co-cycle relation

$$J(ab, z) = J(a, b \cdot z) J(b, z)$$
$$\pi_\alpha(a) f(z) = j_\nu(a^{-1}, z))^{-1} f(a^{-1} \cdot z)$$

a representation and ν big enough it is unitary and corresponds to $\mathbb{A}^2_\alpha$. One also has a reproducing kernel

$$K_\nu(Z, W) = \chi_\nu(\exp(-\bar{W}) \exp Z)^{-1} .$$

and one can start to do the some things as for the ball.

The point is, all of the fact for the unit ball can be generalized to arbitrary bounded symmetric domains. It has simple explanation using the groups structure, but the analysis becomes harder.

PART III

REPRESENTATION THEORY

AND

COMMUTING FAMILIES OF TOEPLITZ OPERATORS

Joint work with M. Dawson and R. Quiroga-Barranco

Toeplitz operators

- For $\varphi \in L^\infty(\mathbf{X})$ define

$$T^\alpha_\varphi : \mathbb{A}^2_\alpha \to \mathbb{A}^2_\alpha,\ \ T^\alpha_\varphi(F)(z) = \int \varphi(w)F(w)K_\alpha(z,w)\,d\mu_\alpha(w)$$

Thus $T^\alpha_\varphi(F)$ is the orthogonal projection of φF onto $\mathbb{A}^2_\alpha$ or $T^\alpha_\varphi = P_\alpha \circ M_\varphi$ where M_φ is the multiplication operator $F \mapsto \varphi F$. T^α_φ is the Toeplitz operator with symbol φ.

Theorem

1) T^α_2 is bounded.

2) The map $\varphi \mapsto T^\alpha_\varphi$ is injective.

Question

Question: For which families $\mathcal{A}$ of symbols is the C^*-algebra generated by $\{T_\varphi^\alpha \mid \varphi \in \mathcal{A}\}$ commutative?

- It turns out that this is related to a very interesting questions in representation theory.

Restriction of representations

Definition

Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be two representation of a group G and let H be a subgroup. Then a linear map $T : \mathcal{H} \to \mathcal{K}$ is H-intertwining if for all $a \in H$ we have $T \circ \pi(a) = \rho(a) \circ T$. If $H = G$ then we say that T is an intertwining operator.

We denote by $\mathcal{I}_H(\pi, \sigma)$ the space of H-intertwining operators. If $\pi = \sigma$ then we write $\mathcal{I}_H(\pi)$ and note that $\mathcal{H}_H(\pi)$ is a $*$-algebra.

• An important question/problem in representation theory is the restriction of representations. Given a representation of G. If H is a closed subgroup, then one can view π as a representation π_H of H by restriction.

Q: How does π_H decompose as a representation of H?

Multiplicity free representations

• Fort nice groups H (of Type I) one has a unique decomposition of π_H as a discrete sum or more generally direct integral. One says that π_H is multiplicity free if every irreducible unitary representation of H occur at most once. This is equivalent to the algebra $\mathcal{I}_H(\pi)$ of H-intertwining operator being commutative. On the spectral site is is in integral of multiplication operators.

• One then uses that as a definition for the general case. A representation is multiplicity free if $\mathcal{I}(\pi)$ is commutative.

- For a closed subgroup $H \subset G$ let

$$L^p(\mathbf{X})^H = \{\varphi \in L^p(\mathbf{X}) \mid (\forall h \in H)\, \varphi(h \cdot z) = \varphi(z)\} \,.$$

- A simple calculation shows that

$$\pi_\alpha(a)(M_\varphi F)(z) = L_a\varphi(z)\pi_\alpha(a)F(z) \,.$$

The injectivity of $\varphi \mapsto T_\varphi^\alpha$ then implies:

Theorem (DÓQ-B)

Let $H \subset G$ and let $\varphi \in L^\infty(\mathbf{X})$. Then T_φ^α is a H-intertwining operators if and only if $\varphi \in L^\infty(\mathbf{X})^H$.

