Estimation of Equilibria in an Advertising Game with Unknown Distribution of the Response to Advertising Efforts

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Abstract. We study a class of discrete-time advertising game with random responses to the advertising efforts made by a duopoly. The firms are assumed to observe the values of the random responses but they do not know their distributions. With the recorded values, firms estimate distributions and play estimated equilibrium strategies. Under suitable assumptions, we prove that the estimated equilibrium strategies converge to equilibria of the advertising game with the true distributions. Our results are numerically illustrated for specific cases.

Keywords: Advertising games · Lanchester model · Markov games · Empirical distribution

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1 Introduction

We consider a dynamic noncooperative game of advertising where the market shares of the firms follow a stochastic difference equation. The stochastic behavior in the market shares comes from the uncertain responses to advertising efforts modeled by a sequence of random variables. Further, we assume that firms can observe the values of such random variables a posteriori but they do not know the distributions. In this sense, by using appropriate statistical estimation methods to approximate the distributions of the random variables, firms can play Nash equilibrium strategies of the estimated games. When these equilibrium strategies converge, the question we aim to answer is whether the limit strategies are equilibria for the game with the true distributions of the responses to advertising efforts.

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The literature about dynamic models of advertising and marketing games is very large; we can mention the papers [4,6,7,21] and the books [2,8]. Most of these references mainly focus on deterministic differential game models; instead there are few works that deal with stochastic differential game models and deterministic discrete-time models, we can cite, for instance, [1,18]. On the other hand, discrete-time stochastic zero-sum games with incomplete information have been studied under several context, see, e.g., [5,10,12–16,22,23], which include the case when the transition law among states is unknown. However, to the best of our knowledge, the only work dealing on estimation problem for nonzero-sum Markov games is [19]. Specifically, in [19] is used the empirical distribution of the disturbance process to obtain an almost surely convergent procedure to approximate Nash equilibria under the discounted criterion.

In this chapter we analyze the stochastic version of the advertising Lanchester model introduced in [1]. Additionally, we assume that the random variables modeling the uncertainty in responses to advertising efforts have unknown distributions. Under this scenario, using the empirical distribution as an estimator and considering finite action sets for players, we apply similar ideas to [19] to simulate values of the advertising responses, estimate equilibrium strategies, and prove that these equilibria converge in some sense to an equilibrium of the advertising game with full information. In order to introduce the model and compare our results, previously we analyze the advertising game with full information, where we numerically compute the Nash equilibria in mixed stationary strategies.

The remaining of the paper is organized as follows. The stochastic advertising game we deal with is described in Sect. 2 as well as the numerical algorithm we use to compute the Nash equilibria. Section 3 is devoted to the stochastic game with unknown distributions of the advertising responses. Finally, in Sect. 4, we give some conclusions.

2 A Discrete-Time Stochastic Game of Advertising

Essentially, Lanchester model is an ordinary-differential-equation model of warfare [11]. Over time, this model has been adapted to study different conflict situations, including advertising models. In this section, we introduce a discrete-time stochastic version of the Lanchester model in the context of the models that appear in [1] and [8, pp. 29–31]. We also give a numerical algorithm to find Nash equilibria in stationary strategies of the proposed model.

2.1 The Advertising Game Model

Consider a duopoly competing for the market share by making advertising efforts. Let x be the market share of Firm 1 and let a and b be the advertising efforts of Firm 1 and Firm 2, respectively, at some decision epoch. The market share of Firm 2 is $1-x$. Then the market share of Firm 1 at the beginning of the next decision epoch is determined by the mapping

$$(x, a, b) \mapsto x + (1-x)d(\xi, a) - xe(\zeta, b) \tag{1}$$

where $d(\xi, a)$ and $e(\zeta, b)$ are the advertising responses to a and b, respectively, and (ξ, ζ) is a pair of random variables. The functions $d(i, \cdot)$ and $e(j, \cdot)$—for fixed values of i and j—are *production functions*, that is, they are increasing, have diminishing marginal effects, and take nonnegative values. Typical advertising responses are

$$d(\xi, a) = \xi\sqrt{a}, \quad e(\zeta, b) = \zeta\sqrt{b}. \tag{2}$$

The evolution of the state system is given by the mapping (1) and has the following interpretation: the advertising of Firm 1 aims to attract customers from Firm 2, thus the increment of the market share is proportional to $(1 - x)$, and analogously for the advertising made by Firm 2.

