$*$-Algebras generated by projections and their representations

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$*$-Algebras generated by projections and families of orthoprojections

Let $\mathcal{P}_n$ be a $*$-algebra generated by n self-adjoint idempotents: $p_1, \ldots, p_n$, $p_j^* = p_j = p_j^2$, $j = 1, \ldots, n$

A representation of $\mathcal{P}_n$ is determined by a collection P_j, $j = 1, \ldots, n$ of orthoprojections on some Hilbert space H

Our task is to describe representations of $\mathcal{P}_n$, i.e., classify n-tuples of projections.

As we will see, for $n > 2$ this problem appears too complicated, and we apply extra conditions on the set of projections, as a rule in the form of algebraic relations between the generators:

$$f_k(p_1, \ldots, p_n) = 0, \quad k = 1, \ldots, m$$

where f_k are some polynomials.

Systems of subspaces of a Hilbert space

Definition

Let $H_j \subset H$, $j = 1, \dots, n$, be closed subspaces of a Hilbert space H. We write

$$S = (H; H_1, \dots, H_n)$$

and say that S is a system of subspaces in H.

For a family of projections P_j, $j = 1, \dots, n$ define $H_j = \operatorname{Im} P_j$, then any representation of $\mathcal{P}_n$ defines a system of subspaces and vice versa, therefore,

the problem of description of systems of subspaces is equivalent to the description of representations of $\mathcal{P}_n$.

Direct sums and indecomposable systems

Definition

Let $S = (H; H_1, \dots, H_n)$,
$S' = (H'; H'_1, \dots, H'_n)$, $S'' = (H'', H''_1, \dots, H''_n)$
be systems of subspaces. S is a direct sum of S' and S'',
$S = S' \oplus S''$ if $H = H' \oplus H''$ and $H_j = H'_j \oplus H''_j$, $j = 1, \dots, n$.

Definition

System S is indecomposable if it cannot be represented as a non-trivial direct sum of system of subspaces.

Theorem

System S is indecomposable if and only if the corresponding representation of $\mathcal{P}_n$ is irreducible.

Unitary equivalent systems

Definition

Let $S' = (H'; H'_1, \dots, H'_n)$, $S'' = (H'', H''_1, \dots, H''_n)$ be systems of subspaces. S' is unitary equivalent to S'' if there exists a unitary operator $U\colon H' \to H''$ such that $H''_j = UH'_j$, $j = 1, \dots, n$.

Systems of subspaces, S' and S'' are unitary equivalent iff the corresponding representations of $\mathcal{P}_n$ are unitary equivalent.

Our task is to classify indecomposable systems of subspaces up to unitary equivalence = classify irreducible representations of $\mathcal{P}_n$ up to unitary equivalence.
Below we assume that $H_1 + \cdots + H_n$ is dense in H (otherwise it has a trivial component as a direct summand)

∗-Tame and ∗-wild problems

In representation theory, some problems have nice explicit solution, while other ones are extremely complicated. E.g., any orthoprojection P up to unitary equivalence is uniquely determined by the dimension and co-dimension of P. On the other hand, there is no satisfactory description for a pair of bounded self-adjoint operators A, B in a Hilbert space H. Moreover, the latter problem contains a subproblem of desctiption of any collections of finite or even countable number of self-adjoint operators.

∗-Finite problem: there exist only finitely many unitary inequivalent irreducible representations.
∗-Tame problem: one can present an explicit list of all, up to unitary equivalence, irreducible representations.
∗-Wild problem: the problem contains the description of pairs of self-adjoint operators.

Example of $*$-wild problem

Theorem

Description, up to unitary equivalnece, of all pairs (P, Q) of idempotents in a Hilbert space H is a $*$-wild problem.

Proof.

Let A, B be bounded self-adjoint operators in H', let $H = H' \oplus H'$. Consider the idempotents in H of the form

$$P = \begin{pmatrix} I & A + iB \\ 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$$

Then the pair (P, Q) in H is irreducible iff the pair (A, B) is irreducible in H'. Two pairs of such form, (P, Q), and (P', Q') are unitary equivalent in H iff the corresponding pairs (A, B) and (A', B') are unitary equivalent in H'. □

Single projection

Description of representations of $\mathcal{P}_1$ is $*$-finite problem.

Any representations of $\mathcal{P}_1$ is determined by a single projection P which is uniquely determined by dimension and co-dimension of its image $\operatorname{Im} P$.

All irreducible representations are one-dimensional:

- $H = \mathbb{C}$, $P = 0$,
- $H = \mathbb{C}$, $P = 1$.

For any projection P, the space H can be uniquely decomposed into invariant w.r.t. P direct sum $H = H_0 \oplus H_1$ so that $P|_{H_0} = 0$ and $P|_{H_1} = I$.

Pair of projections. Irreducible representations

The problem of unitary description of representations of $\mathcal{P}_2$ is tame.

Theorem

Any irreducible representation of $\mathcal{P}_2$ has dimension 1 or 2. All irreducible representations, up to unitary equivalence, are the following.

- *Four one-dimensional, $H = \mathbb{C}$, $P_1, P_2 \in \{0, 1\}$.*
- *One-parameter series of two-dimensional, $H = \mathbb{C}^2$,*

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix},$$

$0 < c < 1$, $s = \sqrt{1 - c^2}$ *(general position representations).*

Collections of projections and systems of subspaces
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Collections of projections with fixed sets of angles
Collections of projections with orthoscalarity conditions
Basic concepts
Families of projections without extra conditions
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Pair of projections. Structure theorem

Theorem

Let P_1, P_2 be projections in a Hilbert space H. Then H uniquely decomposes into direct sum of invariant w.r.t. P_1 and P_2 subspaces,
$H = H_{00} \oplus H_{01} \oplus H_{10} \oplus H_{11} \oplus \mathbb{C}^2 \otimes H_+$,
so that in H_{jk} $P_1 = jI$, $P_2 = kI$, $j, k \in \{0, 1\}$, and in $\mathbb{C}^2 \otimes H_+$

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where C is a self-adjoint operator in H_+, $0 < C < I$, $S = \sqrt{1 - C^2}$.

We say that the projections P_1, P_2 are in general position if $H_{01} = H_{10} = H_{11} = 0$.

Vasyl Ostrovskyi ∗-Algebras generated by projections and their representations

Pair of projections. Case of point spectrum

In the case where the operator C in H_+ has point spectrum, i.e. $C = \sum_k c_k E_k$, E_k are spectral projections of C, one can easily obtain that in $\mathbb{C}^2 \otimes H_+ = \oplus_k (\mathbb{C}^2 \otimes H_{+,k})$ the projections have the form

$$P_1 = \oplus_k \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \oplus_k \begin{pmatrix} c_k^2 I_k & c_k s_k I_k \\ c_k s_k I_k & s_k^2 I_k \end{pmatrix},$$

where c_k, s_k are eigenvalues of C, S, and I_k is the identity operator in the eigenspace $H_{+,k}$

Pair of projections. Angles between subspaces

Given a general position pair of projections in $\mathbb{C}^2$,

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix},$$

the image of P_1 is spanned by the vector $v_1 = (1, 0)$ and the image of P_2 is spanned by the vector $v_2 = (c, s)$, thus $c = \cos\phi$, where ϕ is the angle between v_1 and v_2

The structure theorem states that the general position part splits into (discrete of continuous) direct sum of invariant 2-dimensional planes, such that intersection of each of the subspaces, H_1, H_2 with any plane is a line, with the angle between these lines determined by the corresponding point of $\sigma(C)$.

Angles between subspaces (continued)

Definition

We say that angles between subspaces H_1, H_2 are in set $\{\phi_1, \dots, \phi_m\}$ if the corresponding projections P_1 and P_2 are in general position and $\sigma(C) \subset \{\cos\phi_1, \dots, \cos\phi_m\}$.