• For a subgroup $H \subset G$ let $\mathcal{I}_H^\alpha = \mathcal{I}_H^\alpha(\pi_\lambda|_H)$. We can now state the connection to representation theory in the following way:

Theorem

Let $\mathcal{T}$ be a family of Toeblitz operators generated by a family $\mathcal{A}$ of bounded symbols.

(1) If $H \subset G$ is a closed subgroup such that

(a) $\mathcal{A} \subset L^\infty(\mathbf{D})^H$,

(b) $\pi_\alpha|_H$ *is multiplicity free,*

then $\mathcal{T}$ is commutative. In particular, let

$$H = H_{\mathcal{A}} = \{a \in G \mid L_a\varphi = \varphi \text{ for all } \varphi \in \mathcal{A}\}.$$

If $\pi_\alpha|_H$ is multiplicity free then $\mathcal{T}$ is ablelian.

(2) Let H be as in (a). If H is compact, then $\mathcal{T}$ is commutative if and only if $\pi_\alpha|_H$ is multiplicity free.

The compact case

• There are three reasons that things works better in the compact case than in the general case:

• Every representation of a compact group is a direct sum of irreducible representations and all irreducible representation are finite dimensional. So in the compact case, L compact

$$\mathcal{A}^2_\alpha|_L = \bigoplus m_\alpha(\pi) V_\pi$$

• We can average, i.e., $T \mapsto \int_L \pi_\alpha(h) T \pi_\alpha(h^{-1})\, dh$ is a well defined projection onto the space of intertwining operators.

• A theorem by Englis: "Density of algebras generated by Toeplitz operators on Bergman spaces" which is about finite dimensional approximation.

• This gives several examples of commutative algebras of Toeblitz operators.

Definition

A subgroup $H \subset G$ is symmetric if there exists an involution $\tau : G \to G$ such that $G_o^\tau \subseteq H \subseteq G^\tau$.

Theorem (T. Kobayashi)

If H is a symmetric subgroup of G, then $\pi|_H$ is multiplicity free.

Corollary

Let $K \subset G$ be maximal abelian ($= \mathrm{S}(U(n) \times U(1))$) then the the C^*-algebra generated by K-invariant symbols (=radial symbols) is multiplicity free.

- Unit disk: K and A are symmetric: The circles and the geodesics. But N is not but it is easily seen to be multiplicity free.

The list

$\mathfrak{g}$	$\mathbf{D}^\tau$ complex	$\mathbf{D}^\tau$ totally real
$\mathfrak{su}(n,m)$ $\mathfrak{su}(2n,2m)$	$\mathfrak{s}(\mathfrak{u}(i,j)\times\mathfrak{u}(n-i,m-j))$	$\mathfrak{so}(n,m)$ $\mathfrak{sp}(n,m)$
$\mathfrak{su}(n,n)$	$\mathfrak{so}^*(2n)$ $\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})\times\mathbb{R}$
$\mathfrak{so}^*(2n)$ $\mathfrak{so}^*(4n)$	$\mathfrak{so}^*(2i)\times\mathfrak{so}^*(2(n-i))$ $\mathfrak{u}(i,n-i)$	$\mathfrak{so}(n,\mathbb{C})$ $\mathfrak{su}^*(2n)\times\mathbb{R}$
$\mathfrak{so}(2,n)$ $\mathfrak{so}(2,2n)$	$\mathfrak{so}(2,i)\times\mathfrak{so}(n-i)$ $\mathfrak{u}(1,n)$	$\mathfrak{so}(1,i)\times\mathfrak{so}(1,n-i)$
$\mathfrak{sp}(n,\mathbb{R})$ $\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{u}(i,n-i)$ $\mathfrak{sp}(i,\mathbb{R})\times\mathfrak{sp}(n-i,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})$ $\mathfrak{sp}(n,\mathbb{C})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}^*(10)\times\mathfrak{so}(2)$ $\mathfrak{so}(8,2)\times\mathfrak{so}(2)$ $\mathfrak{su}(5,1)\times\mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{su}(4,2)\times\mathfrak{su}(2)$	$\mathfrak{f}_{4(-20)}$ $\mathfrak{sp}(2,2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-14)}\times\mathfrak{so}(2)$ $\mathfrak{so}(10,2)\times\mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{so}^*(12)\times\mathfrak{su}(2)$ $\mathfrak{su}(6,2)$	$\mathfrak{e}_{6(-26)}\times\mathfrak{so}(1,1)$ $\mathfrak{su}^*(8)$

More about the involutions

- It is know that we can always assume that $\theta\tau = \tau\theta$. We then have $\tau(Z_o) = \pm Z_o$.