For the purposes of this paper, we assume that the triples (x, a, b) belong to a finite set $\mathbb{X} \times \mathbb{A} \times \mathbb{B}$. Thus the image of the mapping (1)—with the advertising responses (2), for instance—is not necessarily a subset of $\mathbb{X}$. In such a case, we map $x+(1-x)d(\xi, a)-xe(\zeta, b)$ to the nearest state in $\mathbb{X}$. Although, for simplicity, we write

$$x_{k+1} = x_k + (1 - x_k)d(\xi_k, a_k) - x_k e(\zeta_k, b_k), \quad k = 0, 1, ..., \tag{3}$$

where $x_0 \in \mathbb{X}$ is given. In addition, the so-called disturbance processes $\{\xi_k\}$ and $\{\zeta_k\}$ consist of independent and identically distributed (i.i.d.) random variables, which take values in the finite sets $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively. The process $\{(\xi_k, \zeta_k)\}$ is defined on some underlying probability space $(\Omega, \mathcal{F}, P)$. The common probability functions of the random variables $\{\xi_k\}$ and $\{\zeta_k\}$ are, respectively, θ and ϑ, that is,

$$\begin{cases} \theta(i) = P[\xi_k = i] & \forall i \in \mathbb{S}_1, k \in \mathbb{N}_0, \\ \vartheta(j) = P[\zeta_k = j] & \forall j \in \mathbb{S}_2, k \in \mathbb{N}_0. \end{cases} \tag{4}$$

We use the notation $\mathbb{K} := \{(x, a, b) : x \in \mathbb{X}, a \in \mathbb{A}, b \in \mathbb{B}\}$. Combining (3) and (4), we obtain the transition law among the states as follows. For each $(x, a, b) \in \mathbb{K}$,

$$P_{x,y}[a, b] := P[x_{k+1} = y \mid x_k = x, a_k = a, b_k = b] = \sum_{(i,j)\in S_F} \theta(i)\vartheta(j), \quad y \in \mathbb{X} \tag{5}$$

where

$$S_F := \{(s, t) \in \mathbb{S}_1 \times \mathbb{S}_2 : x + (1 - x)d(s, a) - xe(t, b) = y\}.$$

Finally, $r_i : \mathbb{K} \to \mathbb{R}$ is the one-stage payoff function for the Firm $i = 1, 2$,

$$\begin{cases} r_1(x, a, b) = p_1 x - a \\ r_2(x, a, b) = p_2(1 - x) - b \end{cases} \tag{6}$$

where p_1 and p_2 are the gross profit rate of Firms 1 and 2 respectively. In what follows, the probability space $(\Omega, \mathcal{F}, P)$ is fixed and *a.s.* means almost surely with respect to P.

Putting together all the elements described above, we define the advertising game model as

$$\mathcal{G}_{\theta,\vartheta} := (\mathbb{X}, \mathbb{A}, \mathbb{B}, \mathbb{S}_1, \mathbb{S}_2, \theta, \vartheta, r_1, r_2) \tag{7}$$

The model is a representation of a dynamic game which is played as follows. At each stage $k \in \mathbb{N}_0$, when the game is in state $x_k \in \mathbb{X}$, the firms independently choose actions $a_k = a \in \mathbb{A}$ and $b_k = b \in \mathbb{B}$. Consequently, the following happens: first, Firm i receives payoffs of $r_i(x, a, b), i = 1, 2$; and second, the system moves to the next state $x_{k+1} \in \mathbb{X}$ according to probability transition (5). Once the system reaches the next state, the process repeats. In addition, the payoffs are accumulated according to a discounted criterion, as we will define below.

Let $P_{\mathbb{A}}$ and $P_{\mathbb{B}}$ consist of the set of all probability functions on $\mathbb{A}$ and $\mathbb{B}$ respectively. That is, $P_{\mathbb{A}}$ is the set of functions $\sigma : A \to [0, 1]$ such that $\sum_{a\in\mathbb{A}} \sigma(a) = 1$. Similarly for $P_{\mathbb{B}}$. By convention, for each $\sigma \in P_{\mathbb{A}}$, $\tau \in P_{\mathbb{B}}$, we denote

$$v(x, \sigma, \tau) := \sum_{a\in\mathbb{A}} \sum_{b\in\mathbb{B}} v(x, a, b)\sigma(a)\tau(b), \quad x \in \mathbb{X} \tag{8}$$

for any function $v : \mathbb{K} \to \mathbb{R}$. Likewise, for $\sigma \in P_{\mathbb{A}}$, $\tau \in P_{\mathbb{B}}$

$$[x+(1-x)d(s, \sigma)-xe(t, \tau)] := \sum_{a\in\mathbb{A}} \sum_{b\in\mathbb{B}} [x+(1-x)d(s, a)-xe(t, b)]\sigma(a)\tau(b), \tag{9}$$

where $x \in \mathbb{X}$, $s \in \mathbb{S}_1$, and $t \in \mathbb{S}_2$.