If there is only one angle between H_1 and H_2, i.e. $\sigma(C) = \tau \in (0,1)$, then

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \tau^2 I & \tau\sqrt{1-\tau^2} I \\ \tau\sqrt{1-\tau^2} I & (1-\tau^2) I \end{pmatrix},$$

and the projections satisfy

$$P_1P_2P_1 = \tau^2 P_1, \quad P_2P_1P_2 = \tau^2 P_2. \tag{1}$$

Conversely, (1) implies that P_1 and P_2 are in general position and $\sigma(C) = \tau$.

Angles between subspaces (continued)

Also, P_1 and P_2 are in general position with angles in $\{\phi_1, \dots, \phi_m\}$ iff

$$\prod_{k=1}^{m}(P_1P_2P_1 - \tau_k^2 P_1) = 0, \quad \prod_{k=1}^{m}(P_2P_1P_2 - \tau_k^2 P_2) = 0.$$

where $\tau_k = \cos\phi_k$, $k = 1, \dots, m$.
Algebras generated by families of projections with relations of such sort were introduced and studied in [N.Popova, A.Strelets, Yu.Samoilenko, 2007-2009]

Triples of projections. Wildness

Theorem (S.A.Kruglyak, Yu.S.Samoilenko, 1980)

The problem of unitary description of representations of $\mathcal{P}_3$ is $*$-wild.

To prove this, one can take a pair (U, V) of unitary operators in H and explicitly construct three projections, P_1, P_2, P_3 in $\tilde{H} = \mathbb{C}^4 \otimes H$ as block matrices whose matix units are expressed via U and V in such a way that the construction preserves irreducibility and unitary equivalence.

Collections of projections and systems of subspaces
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Triples of projections. Continued

Similarly, one has the following.

Theorem (S.A.Kruglyak, Yu.S.Samoilenko, 1980)

The problem of unitary classification of triples of projections, P_1, P_2, P_3, such that $P_1 P_2 = 0$, is $*$-wild.

Proposition (I.Feschenko, A.Strelets, 2012)

- *Let P_1, P_2, P_3 be projections such that $P_1 P_2 = 0$ and $P_1 + P_2 + P_3 \leq I$, then $P_1 P_3 = P_2 P_3 = 0$.*
- *The problem of unitary classification of triples of projections, P_1, P_2, P_3, such that $P_1 P_2 = 0$ and $P_1 + P_2 + P_3 \leq (1 + \epsilon) I$ for any fixed $\epsilon > 0$, is $*$-wild.*

Operator Gram matrix. Construction

Let $P_1, \dots, P_n$ be a family of projections in H, let $H_j = \operatorname{Im} P_j$, $j = 1, \dots, n$. Let $S_j \colon H_j \to H$ be isometric embeddings, so that $S_j S_j^* = P_j$, $S_j^* S_j = I_{H_j}$. Consider space $\tilde{H} = H_1 \oplus \cdots \oplus H_n$ and operator $J = (S_1, \dots, S_n) \colon \tilde{H} \to H$.

Definition (Yu.Samoilenko, A.Strelets, 2009; I.Feschenko, A.Strelets, 2012)

Operator $G = J^* J \colon \tilde{H} \to \tilde{H}$ is called operator Gram matrix, corresponding to the system of subspaces $(H; H_1, \dots, H_n)$.

Block entries of the operator Gram matrix are $(S_j^* S_k)_{j,k=1}^n$, therefore in the case where all P_j are one-dimensional projections we have $H_j = \mathbb{C}\langle e_j \rangle$, $\|e_j\| = 1$ and G is the Gram matrix of the system of vectors $(e_1, \dots, e_n)$.

Operator Gram matrix. Properties

Theorem (I.Feschenko, A.Strelets, 2012)

Operator Gram matrix possesses the following properties.

1. $G = G^*$, $G \geq 0$
2. *Diagonal entries of G are identity operators,* $G_{jj} = I_{H_j}$, $j = 1, \dots, n$.
3. $G_{jk} = 0 \iff H_j \perp H_k$.
4. *H_j and H_k are in general position with set of angles* $(\phi_1, \dots, \phi_m)$ *iff* $\sigma(G_{jk} G_{kj}) = \sigma(G_{kj} G_{jk}) \subset \{\tau_1^2, \dots, \tau_m^2\}$, $\tau_p = \cos \phi_p$, $p = 1, \dots, m$.
5. $\sum_j \alpha_j P_j = I$ *for some* $\alpha_j > 0$, $j = 1, \dots, n$, *iff* DGD *is a projection,* $D = diag(\sqrt{\alpha_1} I_{H_1}, \dots, \sqrt{\alpha_n} I_{H_n})$.

Operator Gram matrix. Properties (continued)

Let $Q_1, \dots, Q_n$ be the projections on H_j in $\tilde{H}$.

Theorem (I.Feschenko, A.Strelets, 2012)

- *Family $(P_1, \dots, P_n)$ in H is irreducible iff the family $(G, Q_1, \dots, Q_n)$ is irreducible in $\tilde{H}$.*
- *Families $(P_1, \dots, P_n)$ and $(P_1', \dots, P_n')$ are unitary equivalent iff the corresponding families $(G, Q_1, \dots, Q_n)$ and $(G', Q_1', \dots, Q_n')$ are unitary equivalent.*

Inverse construction

Above, given a family of projections $P_1, \dots, P_n$ in H, we constructed the corresponding Gram operator $G \geq 0$ together with a family of projections $Q_1, \dots, Q_n$, such that

$$\sum_{j=1}^{n} Q_j = I, \quad Q_j G Q_j = Q_j$$

and showed that they carry information about the initial family.

Assume we have projections $Q_1, \dots, Q_n$, in a Hilbert space $\tilde{H}$, $\sum_{k=1}^{n} Q_k = I$, and bounded $B \geq 0$ in $\tilde{H}$ such that $Q_j B Q_j = Q_j$, $j = 1, \dots, n$. Is it possible to counstruct a family $P_1, \dots, P_n$, for which B would be the Gram operator?

Inverse construction (continued)

Let H' be the closure of $\operatorname{Im} B$ and let $S\colon H' \to \tilde{H}$ be isometric embedding. Define $J = S^*\sqrt{B}\colon \tilde{H} \to H'$. Obviously, $J^*J = B$. Put $P_j' = JQ_jJ^*\colon H' \to H'$, $j = 1, \ldots, n$.

Theorem (I.Feschenko, A.Strelets, 2012)

- *P_j', $j = 1, \ldots, n$ are projections.*
- *Let B be the Gram operator of some family $(P_1, \ldots, P_n)$. The constructed above family $(P_1', \ldots, P_n')$ is unitary equivalent to $(P_1, \ldots, P_n)$.*
- *Let G' be the Gram operator of the constructed family P_j', $j = 1, \ldots, n$, and $Q_1', \ldots, Q_n'$ are the corresponding projections. Then the family $(B, Q_1, \ldots, Q_n)$ is unitary equivalent to $(G', Q_1', \ldots, Q_n')$.*

Posets and their representations

Additional condition: family forms a representation of a finite poset.

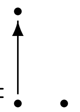
Definition

Let Γ be a finite poset, $S = (H; H_j, j \in \Gamma)$ is a representation of Γ, if $H_j \subset H_k$ as $j < k$.

For the corresponding projections we have $P_jP_k = P_j$ as $j < k$.

Posets can be depicted by Hasse diagrams: points of Γ are represented as vertices and $i \to j$ if $i < j$ and no $k \in \Gamma\colon i < k < j$.

Representations of posets. Examples

- $\Gamma = \bullet$, $S = (H; H_1)$ — $*$-tame problem.
- $\Gamma = \bullet \quad \bullet$, $S = (H; H_1, H_2)$ — $*$-tame problem (P_1, P_2).
- $\Gamma = \bullet \quad \bullet \quad \bullet$, $S = (H; H_1, H_2, H_3)$ — $*$-wild problem (P_1, P_2, P_3).
- $\Gamma =$  , $S = (H; H_1, H_2, H_3)$, $H_1 \subset H_2$ — $*$-wild problem, equivalent to $(P_1 \perp P_2, P_3)$.