(a) If $\tau(Z_o) = Z_o$ then τ induces a holomorphic involution η on $\mathbf{D}$ by $\eta(a \cdot 0) = \tau(a) \cdot 0$.

(b) If $\tau(Z_o) = -Z_o$ then η is a complex conjugation and

$$\mathbf{D}^\eta = H \cdot 0 \subset \mathbf{D} \text{ is a totally real sub-manifold}$$

Example

Take τ the complex conjugation on $\mathrm{SU}(n, 1)$. Then η is the usual complex conjugation on $\mathbb{B}(0)$ and $\mathbf{D}_\mathbb{R}$ is the real ball of radius one.

The restriction map

• Assume that we have one of the bounded domains and a totally real sub manifold $\mathbf{D}_{\mathbb{R}} = H \cdot 0$. Assume that we can find a function φ such that

- For a dense subspace in $\mathbb{A}^2_\alpha$ we have $\varphi F \in L^2(\mathbf{D}_{\mathbb{R}})$.
- The map $F \mapsto \varphi F$ is H-intertwining.

• In our case that is possible. For the circles take $\varphi = 1$ and for $(-1, 1)$ $\cosh(t)^{-1}$. For all totally real symmetric space take $\varphi(h \cdot 0) = \chi_\nu(k_{\mathbb{C}}(h))$.

• Then we define a map $R : \mathbb{A}^2_\alpha \to L^2(\mathbb{D}_{\mathbb{R}})$ by

$$RF(x) = R_\alpha F(x) = \varphi(x) F(x) \ .$$

R is closed and because the polynomials are dense in $\mathcal{A}^2_\alpha$ and the compactly supported continuous functions are dense in L^2 it follows that it has a dense image. Hence $R^* : L^2 \to \mathbb{A}^2_\alpha$ is well defined, injective, closed, and with a dense image.

The restrication map continued

- Let $K_w(z) = K(z,w)$ so that $F(w) = (F, K_w)$ we get

$$R^*f(z) = (R^*f, K_z) = (f, RK_z) = \int f(x)\overline{\varphi(x)K(x,z)}\, d\mu(x)$$

Thus

$$\boxed{RR^*f(y) = \int_{H/H\cap K} f(x)\varphi(y)\overline{\varphi(x)}K(y,x)\, dx}$$

always an integral operators, in fact a convolution operator which is explicitly known. Now write

$R_\alpha^* = U_\alpha (R_\alpha R_\alpha^*)^{1/2}$ Problem: Find $(R_\alpha R_\alpha^*)^{1/2}$.

The Segal-Bargman transform

The map $U_\alpha : L^2(\mathbf{D}_\mathbb{R}, \mu_H) \to \mathbb{A}^2_\alpha$ is called the Segal-Bargman transform because if you do this game for L^2 and the Fock space you get the usual Segal-Bargman transform. If you do this for tube domains and the cone, you get the Laplace transform ($D = 1$).

Theorem (Ó-Ørated,1996)

The map $U_\alpha : L^2(\mathbf{D}_\mathbb{R}, \mu) \to \mathbb{A}^2_\alpha$ is a unitary H-isomorphism. In particular $\pi_\alpha|_H$ is isomorphic to the left regular representation of H on $L^2(H/H \cap K)$ and hence has a multiplicity one decomposition as a representation of H.

- $\pi_{\alpha,H}$ Independent of α
- Well known Helgason Fourier transform on $H/H \cap K$ to find the spectral decomposition of T_φ in this picture.

Why do this?

• Motivation for doing this comes from representation theory and analysis.