A strategy played by Firm 1 is a sequence $\pi = \{\pi_k\}$ where π_k is a probability function over $\mathbb{A}$ conditioned on the history $h_k := (x_0, a_0, b_0, ..., a_{k1}, b_{k1}, x_k)$ That is, for each history h_k, $\pi_k(\cdot|h_k) \in P_{\mathbb{A}}$. The set of all strategies for Firm 1 is denoted by Π. A strategy $\pi \in \Pi$ is said to be a Markov strategy if there is a probability function f_k over $\mathbb{A}$ such that $\pi_k(\cdot|h_k) = f_k(\cdot|x_k)$ for all $k \in \mathbb{N}_0$. Further, a Markov strategy $\pi = \{f_k\}$ is stationary if $f_k = f$ for all $k \in \mathbb{N}_0$; in this case, we use this notation

$$f^\infty := \{f, f, f, ...\}.$$

We denote by Π_M and $\mathbb{F}$ the sets of Markov strategies and stationary strategies, respectively, for Firm 1. The sets Γ, Γ_M, and $\mathbb{G}$ of all strategies, Markov strategies, and stationary strategies for Firm 2 are defined similarly.

Let $\pi = \{\pi_k\} \in \Pi$ and $\gamma = \{\gamma_k\} \in \Gamma$ be a pair of strategies. For each initial state $x \in \mathbb{X}$, we define the discounted criterion, also known as expected discounted payoff, for Firm $i = 1, 2$, as

$$\begin{cases} J_1^{\theta,\vartheta} = E_x^{(\pi,\gamma)} \left[\sum_{k=0}^{\infty} \beta^k \{p_1 x_k - a_k\} \right] \\ J_2^{\theta,\vartheta} = E_x^{(\pi,\gamma)} \left[\sum_{k=0}^{\infty} \beta^k \{p_2(1 - x_k) - b_k\} \right] \end{cases} \tag{10}$$

where $\beta \in (0, 1)$ is the discount factor and $E_x^{(\pi,\gamma)}$ denotes the expectation operator corresponding to the unique probability measure $P_x^{(\pi,\gamma)}$ induced by $x \in \mathbb{X}$ and $(\pi, \gamma) \in \Pi \times \Gamma$, (see [3]).

2.2 Stationary Nash Equilibrium in Discounted Games

Definition 1. *A pair of strategies $(\pi^*, \gamma^*) \in \Pi \times \Gamma$ is a Nash equilibrium if, for all $x \in \mathbb{X}$,*

$$J_1^{\theta,\vartheta}(x, \pi^*, \gamma^*) \geq J_1^{\theta,\vartheta}(x, \pi, \gamma^*), \quad \forall \pi \in \Pi$$

and

$$J_2^{\theta,\vartheta}(x, \pi^*, \gamma^*) \geq J_2^{\theta,\vartheta}(x, \pi^*, \gamma), \quad \forall \gamma \in \Gamma.$$

The equilibrium payoffs of the game, with initial state x, are $J_1^{\theta,\vartheta}(x, \pi^*, \gamma^*)$ and $J_2^{\theta,\vartheta}(x, \pi^*, \gamma^*)$.

The following lemma about the existence of Nash equilibria in Markov strategies for this model is well known. For instance, see [17, Theorem 5.1].

Lemma 1. *The game model, with discounted payoffs $J_1^{\theta,\vartheta}$ and $J_2^{\theta,\vartheta}$, has a Nash equilibrium in stationary strategies. That is, there exists $(f^\infty, g^\infty) \in \mathbb{F} \times \mathbb{G}$ such that for each $x \in \mathbb{X}$,*

$$J_1^{\theta,\vartheta}(x, f^\infty, g^\infty) \geq J_1^{\theta,\vartheta}(x, \pi, g^\infty), \quad \forall \pi \in \Pi$$

and

$$J_2^{\theta,\vartheta}(x, f^\infty, g^\infty) \geq J_2^{\theta,\vartheta}(x, f^\infty, \gamma), \quad \forall \gamma \in \Gamma.$$