Collections of projections and systems of subspaces
Representations of posets
Collections of projections with fixed sets of angles
Collections of projections with orthoscalarity conditions

Finite posets of tame and wild type

Definition

- Finite poset Γ is a chain, if it is linearly ordered, i.e. $\Gamma = a_1 < a_2 < \cdots < a_m$.
- Finite poset Γ is a semi-chain, if it has the form $\Gamma = \Gamma_1 < \Gamma_2 < \cdots < \Gamma_m$ where each Γ_j consists of one or two incomparable elements and $\Gamma_j < \Gamma_{j+1}$ means $a < b$ as $a \in \Gamma_j$, $b \in \Gamma_{j+1}$.

Theorem

Unitary description of representations of Γ is a $*$-finite problem iff Γ is a chain, $*$-tame problem iff Γ is a semi-chain. Otherwise, it is a $*$-wild problem.

Vasyl Ostrovskyi $*$-Algebras generated by projections and their representations

Description of representations for finite chains and semi-chains

Theorem

- *Let* $\Gamma = a_1 < \cdots < a_m$ *be a chain. Any irreducible representataion of* Γ *is one-dimensional and has the form* $(P_1, \ldots, P_m) = (0, \ldots, 0, 1, \ldots, 1)$.
- *Let* Γ *be a semi-chain. Any irreducible representataion of* Γ *is one- or two-dimensional and has the form:*
  - $H = \mathbb{C}$, $(P_1, \ldots, P_m) = (0, \ldots, 0, 1, \ldots, 1)$, $(P_1, \ldots, P_m) = (0, \ldots, 0, \underbrace{1, 0}, 1 \ldots, 1)$ *where underbraced are incomparable.*
  - $H = \mathbb{C}^2$, $(P_1, \ldots, P_m) = (0, \ldots, 0, R, \ldots, R, \underbrace{R, Q}, I \ldots, I)$ $R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$, $\phi \in (0, \pi/2)$, *underbraced are incomparable.*

Class of problems

Condition on the collection: each pair of subspaces are orthogonal or angles between them are in a fixed finite set.
$S = (H, H_1, \ldots, H_n)$
$T_{jk} = \{0 < \tau_{jk}^{(1)} < \cdots < \tau_{jk}^{(m_{jk})} < 1\}$, $\tau_{jk} = \cos^2\phi_{jk}$,
$j, k = 1, \ldots, n$;
to set $H_j \perp H_k$ we assume $m_{jk} = 0$ and $T_{jk} = 0$.

Problem

Describe up to a unitary equivalence irreducible systems, for which angles between H_i and H_j are in T_{ij}

Graph notation

Such systems of subspaces can be described by weighted (Coxeter) graphs. To each H_j we associate a vertex, and connect a pair of vertices j, k, $\bullet\!\overset{r_{jk}}{\text{———}}\!\bullet$ the number r_{jk} depends on the number depends on the number m_{jk} of possible angles in T_{jk} as follows:

2 — no edge: $\bullet \quad \bullet$, projections are orthogonal

3 (not written) — one angle: $\bullet\!\text{——}\!\bullet$, relations $P_iP_jP_i = \tau_{ij}P_i$, $P_jP_iP_j = \tau_{ij}P_j$

4 : $\bullet\!\overset{4}{\text{——}}\!\bullet$, relations $(P_iP_j)^2 = \tau_{ij}(P_iP_j)$, $(P_jP_i)^2 = \tau_{ij}(P_jP_i)$

5 — two angles: $\bullet\!\overset{5}{\text{——}}\!\bullet$, relations
$(P_iP_jP_i - \tau_{ij}^{(1)}P_i)(P_iP_jP_i - \tau_{ij}^{(2)}P_i) = 0$,
$(P_jP_iP_j - \tau_{ij}^{(1)}P_j)(P_jP_iP_j - \tau_{ij}^{(2)}P_j) = 0$

etc. Notice that for even numbers we obtain intermediate class of relations.

$*$-Algebras related to $(\Gamma, \mathbf{T})$, $\mathbf{T} = \{T_{ij}\}$

Define set f of polynomials (assume $f_{jk} = f_{kj}$)
$f_{jk}(x) = (x - \tau_{jk}^{(1)})\dots(x - \tau_{jk}^{(m_{jk})})$, for odd weight,
$f_{jk}(x) = x(x - \tau_{jk}^{(1)})\dots(x - \tau_{jk}^{(m_{jk})})$, for even weight.
Then the relations for the projections for odd and even weights are correspondingly
$f_{jk}(P_jP_k)P_j = f_{jk}(P_kP_j)P_k = 0$,
$f_{jk}(P_jP_k) = f_{jk}(P_kP_j) = 0$.
Such families of projections are representations of $*$-algebra

$$TL_{\Gamma,f,\perp} = \mathbb{C}\langle p_1,\dots,p_n \mid p_j^2 = p_j^* = p_j, j = 1,\dots,n \\ f_{jk}(p_jp_k)p_j^{\sigma_{jk}} = f_{jk}(p_kp_j)p_k^{\sigma_{jk}}, j \neq k\rangle$$

we call $TL_{\Gamma,f,\perp}$ the Tempreley–Lieb type algebra corresponding to Γ, f with orthogonality.

Representations of $TL_{\Gamma,f,\perp}$

Let $P_1, \dots, P_n$ be a representation of $TL_{\Gamma,f,\perp}$, and let G be the corresponding Gram matrix.
Since $\sigma(G_{jk}G_{kj}) = \sigma(G_{kj}G_{jk}) \subset T_{jk}$, $j, k = 1, \dots, n$, the condition $G \geq 0$ imposes conditions on the sets T_{jk}, $j, k = 1, \dots, n$ for a representation to exist.
For various classes and examples of graphs such conditions were studied in details.

Simple systems with orthogonality

Condition: each T_{ij} is either 0 or $\tau_{ij} < 1$

$\Gamma = (V_\Gamma, E_\Gamma)$ — simple connected graph, $V_\Gamma = \{1, \dots, n\}$,
$\tau = (\tau_{ij})_{i,j=1}^n$, $(i,j) \in E_\Gamma \iff \tau_{ij} > 0$

Theorem (N.Popova, 2001,2002; M.Vlasenko, 2004; Yu.Samolenko, A.Strelets, 2009)

If Γ is a tree, the classification of all irreps is a $*$-finite problem.
If Γ has unique cycle, the classification of all irreps is a $*$-finite or $*$-tame problem (depends on τ).
If Γ has n cycles, $n \geq 2$, there exists τ, for which the classification of all irreps is a $*$-wild problem.

Simple systems with orthogonality (continued)

Sketch of the proof

1. Γ is connected, all H_j, $j = 1, \dots, n$, have the same dimension.
2. The entries G_{jk} of the Gram matrix are $\sqrt{\tau_{jk}} U_{jk}$, U_{jk} are unitary, $j, k = 1, \dots, n$.
3. Passing to unitary equivalent system one can assume $U_{jk} = I$ for all but $e - v + 1$ operators.
4. Invariant subspace for the remaining U_{jk} gives rise to invariant subspace of the whole family.
5. For a tree, all $U_{jk} = I$, so there is at most one representation.
6. For the case of a single cycle, there can be a family of irreps parametrized by points of a circle (provided $G \geq 0$), or its subset, including single point or $\emptyset$.
7. For the case of 2 or more cycles we have two or more unitaries and $G \geq 0$ for $\tau_{ik} = \tau < (ind \Gamma)^{-2}$

Γ is a tree

As noticed above, in the case where Γ is a tree, there can be at most one representation,
and in this case $\dim ImP_j = 1$, $j = 1, \dots, n$. The Gram matrix is

$$G = \begin{pmatrix} 1 & & & \\ & \ddots & \sqrt{\tau_{ij}} & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

A representation exists iff $G \geq 0$, in this case G is a Gram matrix for some vectors $e_1, \dots, e_n$, $P_j = P_{e_j}$, $j = 1, \dots, n$.
$\dim H = n$ ($G > 0$) or $n - 1$ ($\ker G \neq \{0\}$).
In the case $\tau_{jk} = \tau$, $j, k = 1, \dots, n$, we have $G \geq 0$ iff
$I + \sqrt{\tau} A_\Gamma \geq 0$, $\tau \leq \frac{1}{(ind\Gamma)^2}$, A_Γ — adjancency matrix of Γ.