- Generalization of the “classical” Segal-Bargmann transform.
- Application The Hermite functions are image of the inverse of the Segal-Bargmann transform of the standard basis for the Fock-spaces. In several articles around 2004/2005 M. Aristidou, M. Davidson and Ó used this to study Laguerre functions on cones. One can also define special function for the bounded realization.
- Has been used in representation to study spaces of functions on the boundary (Gindikin, Krörz, Ó).

PART IV

ATOMIC DECOMPOSITION OF BERGMAN SPACES USING REPRESENTATION THEORY

COORBIT THEORY

Joint work J. Christensen and Karlheinz Gröchenig

- In this last part of my talks we discuss the use of representation theory to construct and understand Banach spaces of functions and "distributions". The underlying theory is called Cooribt Theory.

- This theory goes back the the work of Feichtinger and Gröchening around 1988. We will give a short description of their work and then discuss some generalizations mainly generalizations due to J. Christensen, partly in joint project with myself.

- The discretization part - as we use it - is mostly due to Christensen, but for sure discretization results for coorbit spaces goes back all the way to the original work of Feichtinger and Gröchening.

The Coorbit Theory-Feichtinger and Gröchenig

- The original work of Feichtinger and Gröchening used

irreducible, unitary and integrable representations of locally compact Hausdorff topological groups to define Banach spaces of functions/distributions.

- Developed a unified theory to use the group structure, covering arguments, partition of unity, . . . to give atomic decomposition.

Basic Idea-skipping details

- Let G be a LCHTG and let $(\pi, \mathcal{H})$ be an irreducible & integrable unitary representation and let $u \in \mathcal{H}$, $u \neq 0$, be such that

$$V_u(u)(x) = (u, \pi(x)u) \in L^1(G, w(x)d\mu_G(x)) = L^1_w$$

where μ_G is the (up to positive constant) unique left invariant measure on G and $w(x)$ is a submultiplicative weight function which.

- Let $\mathcal{S} = \{v \in \mathcal{H} \mid (v, \pi(\,\cdot\,)u) \in L^1_w\}$ with the norm

$$\|v\|_{1,w} = \|V_u(v)\|_{L^1_w} .$$

A Banach space.

- We have linear embeddings $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}^*$ with dense image.

The Coorbit Spaces

• A Banach space $\mathbf{Y}$ of functions on G is said to be G-invariant if for $F \in \mathbf{Y}$ and $a \in G$ we have $L_a F : x \mapsto F(a^{-1}x)$ is in $\mathbf{Y}$ and for each compact set $L \subset G$ there exists $C_L > 0$ such that

$$\sup_{a \in L} \|L_a\| \leq C_L \, .$$

• Is it solid if $f \in \mathbf{Y}$ and g measurable with $|g| \leq |f|$ implies $g \in \mathbf{Y}$.

• From now on $\mathbf{Y}$ is a G-invariant solid Banach space of functions on G Th coorbit space $\mathrm{Co}_{\pi,u}(\mathbf{Y})$ is defined by

$$\boxed{\mathrm{Co}_{\pi,u}(Y) = \{\varphi \in \mathcal{S}^* \mid V_u(\varphi) \in \mathbf{Y}\}.}$$

• $\mathrm{Co}_{\pi,u}(\mathbf{Y})$ is a Banach space with norm $\|f\| = \|V_u(f)\|_{\mathbf{Y}}$.

Reproducing formula

- The irreducibility gives the important reproducing formula which is in the center of the theory: We can normalize u such that

$$V_u(u) * V_u(u) = V_u(u) \, .$$

- This also gives a certain natural subspace of $\mathbf{Y}$. Let

$$\mathbf{Y}_u = \{F \in \mathbf{Y} \mid F * V_u(u) = F\} \, .$$

Then $\mathbf{Y}_u$ is a closed subspace of Y with continuous point evaluation and "reproducing kernel" $V_u(u)(x^{-1}y)$.

> Note there is no assumption that $V_u(u)$ is in $\mathbf{Y}_u$.