Observe that once $f^\infty \in \mathbb{F}$ and $g^\infty \in \mathbb{G}$ are fixed,

$$\overline{J}_1(x, \pi) := J_1^{\theta,\vartheta}(x, \pi, g^\infty), \quad \pi \in \Pi, \quad x \in \mathbb{X}$$

and

$$\overline{J}_2(x,\gamma) := J_2^{\theta,\vartheta}(x, f^\infty, \gamma), \quad \gamma \in \Gamma, \quad x \in \mathbb{X}$$

constitute performance indices, where each of them corresponds to an optimal control problem. Hence, the *value functions*

$$V(x) := \max_{\pi \in \Pi} \overline{J}_1(x,\pi), \quad x \in \mathbb{X} \tag{11}$$

and

$$W(x) := \max_{\gamma \in \Gamma} \overline{J}_2(x,\gamma), \quad x \in \mathbb{X}, \tag{12}$$

satisfy, respectively, the Dynamic Programming equations

$$V(x) = \max_{\mu \in P_{\mathbb{A}}} \left[[p_1 x - \mu] + \beta \sum_{(i,j) \in \mathbb{S}_1 \times \mathbb{S}_2} V[x + (1-x)d(i,\mu) - xe(j,g)]\theta(i)\vartheta(j) \right] \tag{13}$$

$$= [p_1 x - f] + \beta \sum_{(i,j) \in \mathbb{S}_1 \times \mathbb{S}_2} V[x + (1-x)d(i,f) - xe(j,g)]\theta(i)\vartheta(j), \quad \forall x \in \mathbb{X}, \tag{14}$$

and

$$W(x)$$
$$= \max_{\lambda \in P_{\mathbb{B}}} \left[[p_2(1-x) - \lambda] + \beta \sum_{(i,j) \in \mathbb{S}_1 \times \mathbb{S}_2} W[x + (1-x)d(i,f) - xe(j,\lambda)]\theta(i)\vartheta(j) \right] \tag{15}$$
$$= [p_2(1-x) - g] + \beta \sum_{(i,j) \in \mathbb{S}_1 \times \mathbb{S}_2} W[x + (1-x)d(i,f) - xe(j,g)]\theta(i)\vartheta(j), \forall x \in \mathbb{X}. \tag{16}$$

Remark 1. By considering standard dynamic programming arguments, if there are functions V and W and a pair (f,g) satisfying (13)–(16), then $(f^\infty, g^\infty) \in \mathbb{F} \times \mathbb{G}$ is a stationary Nash equilibrium for the game with discounted payoffs (10). Further, the equilibrium payoffs are $J_1^{\theta,\vartheta}(x, f^\infty, g^\infty) = V(x)$ and $J_2^{\theta,\vartheta}(x, f^\infty, g^\infty) = W(x)$.

2.3 Numerical Examples

We compute the equilibria in Markov strategies for an advertising game with the data of Table 1.

The equilibrium strategies are found using and adaptation of the well-known value iteration algorithm from discounted dynamic programming. In each

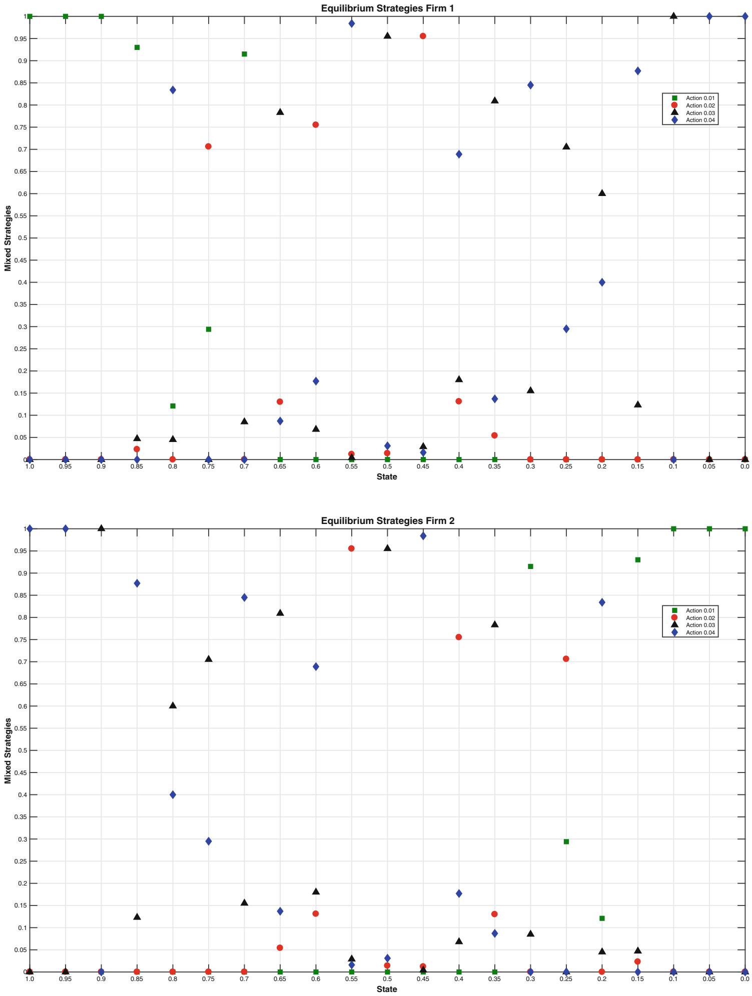

Fig. 1. Equilibrium strategies f and g in the full-information game with data of Table 1. The height of each action is the probability it is played with.