Example: All but one collections

Consider $*$-algebra $\mathcal{P}_{abo,n}$ with generators $q, p_1, \ldots, p_n$ and relations

$$q^2 = q^* = q, \quad p_j^2 = p_j^* = p_j, \quad j = 1, \ldots, n,$$
$$p_1 + \cdots + p_n = e.$$

[N.Vasilevski, 1998]
A representation of this algebra is a family of projections $P_0, P_1, \ldots, P_n$ with $P_1 + \cdots + P_n = I$. For $n \geq 2$ the problem of unitary description of all representations is $*$-wild.

All but one collections. One-dimensional projections

However, under additional condition that

$$\dim \operatorname{Im} P_j = 1, \quad j = 0, \ldots, n$$

the description of all irreducible representations is $*$-tame problem.
In the case of general position, i.e. if each pair (P_j, P_0), $j = 1, \ldots, n$ is in general position, any representation of $\mathcal{P}_{abo,n}$ is a representation of $TL_{\Gamma_{n,\tau}\perp}$, with

$$\tau_{01} + \cdots + \tau_{0n} = 1$$

Case of single cycle

If Γ has single cycle, for any irreducible representation of $TL_{\Gamma,\tau,\perp}$ we again have that

$$\dim \operatorname{Im} P_j = 1, \quad j = 1, \dots, n.$$

The correspondig Gram matrix can be reduced to the form

$$G_{\Gamma,\phi} = \begin{pmatrix} 1 & & & e^{i\phi}\sqrt{\tau_{1n}} \\ & \ddots & \sqrt{\tau_{jk}} & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \phi \in [0, 2\pi)$$

$\dim H \in \{n, n-1, n-2\}$

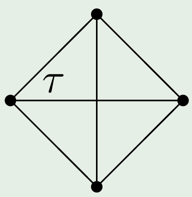
The condition $G \geq 0$ can imply further restrictions on ϕ: explicit examples show that for some $\{\tau_{jk}\}$ ϕ can be arbitrary value in $[0, 2\pi)$, for other the matrix is nonnegative as ϕ is in some segment of a circle, single point or even $\emptyset$

Case of multiple cycles

For multiple cycles, the complete description is $*$-wild problem.
A class of representations with $\dim ImP_j = 1$

$$G_\Gamma = \begin{pmatrix} 1 & & & \\ & \ddots & e^{i\phi_{jk}}\sqrt{\tau_{jk}} & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \geq 0, \quad \phi_{jk} \in [0, 2\pi)$$

Example

$P_k P_j P_k = \tau P_k$, $k \neq j = 1,\ 2,\ 3,\ 4$. The graph is K_4: 

,

The sufficient condition for $G \geq 0$ is $I + \sqrt{\tau} A_{K_4} \geq 0$
$\sigma(A_{K_4}) = \{3, -1, -1, -1\}$, so we have $\tau \leq 1/9$

Systems related to Coxeter graphs

Theorem (N.Popova, Yu.Samoilenko. A.Strelets, 2008)

- *If* Γ *is a tree, at most one edge* (j,k) *has* $r_{jk} > 3$*, then the description of irreps of* $TL_{\Gamma,f,\perp}$ *is* $*$*-finite.*
- *If* Γ *is a tree with two edges*

  $\ldots \bullet\overset{4}{—}\bullet \ldots \bullet\overset{4}{—}\bullet \ldots$

  $\ldots \bullet\overset{4}{—}\bullet \ldots \bullet\overset{5}{—}\bullet \ldots$

  $\ldots \bullet\overset{5}{—}\bullet \ldots \bullet\overset{5}{—}\bullet \ldots,$

  then the description of irreps of $TL_{\Gamma,f,\perp}$ *is* $*$*-tame.*
- *In the rest cases there exist such collections of angles that the description of irreps of* $TL_{\Gamma,f,\perp}$ *is* $*$*-wild.*

All but one revisited

Recall that a representation of $\mathcal{P}_{abo,n}$ is a family of projections $P_0, P_1, \ldots, P_n$ with $P_1 + \cdots + P_n = I$.
To extend the case of $\dim \operatorname{Im} P_j = 1$, $j = 0, \ldots, n$ discussed above, we assume that there can be m_j fixed angles between P_0 and P_j, $j = 1, \ldots, n$, i.e.

$$\prod_{k=1}^{m_j} (P_j P_0 - \tau_j^{(k)}) P_j = \prod_{k=1}^{m_j} (P_0 P_j - \tau_j^{(k)}) P_0 = 0,$$

$j = 1, \ldots, n$; here $\tau_j^{(k)} \in (0,1)$ are fixed parameters corresponding to the angles between the subspaces [I.Feschenko, A.Strelets, to appear].

All but one revisited (continued)

Let G be the Gram matrix of $(P_0, P_1, \ldots, P_n)$, and let $Q_0, Q_1, \ldots, Q_n$ be the corresponding projections defining its block decomposition.

- The condition on angles implies that each pair (P_0, P_j) is in general position, i.e. all blocks have the same dimension.
- Passing to unitary equivalent collection one can assume that all $G_{jk} \geq 0$, $j, k = 0, \ldots, n$.
- $\sigma(G_{0j}^2) \subset \{\tau_j^{(1)}, \ldots, \tau_j^{(m_j)}\}$, $j = 1, \ldots, n$.
- Since $G_{0j} = S_0^* S_j$, $P_j = S_j S_j^*$ we have

$$\sum_{j=1}^{n} G_{0j}^2 = \sum_{j=1}^{n} G_{0j} G_{j0} = \sum_{j=1}^{n} S_0^* S_j S_j^* S_0 = S_0^* \sum_{j=1}^{n} P_j S_0 = I$$

Collections of projections and systems of subspaces
Representations of posets
Collections of projections with fixed sets of angles
Collections of projections with orthoscalarity conditions
Preliminaries
Case of single angle
Case of multple angles

All but one revisited (continued)

The problem of description of collections of projections $P_0, P_1, \ldots, P_n$ with $P_1 + \cdots + P_n = I$ and

$$\prod_{k=1}^{m_j} (P_j P_0 - \tau_j^{(k)}) P_j = \prod_{k=1}^{m_j} (P_0 P_j - \tau_j^{(k)}) P_0 = 0,$$

$j = 1, \ldots, n$, is equivalent to the description of collections

$$A_j = A_j^*, \quad \sigma(A_j) \subset \{\tau_j^{(1)}, \ldots, \tau_j^{(m_j)}\}, \quad j = 1, \ldots, n$$

with

$$\sum_{j=1}^{n} A_j = I$$

Vasyl Ostrovskyi *-Algebras generated by projections and their representations

Families of operators

Object

Families $A_1, \ldots, A_n$,

$$\sum_{k=1}^{n} A_k = \gamma I,$$

$A_k = A_k^*$, $\sigma(A_k) \subset M_k$, $k = 1, \ldots, n$.