- Furthermore $V_u : \mathrm{Co}_{\pi,u}(\mathbf{Y}) \to \mathbf{Y}_u$ is a Banach space isomorphism with inverse

$$f \mapsto \pi^*(f)u = \int_G f(x)\pi^*(x)u \, du \, .$$

Example: The Schrödinger Representation

- Let $G = \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{T}$ be the (reduced) Heisenberg group again and π theSchr̈dinger representation acting on $L^2(\mathbb{R}^n)$ by translation and modulation.

- In this case the smooth vectors are exactly the rapidly decreasing functions on $\mathbb{R}^n$. Moreover, if $u \in \mathcal{S}(\mathbb{R}^n)$ then

$$V_u(f)(x,y) = \int f(t)\overline{u(t-x)}e^{-2\pi i t\cdot y}\,dt \in L^1(G)$$

as mentioned earlier. It is a simple fact that the Schr̈dinger representation is integrable. (just use Fubini on the above formula).

Modulation Spaces

$$\mathbf{Y} = L^{p,q} = \left\{ f \;\middle|\; \|f\|_{p,q} = \left(\int_{R^n}\left(\int_{\mathbb{R}^n} |f(x,y)|^q dy\right)^{p/q} dx\right)^{1/p} < \infty \right\}.$$

- The Modulation spaces $M^{p,q}$ are the coorbit spaces

$$M^{p,q} = \mathrm{Co}_{u,\text{Schrödinger}}(L^{p,q}(\mathbb{R}^n \times \mathbb{R}^n))\,.$$

Used in the theory of pseudo-differential operators and Gabor analysis.

They are independent of the window $u \in \mathcal{S}(\mathbb{R}^n)$.

Generalizations

• Since then several generalizations, refinements and other applications of representation theory. Too much to list all: Students of Feichtinger, recent article by Dahlke, F. De Mari, E. De Vito, S. Häuser, G. Steidl, G. Teschke (use $\mathcal{S}$ as the intersection over all the L^p-spaces $1 < p \leq 2$).

Also Coorbit spaces based on homogeneous spaces instead of groups. H. Führ, B. Currey, A. Mayeli just to name few.

• I will describe the one developed by J. Christensen in his Thesis and then joint articles. Idea: Replace irreducible, integrable, ... with as weak axioms as "possible". Thus Replace the Gelfand-tripples in the Feichtinger-Gröchening Theory with

Duality, a weakly cyclic representations $(\pi, \mathcal{S})$ as in the beginning of this talk and assume the reproducing formula.

The Conditions R1& R2

- Let **Y** be a left invariant Banach space of functions on G such that convergence in **Y** implies convergence in measure on G (all L^p-spaces). We assume that there exists a weakly cyclic vector $u \in \mathcal{S}$ such that

(R1) Reproducing formula $V_u(v) * V_u(u) = V_u(v)$;

(R2) $Y \times S \to \mathbb{C}\,, \quad (f, v) \mapsto \int_G f(x) V_u(v)(x^{-1})\, d\mu(x)$ is continuous;

Is more general but enough in all examples that we have considered so far.

- We now fix $\mathbf{Y}$ and u and assume (R1)-(R2) and that is enough to get the construction working as before. In particular:

Lemma

Let $\mathbf{Y}_u = \{F \in \mathbf{Y} \mid F = F * V_u(u)\}$. Then $\mathbf{Y}_u$ is closed in $\mathbf{Y}$ and has a "reproducing kernel" $K(x, y) = V_u(u)(x^{-1}y)$.

- As before define

$$\mathrm{Co}^u_{\mathcal{S}}(\mathbf{Y}) := \{\eta \in \mathcal{S}^* \mid V_u(\eta) \in Y\}.$$

Banach space with norm $\|\eta\| = \|V_u(\eta)\|_{\mathbf{Y}}$ and $\mathrm{Co}^u_{\mathcal{S}}(\mathbf{Y}) \simeq \mathbf{Y}_u$.

- The space $\mathrm{Co}^u_{\mathcal{S}}(Y)$ is called a Coorbit space.