iteration we get the equilibrium by minimizing McKelvey's function, see [9, p. 133]. For the parameters given above, the iteration algorithm converges. The algorithm is implemented in Python and the code is available at

https://github.com/adra1973/

The limit strategies (f, g), that form the stationary equilibrium (f^∞, g^∞), are plotted in Fig. 1 and 2. Since we are using exactly the same parameters for both firms, in Fig. 1 we can observe for each state an effect of "mirror" in the strategies for both firms.

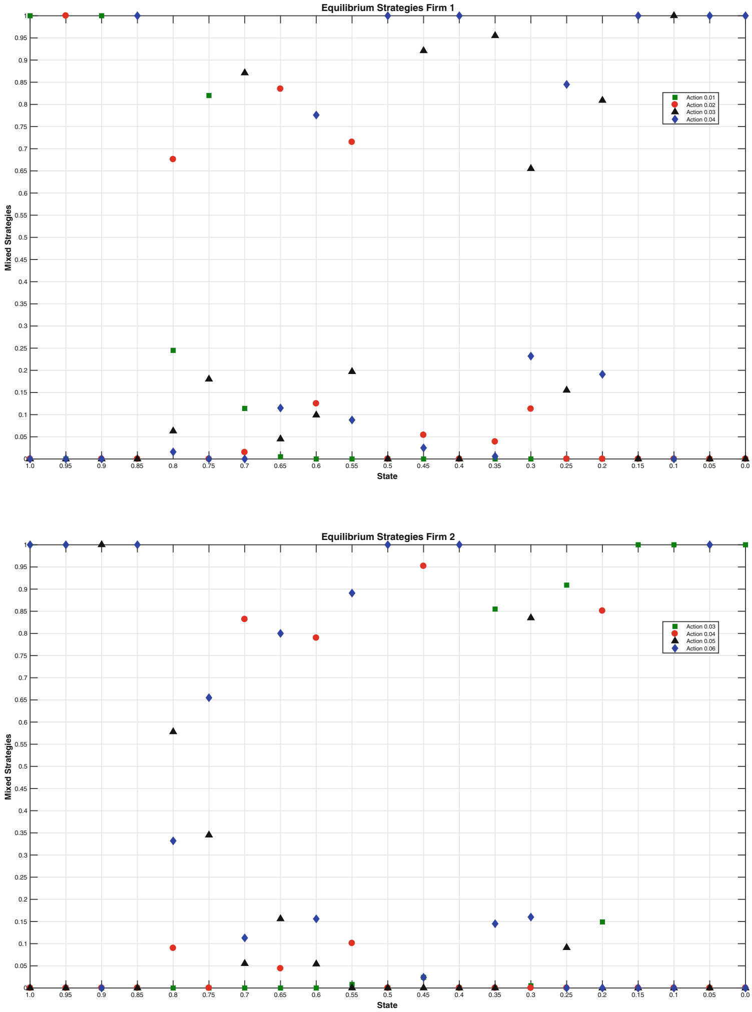

Fig. 2. Equilibrium strategies f and g in the full-information game with data from Table 1 but the set of actions for Firm 2 is replaced by (17).

In Fig. 2 we plot the equilibrium strategies for the game with the same data of Table 1 but the set of actions for Firm 2 now is

$$\mathbb{B} = \{0.03, 0.04, 0.05, 0.06\} \tag{17}$$

and thus the behavior of the strategies breaks the "mirror" observed before.

Table 1. Data for the advertising game.

Variable	Description
$\mathbb{X}$	Space of 21 states of market shares, $\{0.0, 0.05, 0.1, 0.15, 0.2, ..., 0.8, 0.85, 0.9, 0.95, 1.0\}$,
$\mathbb{A}$	Set of 4 actions for advertising effort of Firm 1, $\mathbb{A} = \{0.01, 0.02, 0.03, 0.04\}$,
$\mathbb{B}$	Set of 4 actions for advertising effort of Firm 2, $\mathbb{B} = \{0.01, 0.02, 0.03, 0.04\}$,
$\mathbb{S}_1$	Set of 10 values of Firm 1, $\mathbb{S}_1 = \{0.95, ..., 1.05\}$
$\mathbb{S}_2$	Set of 10 values of Firm 2, $\mathbb{S}_2 = \{0.95, ..., 1.05\}$
ξ	Random variable of Firm 1 that take values in $\mathbb{S}_1$ with probability $\theta(i)$, $i \in \mathbb{S}_1$, $\xi \sim Binomial(10, 0.4)$
ζ	Random variable of Firm 2 that take values in $\mathbb{S}_2$ with probability $\vartheta(j)$, $j \in \mathbb{S}_2$, $\zeta \sim Binomial(10, 0.4)$.
d	Advertising response function of Firm 1, $d(\xi, a) = \xi\sqrt{a}, \quad a \in \mathbb{A}$
e	Advertising response function of Firm 2, $e(\zeta, b) = \zeta\sqrt{b}, \quad b \in \mathbb{B}$
p_1	Gross profit for each product sold by Firm 1, $p_1 = 1.2$
p_2	Gross profit for each product sold by Firm 2, $p_2 = 1.2$
β	The discount factor $\beta = 0.95$