Problems

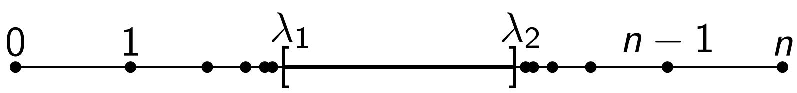
- Sets of parameters, for which a solution exists
- Structure of the operators $A_1, \ldots, A_n$

Example

Sums of projections $P_1 + \cdots + P_n = \gamma I$.
$\Sigma_n = \{\gamma \in \mathbb{R} \mid \exists\ P_1 + \cdots + P_n = \gamma I\}$

Theorem (S.Kruglyak, V.Rabanovich, Yu.Samolienko, 2002)

$\Sigma_n = \Lambda_n \cup [\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})] \cup n - \Lambda$, *where*
$\Lambda_n = \{\frac{1}{2} \operatorname{cth}(k \operatorname{arcth}(\frac{1}{2}\sqrt{n}))(n - \sqrt{n^2 - 4n}), k \in \mathbb{N}\}$



$$\lambda_{1,2} = \frac{n \pm \sqrt{n^2 - 4n}}{2}$$

Four-tuples of projections with scalar sum

Let P_1, P_2, P_3, P_4 be projections satisfying

$$P_1 + P_2 + P_3 + P_4 = \lambda I, \quad \lambda \in \mathbb{R}.$$

Theorem (V.O., Yu.Samoilenko, 1998)

- *A solution exists for* $\lambda \in \{2 \pm \frac{1}{k+s} \mid s \in \{1/2, 1\}, k = 0, 1, \dots\} \cup \{2\}$;
- *For* $\lambda = 2 \pm \frac{2}{2k+1}$, $k \geq 0$, *there exists one irrep of dimension* $2k+1$;
- *For* $\lambda = 2 \pm \frac{1}{k+1}$, $k \geq 0$, *there exists 4 irreps of dimension* $k+1$;
- *For* $\lambda = 2$ *there exist two-parametric family of irreps of dimension 2 and 6 irreps of dimension 1.*

Four-tuples of projections with scalar sum, $\lambda = 2$

Theorem (V.O., Yu.Samoilenko, 1998)

Any irreducible four-tuple of projections $P_1 + P_2 + P_3 + P_4 = 2I$ *is unitary equivalent to one of the listed four-tuples:*

- $H = \mathbb{C}$, $P_j = p_j \in \{0, 1\}$, $j = 1, 2, 3, 4$ *(total 6 irreps).*
- $H = \mathbb{C}^2$,

$$P_1 = \frac{1}{2}\begin{pmatrix} 1+a & -b-ic \\ -b+ic & 1-a \end{pmatrix}, \quad P_2 = \frac{1}{2}\begin{pmatrix} 1-a & b-ic \\ b+ic & 1+a \end{pmatrix},$$
$$P_3 = \frac{1}{2}\begin{pmatrix} 1+a & -b+ic \\ -b-ic & 1-a \end{pmatrix}, \quad P_4 = \frac{1}{2}\begin{pmatrix} 1-a & b+ic \\ b-ic & 1+a \end{pmatrix},$$

$a^2 + b^2 + c^2 = 1$, *and either* $a > 0, b > 0, c \in (-1, 1)$, *or* $a = 0, b > 0, c > 0$, *or* $a > 0, b = 0, c > 0$.

Four-tuples of projections with scalar sum, $\lambda \neq 2$

Let

$$Q_{l,m} = \frac{1}{m}\begin{pmatrix} l & \sqrt{l(m-l)} \\ \sqrt{l(m-l)} & m-l \end{pmatrix},$$

$$R_{l,m} = \frac{1}{m}\begin{pmatrix} l & -\sqrt{l(m-l)} \\ -\sqrt{l(m-l)} & m-l \end{pmatrix}, \quad 0 \le l \le m.$$

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Four-tuples of projections with scalar sum, $\lambda = 2 \pm \frac{2}{2k+1}$

Theorem (V.O., Yu.Samoilenko, 1998)

Any irreducible four-tuple of projections $P_1 + P_2 + P_3 + P_4 = 2 - \frac{2}{2k+1} I$, *is unitary equivalent to*

$$P_1 = Q_{n-1,n} \oplus Q_{n-3,n} \oplus \cdots \oplus Q_{2,n} \oplus 0,$$

$$P_2 = R_{n-1,n} \oplus R_{n-3,n} \oplus \cdots \oplus R_{2,n} \oplus 0, \quad H = \underbrace{\mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2}_{k} \oplus \mathbb{C}^1,$$

$$P_3 = 0 \oplus Q_{n-2,n} \oplus Q_{n-4,n} \oplus \cdots \oplus Q_{1,n},$$

$$P_4 = 0 \oplus R_{n-2,n} \oplus R_{n-4,n} \oplus \cdots \oplus R_{1,n}, \quad H = \mathbb{C}^1 \oplus \underbrace{\mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2}_{k},$$

Any irreducible four-tuple $P'_1 + P'_2 + P'_3 + P'_4 = 2 + \frac{2}{2k+1} I$, *is* $P'_j = I - P_j$, *where* $P_1 + P_2 + P_3 + P_4 = 2 - \frac{2}{2k+1} I$ *are as described above.*

Four-tuples of projections with scalar sum, $\lambda = 2 \pm \frac{1}{2k}$

Any irreducible four-tuple of projections
$P_1 + P_2 + P_3 + P_4 = 2 - \frac{1}{2k} I$, is unitary equivalent to

$$P_1 = 0 \oplus Q_{2k-2,4k} \oplus \dots \oplus Q_{2,4k} \oplus 0,$$
$$P_2 = 1 \oplus R_{2k-2,4k} \oplus \dots \oplus R_{2,4k} \oplus 0, \quad H = \mathbb{C}^1 \oplus \underbrace{\mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2}_{k-1} \oplus \mathbb{C}^1,$$

$$P_3 = Q_{2k-1,4k} \oplus Q_{2k-3,4k} \oplus \dots \oplus Q_{1,4k},$$
$$P_4 = R_{2k-1,4k} \oplus R_{2k-3,4k} \oplus \dots \oplus R_{1,4k}, \quad H = \underbrace{\mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2}_{k},$$

and three more reps obtained by cyclic permutations of P_j. Any irreducible four-tuple $P_1' + P_2' + P_3' + P_4' = 2 + \frac{1}{2k} I$, is $P_j' = I - P_j$, where $P_1 + P_2 + P_3 + P_4 = 2 - \frac{1}{2k} I$ are as described above.

Four-tuples of projections with scalar sum, $\lambda = 2 \pm \frac{1}{2k+1}$

Any irreducible four-tuple of projections
$P_1 + P_2 + P_3 + P_4 = 2 - \frac{1}{2k+1} I$, is unitary equivalent to

$$P_1 = 1 \oplus Q_{2k-1,4k+2} \oplus \dots \oplus Q_{1,4k+2},$$
$$P_2 = 0 \oplus R_{2k-1,4k+2} \oplus \dots \oplus R_{1,4k+2}, \quad H = \mathbb{C}^1 \oplus \underbrace{\mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2}_{k},$$

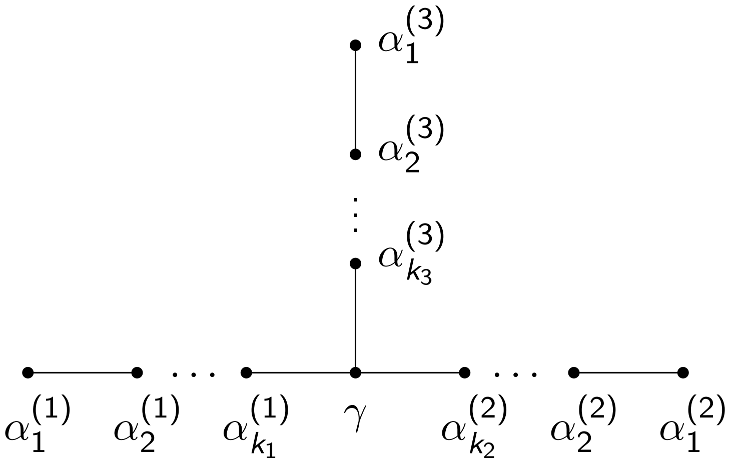
$$P_3 = Q_{2k,4k+2} \oplus \dots \oplus Q_{2,4k+2} \oplus 0,$$
$$P_4 = R_{2k,4k+2} \oplus \dots \oplus R_{2,4k+2} \oplus 0, \quad H = \underbrace{\mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2}_{k} \oplus \mathbb{C}^1.$$

and three more reps obtained by cyclic permutations of P_j. Any irreducible four-tuple $P_1' + P_2' + P_3' + P_4' = 2 + \frac{1}{2k+1} I$, is $P_j' = I - P_j$, where $P_1 + P_2 + P_3 + P_4 = 2 - \frac{1}{2k+1} I$ are as described above.