Coorbit Spaces

Theorem (C-Ó)

Assume (R1)+(R2). Then

(1) $\mathrm{Co}_{\mathcal{S}}^{u}\mathbf{Y}$ *is a* π^**-invariant Banach space.*

(2) π^* *acts continuously on* $\mathrm{Co}_{\mathcal{S}}^{u}\mathbf{Y}$.

(3) $V_u : \mathrm{Co}_{\mathcal{S}}^{u}\mathbf{Y} \to \mathbf{Y}_u$ *is an isometric isomorphism which intertwines* π^* *and left translation.*

(4) $\mathrm{Co}_{\mathcal{S}}^{u}\mathbf{Y} = \{\pi^*(f)u \mid f \in \mathbf{Y}_u\}$.

Dependence on the analyzing vector

Theorem

Let u_1 and u_2 be two vectors satisfying (R1) and (R2). If

$$\boxed{V_{u_1}(v) * V_{u_2}(u_1) = cV_{u_2}(v)}$$

for some constant $c \neq 0$ then $\mathrm{Co}_{\mathcal{S}}^{u_1}\mathbf{Y} \simeq \mathrm{Co}_{\mathcal{S}}^{u_2}\mathbf{Y}$.

Note that this is always valid if π is an irreducible square integrable representation.

- One can also give condition on the Fréchet space $\mathcal{S}$ and $\mathcal{T}$ so that

$$\mathrm{Co}_{\mathcal{S}}^{u} Y \simeq \mathrm{Co}_{\mathcal{T}}^{v} Y\,.$$

There are also condition that imply the duality

$$\mathrm{Co}_{\mathcal{S}}^{u} Y^{*} = [\mathrm{Co}_{\mathcal{S}}^{u} Y]^{*}\,.$$

Some examples

- The Feichtinger theory with $\mathcal{S} = \mathcal{H}_1$.

- Hilbert spaces of bandlimited functions on $\mathbb{R}^n$ and commutative homogeneous spaces (Christensen+Ó). The representation is NOT irreducible but cyclic

- Besov spaces on stratified Lie groups (Christensen+Mayeli+Ó).

- Besov spaces on symmetric cones (Christensen).

- Bergman spaces on bounded domain (i.e., the unit ball in $\mathbb{C}^n$)

Why the Bergman spaces on the ball?

- First the question:

- People understand the Bergman spaces on the unite ball quite well using complex analysis, so why use coorbit theory?

- Use general theory of discretiastion of coorbit spaces to derive atomic decomposition/discretisation of the Bergman spaces.

- Understand the construction well enough to generalize it to all bounded symmetric domains, something that is less well understood using methods from complex analysis.

- So, long time goal: Bergman spaces for all bounded symmetric domains (parts well understood already). Some steps have already been taken!

• Note that this is strictly speaking not a representation of G. If α is not an integer then one has to go to a finite covering which depends on α

• Or use that G can be decomposed into KS where $K \simeq U(n)$ and S is a stratified Lie group isomorphic to $\mathbb{R}_+ \ltimes H_n$ and one can restrict the representation to S ($\leftarrow$ Besov spaces on the ball!)

• For $\mathrm{SU}(1,1)$ this is:

$$S = \left\{ g(a,b) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha = a + a^{-1} + ib \ \&\ \beta = b + 2i\sinh(t) \right\}$$

Theorem

$\pi_\alpha|_S$ *is cyclic.*

- For a analyzing vector one start by taking the simplest possible choice:

$$u_\alpha(z) = 1$$

- Then $|V_u(u)|$ is right K-invariant and hence a function on $\mathbb{B}^n$ given by

$$|V_u(u)(\mathbf{z})| = c(1 - \|\mathbf{z}\|^2)^{-\alpha/2}\,.$$

- Let $w_r(\mathbf{z}) = (1 - \|\mathbf{z}\|^2)^{r/2}$ and

$$L_r^2 = \left\{ f \;\middle|\; \|f\|_{L_r^p} = \left(\int |fw_r|^p \right)^{1/p} < \infty \right\}\,.$$

Sloppy here, because we can integrate over the full group, the smaller group S or even use the ball with the invariant measure.