3 The Advertising Game with Unknown Distribution

In this section, we study the advertising game when the distributions of the random variables (ξ, ζ) are unknown for the players. We assume that, after the $n-$th stage, players have recorded the values $\overline{\xi}_n := (\xi_0, \xi_1, ..., \xi_n)$ and $\overline{\zeta}_n := (\zeta_0, \zeta_1, ..., \zeta_n)$ and use the empirical distributions

$$\theta_n(i) := \frac{1}{n}\sum_{t=0}^{n-1} \mathbf{1}_i(\xi_t), \quad i \in \mathbb{S}_1, \quad n \in \mathbb{N}$$

and

$$\vartheta_n(j) := \frac{1}{n}\sum_{t=0}^{n-1} \mathbf{1}_j(\zeta_t), \quad j \in \mathbb{S}_2, \quad n \in \mathbb{N}$$

to estimate equilibrium strategies. More precisely, for each $n \in \mathbb{N}$, consider the *empirical advertising game*

$$\mathcal{G}_{\theta_n,\vartheta_n} := (\mathbb{X}, \mathbb{A}, \mathbb{B}, \mathbb{S}_1, \mathbb{S}_2, \theta_n, \vartheta_n, r_1, r_2) \tag{18}$$

with dynamics (3) and payoffs (10), where θ and ϑ are replaced by θ_n and ϑ_n, respectively. Given a stationary Nash equilibrium (f_n^∞, g_n^∞) for the empirical advertising game (18), by well-known dynamic programming results, there exist functions V_n and W_n that satisfy the optimality equations

$$V_n(x) = \max_{\mu \in P_{\mathbb{A}}} \left[[p_1 x - \mu] + \beta \sum_{i,j} V_n[x + (1-x)d(i,\mu) - xe(j,g_n)]\theta_n(i)\vartheta_n(j) \right] \tag{19}$$
$$= [p_1 x - f_n] + \beta \sum_{i,j} V_n[x + (1-x)d(i,f_n) - xe(j,g_n)]\theta_n(i)\vartheta_n(j), \quad x \in \mathbb{X},$$

and

$$W_n(x)$$
$$= \max_{\lambda \in P_{\mathbb{B}}} \left[[p_2(1-x) - \lambda] + \beta \sum_{i,j} W_n[x + (1-x)d(i,f_n) - xe(j,\lambda)]\theta_n(i)\vartheta_n(j) \right] \tag{20}$$
$$= [p_2(1-x) - g_n] + \beta \sum_{i,j} W[x + (1-x)d(i,f_n) - xe(j,g_n)]\theta_n(i)\vartheta_n(j), \ x \in \mathbb{X}.$$

Remark 2. Notice that V_n and W_n are defined on $\mathbb{X} \times \Omega$, thus $V_n(x)$ and $W_n(x)$ are random variables for each $x \in \mathbb{X}$. The strategies f_n and g_n are also random vectors.

The following proposition is based on [19]; for completeness, we outline a proof in the scenario of the present work.

Proposition 1. *For each $n \in \mathbb{N}$, let f_n, g_n, V_n, and W_n satisfy (19) and (20). If*

$$\lim_{n\to\infty} (f_n, g_n) = (f, g) \quad P - a.s. \tag{21}$$

and

$$\lim_{n\to\infty} (V_n, W_n) = (V, W) \quad P - a.s., \tag{22}$$

then (f^∞, g^∞) is $P - a.s.$ a Nash equilibrium for the advertising game with dynamics (3) and payoffs (10).