Star-shaped graphs and weights

Let Γ be a star-shaped graph. A *weight* on the graph:

$$\chi = (\alpha_1^{(1)}, \ldots, \alpha_{k_1}^{(1)}; \ldots; \alpha_1^{(n)}, \ldots, \alpha_{k_n}^{(n)}; \gamma),$$



Algebra related to a star-shaped graph and a weight

- $*$-Algebra $\mathcal{A}_{\Gamma,\chi}$ is generated by self-adjoint elements a_l, $l = 1$, $\ldots$, n, which satisfy relations

$$p_l(a_l) = 0, \quad \sum_{l=1}^{n} a_l = \gamma e,$$

  where $p_l(x) = x(x - a_1^{(l)}) \ldots (x - a_{k_l}^{(l)})$, $l = 1, \ldots, n$.
- $*$-Representations of this algebra are n-tuples $A_1, \ldots, A_n$ with $A_1 + \cdots + A_n = \gamma I$ and the spectrum of each A_l is contained in $\{0, a_1^{(l)}, \ldots, a_{k_l}^{(l)}\} = M_l$.
- *Problems*:
  — For which χ there exist $*$-representations of $\mathcal{A}_{\Gamma,\chi}$?
  — What is the structure of $*$-representations?

Locally scalar (orthoscalar) representations of a graph

Let Γ be a sipmly laced nonoriented graph.

Definition (S.A.Kruglyak, A.V.Roiter (2005))

Locally scalar representation of a graph Γ: $\Gamma_v \ni \alpha \mapsto H_\alpha$, $\Gamma_e \ni (\alpha, \beta) \mapsto A_{\alpha,\beta}\colon H_\alpha \to H_\beta$, $A_{\beta,\alpha}\colon H_\beta \to H_\alpha$ such that $A^*_{\alpha,\beta} = A_{\beta,\alpha}$ and $\forall \alpha \in \Gamma_v$

$$\sum_{\beta\colon (\alpha,\beta)\in\Gamma_e} A^*_{\alpha,\beta} A_{\alpha,\beta} = x_\alpha I.$$

$u = (x_\alpha)_{\alpha\in\Gamma_v}$ is a *character* of the representation.
Morphism is a collection of unitary operators $U_\alpha\colon H_\alpha \to H'_\alpha$ such that $\forall(\alpha, \beta) \in \Gamma_e$ $U_\beta A_{\alpha,\beta} = A'_{\alpha,\beta} U_\alpha$.

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Representations of $\mathcal{A}_{\Gamma,\chi}$ and l.s.r. of Γ

Theorem (S.A.Kruglyak, S.V.Popovych, Yu.S.Samoilenko. (2005))

The categories of non-degenerate irreducible $*$-representations of $\mathcal{A}_{\Gamma,\chi}$ and non-degenerate locally scalar representations of Γ with character u are equivalent.

Here $x_{root} = \gamma$ and $x^{(l)}_{k_l} = \alpha^{(l)}_{k_l}$, $x^{(l)}_{k_l-2} = \alpha^{(l)}_{k_l-1} - \alpha^{(l)}_1$, $x^{(l)}_{k_l-3} = \alpha^{(l)}_{k_l-1} - \alpha^{(l)}_2$, $x^{(l)}_{k_l-4} = \alpha^{(l)}_{k_l-2} - \alpha^{(l)}_2$, $x^{(l)}_{k_l-5} = \alpha^{(l)}_{k_l-2} - \alpha^{(l)}_3$, etc.

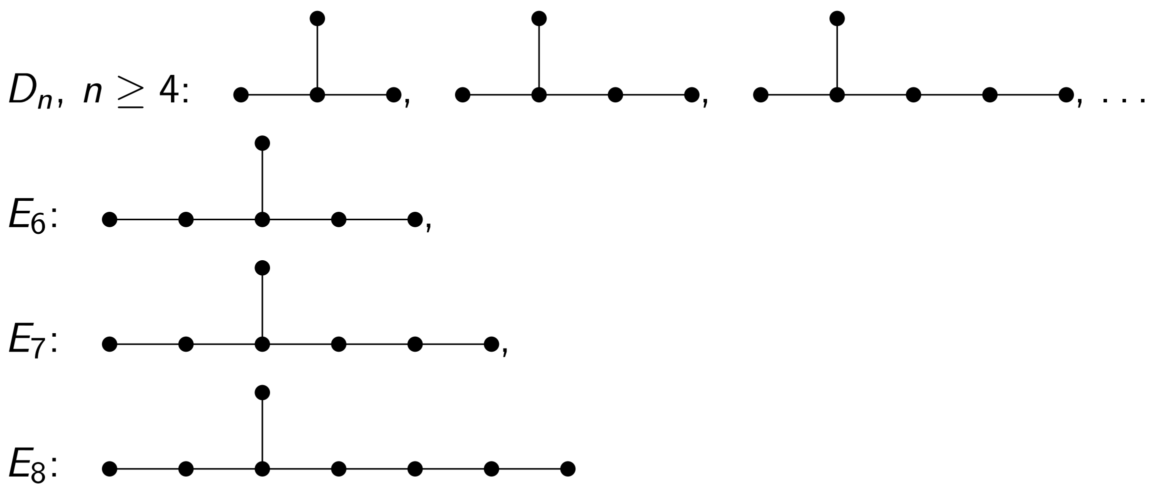
Deformed preprojective algebras

Γ star-shaped, Q related quiver, all arrows directed to the root, $\lambda = (\lambda_g)_{g \in \Gamma_v}$. The correspondig *deformed preprojective algebra* (see W.Crawley-Boevey, M.P.Holland. (1998)) is $\Pi^{\lambda}(Q) = \mathbb{C}\bar{Q}/\mathcal{J}$, $\mathcal{J}$ is generated by $\sum_{a \in Q_e}[a, a^*] - \sum_{g \in Q_v} \lambda_g e_g$; $\mathbb{C}\bar{Q}$ is the path algebra of the doubled quiver $\bar{Q}$, e_g is the idempotent corresponding to g.
Let for l-th branch λ is $(\alpha^{(l)}_{k_l - 1} - \alpha^{(l)}_{k_l}, \ldots, \alpha^{(l)}_1 - \alpha^{(l)}_2, -\alpha^{(l)}_1, \gamma)$, γ is at root.