The Theorem

Theorem

We have

(1) $f \mapsto f * |V_{\pi_s(X^\alpha)u}(u)| (= \int f(z) \frac{1}{(1-\|\mathbf{z}\|^2)^{\alpha/2+n+1}}\, dz = Sf(z))$ *is continuous* $L^p_r \to L^p_r$ *for* $s > r + 2/p$.

(2) *For* $1 < (s-r)p/2 < (s-1)p+1$ *we have* $\mathbb{A}^p_{(s-r)p/2} \simeq Co^u_{\mathcal{H}^\infty_s} L^p_r$. *This is correct using the group* S *and a suitable finite covering* $\tilde{G}$ *of* $\mathrm{SU}(n,1)$.

(3) $Co^u_{\mathcal{H}^\infty_s} L^p_r$ *is independent of the analyzing vector* u. *This is for the group* $\tilde{G}$.

(4) *This gives frames for the Bergman spaces and atomic decomposition.*

Final Comments

• No use of integrability, which allows us to treat the case $1 < s \leq 2$ (a bigger interval in higher dimensions).

• The last statement comes from the orthogonality relations for the square integrable representations + continuity arguments which implies that

$$V_{u_1}(f) * V_u(u_1) = cV_u(v)\,.$$

Atomic Decomposition

Definition

If B is a Banach space and $B_d(I)$ is a sequence space for some index set I, then the vectors $\phi_i \in B$ and the linear functionals a_i are said to form an atomic decomposition for B if

1. for $f \in B$ the $\|a_i(f)\|_{B^\#} \leq C\|f\|_B$,
2. for $a_i \in B_d$ the sum $f = \sum_i a_i\phi_i$ is in B and $\|f\|_B \leq C\|a_i\|_{B_d}$,
3. $f = \sum_i a_i(f)\phi_i$ with convergence in an appropriate topology on B.

Atomic decomposition

Definition

If B is a Banach space and $B_d(I)$ is a sequence space for some index set I, then the vectors $\phi_i \in B^*$ forms a frame for B if there are positive constants C_1 and C_2 such that

$$C_1\|f\|_B \leq \|\{\langle f, \phi_i\rangle\}\|_{B_d} \leq C_2\|f\|_B$$

and there is an operator $R : B_d \to B$ such that $R(\{\langle f, \phi_i\rangle\}) = f$.

Atomic Decomposition

- One reason to identify function spaces as Coorbit spaces is that well developed Theory of atomic decomposition of those spaces is available (Christensen, Dahlke, Feichtinger, Fournasier, Gröchenig, ...)

- Fix $\epsilon > 0$ and let $U = U_\epsilon = \{\exp(t_1X_1)\cdots\exp(t_kX_k) \mid |t_j| \leq \epsilon\}$.

A sequence of non-negative functions $\{\psi_i\}$ is called a *bounded uniform partition of unity* subordinate to U (or U-BUPU), if there is a sequence $\{x_j\}$ in L such that $L \subseteq \bigcup_i x_iU$ and there exists $N \in \mathbb{N}$ such that

$$\sup_i(\#\{j \mid x_iU \cap x_jU \neq \emptyset\}) \leq N$$

and $\mathrm{supp}(\psi_i) \subseteq x_iU$ and $\displaystyle\sum_i \psi_i = 1$ and note that for a given $x \in L$ this sum finite.

Discretization

- Recall the derived representation:

$$\pi(X)u = \lim_{t\to 0} \frac{\pi(\exp tX)u - u}{t}$$

and then $\pi(X^\alpha) = \pi(X_1^{\alpha_1}) \cdots \pi(X_s^{\alpha_s})u$, where $X_1, \ldots, X_s$ is a basis. As an example what one can prove:

Theorem (Christensen)

If $f \mapsto f * |V_{\pi(X^\alpha)u}(u)|$ is continuous on a solid Banach space $\mathbf{B}$ for all $|\alpha| \leq s$, then

$$Tf = \sum_j f(x_j)\psi_j * V_u(u), \quad T : Y_u \to Y_u$$

is invertible if $\{x_j\}$ well spread and sufficiently close.