Proof. It is well known that from the strong law of large numbers,

$$(\theta_n, \vartheta_n) \to (\theta, \vartheta) \quad P - a.s. \tag{23}$$

Now, fix ω in Ω such that the convergence in (21), (22), and (23) holds. Then, for each $\mu \in P_{\mathbb{A}}$, $x \in \mathbb{X}$, and $n \in \mathbb{N}$,

$$\sum_{i,j} \Big| V_n[x + (1-x)d(i,\mu) - xe(j,g_n)]$$
$$- V[x + (1-x)d(i,\mu) - xe(j,g_n)] \Big| \theta_n(i)\vartheta_n(j)$$
$$\leq \sum_{i,j} \max_{x\in\mathbb{X}} \Big| V_n(x) - V(x) \Big| \theta_n(i)\vartheta_n(j)$$
$$\leq \max_{x\in\mathbb{X}} \Big| V_n(x) - V(x) \Big|. \tag{24}$$

and

$$\begin{aligned}
&\sum_{i,j} \Big| V[x+(1-x)d(i,\mu)-xe(j,g_n)] \\
&\quad - V[x+(1-x)d(i,\mu)-xe(j,g)] \Big| \theta_n(i)\vartheta_n(j) \\
&\le \sum_{i,j}\sum_{b\in\mathbb{B}} \Big| V[x+(1-x)d(i,\mu)-xe(j,b)] \Big| \Big| g_n(b|x)-g(b|x) \Big| \theta_n(i)\vartheta_n(j) \\
&\le \max_{x\in\mathbb{X}} |V(x)| \sum_{b\in\mathbb{B}} \Big| g_n(b|x)-g(b|x) \Big| \qquad (25)
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{i,j} \Big| V_n[x+(1-x)d(i,\mu)-xe(j,g_n)]\theta_n(i)\vartheta_n(j) \\
&\quad - V[x+(1-x)d(i,\mu)-xe(j,g)]\theta(i)\vartheta(j) \Big| \\
&\le \sum_{j\in\mathbb{S}} \Big| V_n[x+(1-x)d(i,\mu)-xe(j,g_n)]\theta_n(i)\vartheta_n(j) \\
&\quad - V[x+(1-x)d(i,\mu)-xe(j,g_n)]\theta_n(i)\vartheta_n(j) \Big| \\
&\quad + \sum_{j\in\mathbb{S}} \Big| V[x+(1-x)d(i,\mu)-xe(j,g_n)]\theta_n(i)\vartheta_n(j) \\
&\quad - V[x+(1-x)d(i,\mu)-xe(j,g)]\theta_n(i)\vartheta_n(j) \Big| \\
&\quad + \sum_{j\in\mathbb{S}} \Big| V[x+(1-x)d(i,\mu)-xe(j,g)]\theta_n(i)\vartheta_n(j) \\
&\quad - V[x+(1-x)d(i,\mu)-xe(j,g)]\theta(i)\vartheta(j) \Big|.
\end{aligned}$$

Then, (24), (25), and (23) imply

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i,j} V_n[x+(1-x)d(i,\mu)-xe(j,g_n)]\theta_n(i)\vartheta_n(j) \\
&\qquad = \sum_{i,j} V[x+(1-x)d(i,\mu)-xe(j,g)]\theta(i)\vartheta(j) \qquad P-a.s. \qquad (26)
\end{aligned}$$

for each $\mu \in P_{\mathbb{A}}$ and $x \in \mathbb{X}$. We can also show that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i,j} V_n[x+(1-x)d(i,f_n)-xe(j,g_n)]\theta_n(i)\vartheta_n(j) \\
&\qquad = \sum_{i,j} V[x+(1-x)d(i,f)-xe(j,g)]\theta(i)\vartheta(j) \qquad P-a.s. \qquad (27)
\end{aligned}$$

On the other hand, from (24) and (26), we have

$$V_n(x) \geq [p_1 x - \mu] + \beta \sum_{i,j} V_n[x + (1-x)d(i,\mu) - xe(j,g_n)]\theta(i)\vartheta(j) \quad \forall \mu \in P_{\mathbb{A}}$$

and hence, by letting $n \to \infty$,

$$V(x) \geq [p_1 x - \mu] + \beta \sum_{i,j} V[x + (1-x)d(i,\mu) - xe(j,g)]\theta(i)\vartheta(j) \quad \forall \mu \in P_{\mathbb{A}}.$$

Furthermore, the second equality in (24) and (27) yield

$$\begin{aligned} V(x) &= \max_{\mu \in P_{\mathbb{A}}} \left[[p_1 x - \mu] + \beta \sum_{i,j} V[x + (1-x)d(i,\mu) - xe(j,g)]\theta(i)\vartheta(j) \right] \\ &= [p_1 x - f] + \beta \sum_{i,j} V[x + (1-x)d(i,f) - xe(j,g)]\theta(i)\vartheta(j), \quad P-a.s. \end{aligned}$$