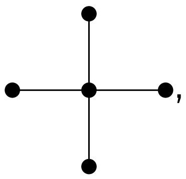
Theorem (A.S.Mellit, Yu.S.Samoilenko, M.A.Vlasenko (2005))

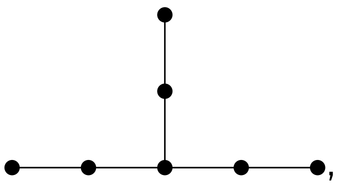
$\mathcal{A}_{\Gamma,\chi}$ is isomorphic to $e_c \Pi^{\lambda} e_c$, c is the root vertex.
$\dim \mathcal{A}_{\Gamma,\chi} < \infty \iff \dim \Pi^{\lambda} < \infty$. *$\mathcal{A}_{\Gamma,\chi}$ and Π^{λ} have the same growth order.*

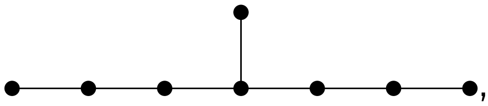
Star-shaped Dynkin graphs

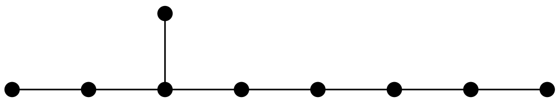


Star-shaped extended Dynkin graphs

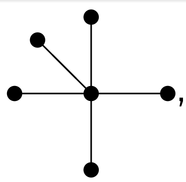
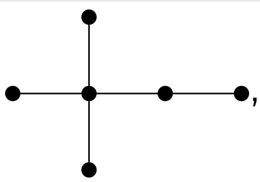
$\tilde{D}_4$: 

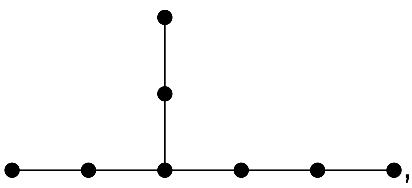
$\tilde{E}_6$: 

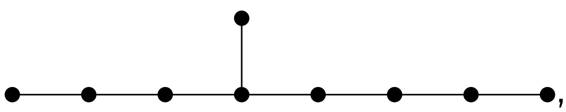
$\tilde{E}_7$: 

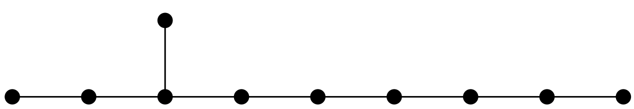
$\tilde{E}_8$: 

Star-shaped critical graphs

$\Gamma_{(1,1,1,1,1)}$:  $\Gamma_{(1,1,1,2)}$: 

$\Gamma_{(2,2,3)}$: 

$\Gamma_{(1,3,4)}$: 

$\Gamma_{(1,2,6)}$: 

Any graph which properly contains an extended Dynkin graph, contains on of the listed above graphs

Questions to study

If Γ is a Dynkin graph, the correponding algebra $\mathcal{A}_{\Gamma,\chi}$ is finite-dimensional for any χ, it has finite number of irreducible $*$-representations, and they are finite-dimensional.

- If Γ is an extended Dynkin graph, there exist χ, such that $\mathcal{A}_{\Gamma,\chi}$ has infinite series of finite-dimensional $*$-representations. Are all irreducible representations finite-dimensional for any χ?
- For $\Gamma = \Gamma_{(1,1,1,1,1)}$ and $\Gamma = \Gamma_{(1,1,1,2)}$ exist weights for which $\mathcal{A}_{\Gamma,\chi}$ has infinite-dimensional irreducible representations. Is this true for any Γ which properly contains an extended Dynkin graph?

In other terms these questions were formulated in [I.K.Redchuk, A.V.Roiter (2003)] for locally scalar representations of graphs

The answers are: Yes, Yes.

Coxeter functors

Coxeter functors
$S\colon \operatorname{Rep}\mathcal{A}_{\Gamma,\chi} \to \operatorname{Rep}\mathcal{A}_{\Gamma,\chi'}$, $\quad T\colon \operatorname{Rep}\mathcal{A}_{\Gamma,\chi} \to \operatorname{Rep}\mathcal{A}_{\Gamma,\chi''}$.
$T^2 = S^2 = \mathrm{Id}$, but $TS \neq ST$.

The action of these functors on $*$-representations gives rise to the action on the set of weights, $S\colon \chi \mapsto \chi'$, $T\colon \chi \mapsto \chi''$. So we have a dynamical system on the set of weights generated by this action.

Basic idea: Take χ, for which representations of $\mathcal{A}_{\Gamma,\chi}$ can be studied easily, apply S, T, TS, $ST, \dots$, and get a series of $\mathcal{A}_{\Gamma,\chi_k}$, for which representations can be described.

Example: Sums of projections

Coxeter functors. Construction of “linear” mapping

Let $A_1 + \cdots + A_n = \gamma I$, $\sigma(A_j) \subset \{0, \alpha_1^{(j)}, \ldots, \alpha_{k_j}^{(j)}\}$, $j = 1, \ldots, n$. Define $S\colon A_j \mapsto A'_j = \lambda_j I - A_j$, $j = 1, \ldots, n$, where $\lambda_j = \alpha_{k_j}^{(j)}$ is the maximal eigenvalue of A_j. Then

$$A'_1 + \cdots + A'_n = (\lambda_1 + \cdots + \lambda_n - \lambda)I$$

and $\sigma(A'_j) = \lambda_j - \sigma(A_j)$, $j = 1, \ldots, n$. Therefore for

$$\chi = (\alpha_1^{(1)}, \ldots, \alpha_{k_1}^{(1)}; \ldots; \alpha_1^{(n)}, \ldots, \alpha_{k_n}^{(n)}; \gamma)$$

we have

$$S\chi = (\alpha_{k_1}^{(1)} - \alpha_{k_1-1}^{(1)}, \ldots, \alpha_{k_1}^{(1)}; \ldots; \alpha_{k_n}^{(n)} - \alpha_{k_n-1}^{(n)}, \ldots, \alpha_{k_n}^{(n)}; \sum_{j=1}^{n} \alpha_{k_j}^{(j)} - \gamma)$$

Coxeter functors. Construction of “hyperbolic” mapping

Let $A_1 + \cdots + A_n = \gamma I$, $\sigma(A_j) \subset \{0, \alpha_1^{(j)}, \ldots, \alpha_{k_j}^{(j)}\}$, $j = 1, \ldots, n$, so that

$$\sum_{j=1}^{n} \sum_{m=1}^{k_j} \alpha_m^{(j)} P_m^{(j)} = \gamma I,$$

where $P_m^{(j)}$ are the spectral projections of A_j, $j = 1, \ldots, n$. Let G be the Gram matrix of $(P_1^{(1)}, \ldots, P_{k_1}^{(1)}, \ldots, P_1^{(n)}, \ldots, P_{k_n}^{(n)})$. Let

$$D = \frac{1}{\gamma} diag(\alpha_1^{(1)}, \ldots, \alpha_{k_1}^{(1)}, \ldots, \alpha_1^{(n)}, \ldots, \alpha_{k_n}^{(n)}),$$

then $D^{1/2} G D^{1/2}$ is a projection, and $I - D^{1/2} G D^{1/2}$ is also a projection.

Construction of "hyperbolic" mapping continued

Let

$$D' = \frac{1}{\gamma} diag(\gamma - \alpha_1^{(1)}, \dots, \gamma - \alpha_{k_1}^{(1)}, \dots, \gamma - \alpha_1^{(n)}, \dots, \gamma - \alpha_{k_n}^{(n)}),$$

then $(D')^{-1/2}(I - D^{1/2}GD^{1/2})(D')^{-1/2}$ is a Gram matrix of some family $(P'^{(1)}_1, \dots, P'^{(1)}_{k_1}, \dots, P'^{(n)}_1, \dots, P'^{(n)}_{k_n})$, for which

$$\sum_{j=1}^{n} \sum_{m=1}^{k_j} (\gamma - \alpha_m^{(j)}) P'^{(j)}_m = \gamma I, \quad P'^{(j)}_l P'^{(j)}_m = 0, \ l \neq m, \ j = 1, \dots, n.$$

For $TA_j = A'_j = \sum_{m=1}^{k_j} (\gamma - \alpha_m^{(j)}) P'^{(j)}_m$ we have $\sum_{j=1}^{n} A'_m = \gamma I$, and

$$T\chi = (\gamma - \alpha_{k_1}^{(1)}, \dots, \gamma - \alpha_1^{(1)}; \dots; \gamma - \alpha_{k_n}^{(n)}, \dots, \gamma - \alpha_1^{(n)}; \gamma).$$

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Evolution of weights: Case of extended Dynkin graphs. Invariant weights

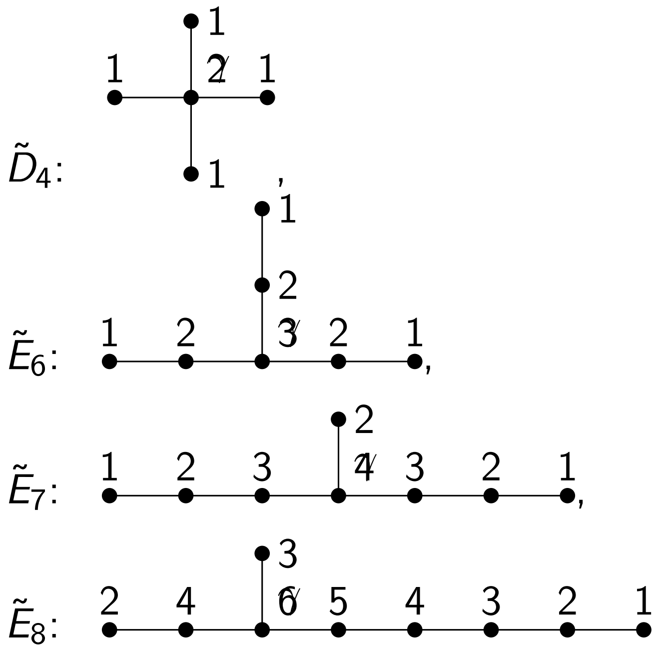
Theorem (Known fact)

If Γ is an extended Dynkin graph, there exists unique (up to a multiplier) weight χ_Γ, invariant wrt S, T.

$\chi_{\tilde{D}_4} = (1; 1; 1; 1; 2)$
$\chi_{\tilde{E}_6} = (1, 2; 1, 2; 1, 2; 3)$
$\chi_{\tilde{E}_7} = (1, 2, 3; 1, 2, 3; 2; 4)$
$\chi_{\tilde{E}_8} = (1, 2, 3, 4, 5; 2, 4; 3; 6)$

Notice that for the corresponding algebras the Coxeter functors technique is not applicable.

Evolution of weights: Case of extended Dynkin graphs



Evolution of weights: Case of extended Dynkin graphs

Let $\chi_{\Gamma,\gamma}$ be like χ_Γ but with γ at the root vertex.

Theorem

$(TS)^m \chi_{\Gamma,\gamma} = \chi_{\Gamma,\gamma'}$, where
$m = 1$ for $\tilde{D}_4$, $m = 2$ for $\tilde{E}_6$, $m = 3$ for $\tilde{E}_7$, $m = 5$ for $\tilde{E}_8$
and γ' is some number, which depends on Γ.
If a weight χ is invariant in such wide sense, it is $\chi_{\Gamma,\gamma}$ for some γ.

Theorem

Let Γ be an extended Dynkin graph. For any $k = 1, 2, \dots,$ and any χ

$$(TS)^{\omega_\Gamma(\omega_\Gamma - 1)k} \chi = \chi - k\omega_\Gamma(\omega_\Gamma - \gamma)\,\chi_\Gamma,$$

where $\omega_\Gamma = 2, 3, 4$ and 6 for $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$, γ is a value of χ at the root vertex.

Sets of parameters, for which there exist representations. Example: Families related to $\tilde{E}_6$

Theorem

$A_1 + A_2 + A_3 = \gamma I$, $\sigma(A_k) \subset \{0, 1, 2\}$ have solutions iff

$$\gamma \in W_{\tilde{E}_6} = \left\{3 \pm \frac{1}{k+s} \mid k = 0, 1, \dots; s \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}\right\} \cup \{3\}$$

Similar theorems hold for all extended Dynkin graphs.

Collections of projections and systems of subspaces
Representations of posets
Collections of projections with fixed sets of angles
Collections of projections with orthoscalarity conditions
Algebras related to graph
$*$-Representations of $\mathcal{A}_{\Gamma,\chi}$. Γ is ED graph
$*$-Representations of $\mathcal{A}_{\Gamma,\chi}$. Γ contains ED graph properly

Finiteness of dimensions of irreducible representations

Theorem (V.O. (2005))

Let Γ be an extended Dynkin graph. For any χ, all irreducible representations of $\mathcal{A}_{\Gamma,\chi}$ are finite-dimensional.

Steps of the proof

- For weights which have non-trivial evolution, we apply the Coxeter functors technique
- For limit cases (stationary or periodic cases) a polynomial identity in the algebra holds [A.Mellit (2005)]

Vasyl Ostrovskyi $*$-Algebras generated by projections and their representations

Case of graphs containing ED graph

For algebras $\mathcal{A}_{\Gamma,\chi}$ where Γ properly contains an extended Dynkin graphs, only partial examples are studied, and only partial results are known.
We obtained some results about representations of these algebras which can be used as a starting point for their systematic study.

Invariant weights. General case

For graphs which are not Dynkin graphs, we introduce χ_Γ:

$$\chi_\Gamma = (\alpha_1^{(1)}, \dots, \alpha_{k_1}^{(1)}; \dots; \alpha_1^{(n)}, \dots, \alpha_{k_1}^{(n)}; 1),$$

$$\alpha_j^{(l)} = \frac{1 + t + \dots + t^{j-1}}{1 + t + \dots + t^{k_l}},$$

where $t + t^{-1} + 2 = r^2$, r is the index of Γ.
For extended Dynkin graph such weight is unique ($t = 1$) and coincides with χ_Γ introduced for ED graphs.
For any graph containing extended Dynkin graph properly there are two special weights corresponding to two solutions for t.

Theorem

$(TS)\chi_\Gamma = t^{-1}\chi_\Gamma$.

Notice that the algebras corresponding to χ_Γ and $t\chi_\Gamma$ are $*$-isomorphic

Odd and even parts of χ_Γ and their evolution

Let χ_Γ be the invariant weight on Γ.
We decompose χ_Γ into the sum of "odd" and "even" parts, $\chi_\Gamma = \overset{\bullet}{\chi}_\Gamma + \overset{\circ}{\chi}_\Gamma$ in some special way following [I.K.Redchuk, A.V.Roiter (2003)].

Theorem (V.O. (2005))

$$(TS)^j \overset{\bullet}{\chi}_\Gamma = \frac{1}{t^j(1-t)}\left((1-t^{2j+1})\overset{\bullet}{\chi}_\Gamma + t(1-t^{2j})\overset{\circ}{\chi}_\Gamma\right),$$
$$(ST)^j \overset{\circ}{\chi}_\Gamma = \frac{1}{t^j(1-t)}\left((1-t^{2j})\overset{\bullet}{\chi}_\Gamma + (1-t^{2j+1})\overset{\circ}{\chi}_\Gamma\right),$$

Existence of infinite-dimensional representations

Theorem (S.Albeverio, V.O., Yu.Samoilenko. (2007))

Let Γ contains an extended Dynkin graph properly. There exists a weight χ such that $\mathcal{A}_{\Gamma,\chi}$ has infinite-dimensional irreducible $*$-representation.

Steps of the proof

- Consider only critical graphs. Representation can be extended to larger one
- Construct representation for $\mathcal{A}_{\Gamma,\overset{\bullet}{\chi}_\Gamma}$ and apply $(TS)^k$, $k \geq 1$, to construct representations of $\mathcal{A}_{\Gamma,\chi_k} \to \mathcal{A}_{\Gamma,\chi_\Gamma}$
- Apply Shulman's thorem [V.S.Shulman (2002)] to construct representation of $\mathcal{A}_{\Gamma,\chi_\Gamma}$
- For a finite-dimensional irreducible representation, the trace equality would hold. We show it fails.