The following equalities are analogously proved

$$\begin{aligned} W(x) &= \max_{\lambda \in P_{\mathbb{B}}} \left[[p_2(1-x) - \lambda] + \beta \sum_{i,j} W[x + (1-x)d(i,f) - xe(j,\lambda)]\theta(i)\vartheta(j) \right] \\ &= [p_2(1-x) - g] + \beta \sum_{i,j} W[x + (1-x)d(i,f) - xe(j,g)]\theta(i)\vartheta(j), \quad P-a.s. \end{aligned}$$

These optimality equations prove that (f^∞, g^∞) is a stationary Nash equilibrium $P-a.s.$ for the advertising game. □

3.1 Numerical Examples for the Empirical Game Model

In order to generate simulations of the empirical games $\mathcal{G}_{\theta_m,\vartheta_m}$, we use the algorithm in [20, p. 56] to produce values from a Binomial random variable. All parameters are exactly the same as in Table 1 but the pair (θ, ϑ) is replaced by (θ_m, ϑ_m). As in Subsection 2.3, we compute the stationary Nash equilibrium (f_m^∞, g_m^∞) for each empirical game $\mathcal{G}_{\theta_m,\vartheta_m}$, with $m \in \mathbb{N}_0$.

For a realization $\omega \in \Omega$ and different values of m, the equilibrium strategies (f_m, g_m) are plotted in Fig. 3 and 4, and equilibrium payoffs (V_m, W_m) are shown in Fig. 5 and 6. By looking at the proof of Proposition 1, if (21) and (22) hold for a given value of ω, then the limit strategy pair (f, g) determines a stationary Nash equilibrium of the full information game. The equilibrium strategy (f, g) or equilibrium payoffs (V, W) for the full-information model (7) are also plotted on the right of each figure.

A numerical validation of the hypotheses in Proposition 1 would consist in simulating empirical games for infinitely many realizations of ω, computing the equilibria along with the payoffs, and verifying (21) and (22). From a practical point of view, however, firms record the values of the random variables—and

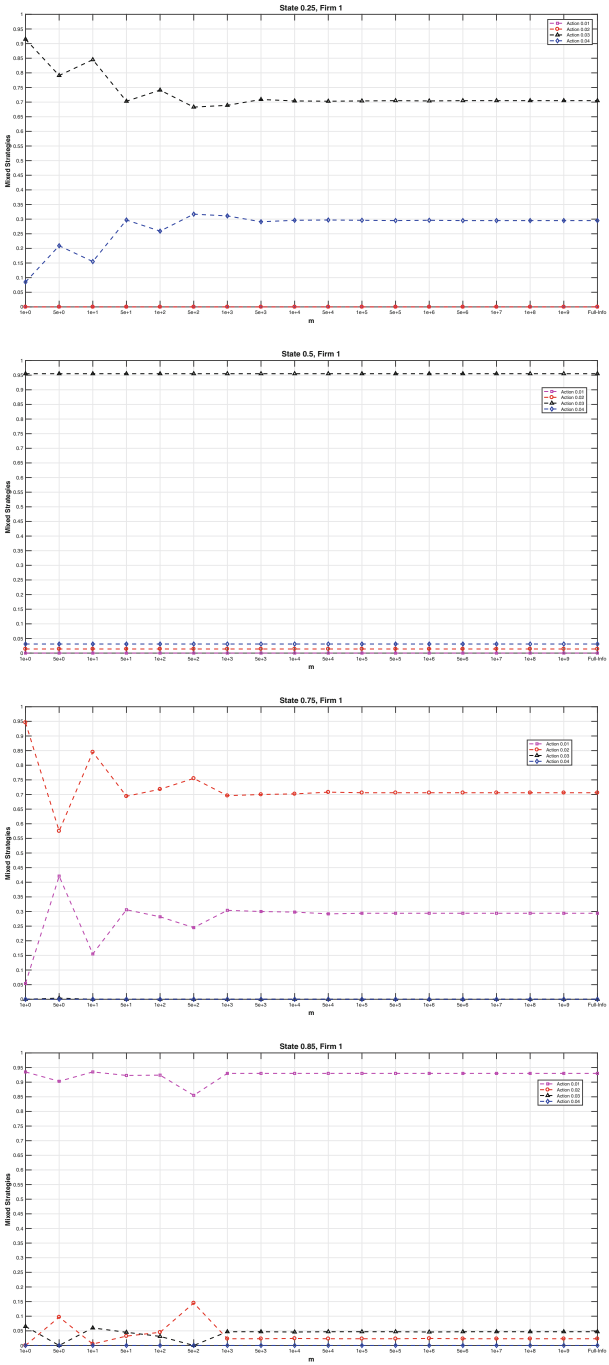

Fig. 3. Estimated equilibrium strategies of Firm 1 for different values of m at the states 0.25, 0.5, 0.75, and 0.85.

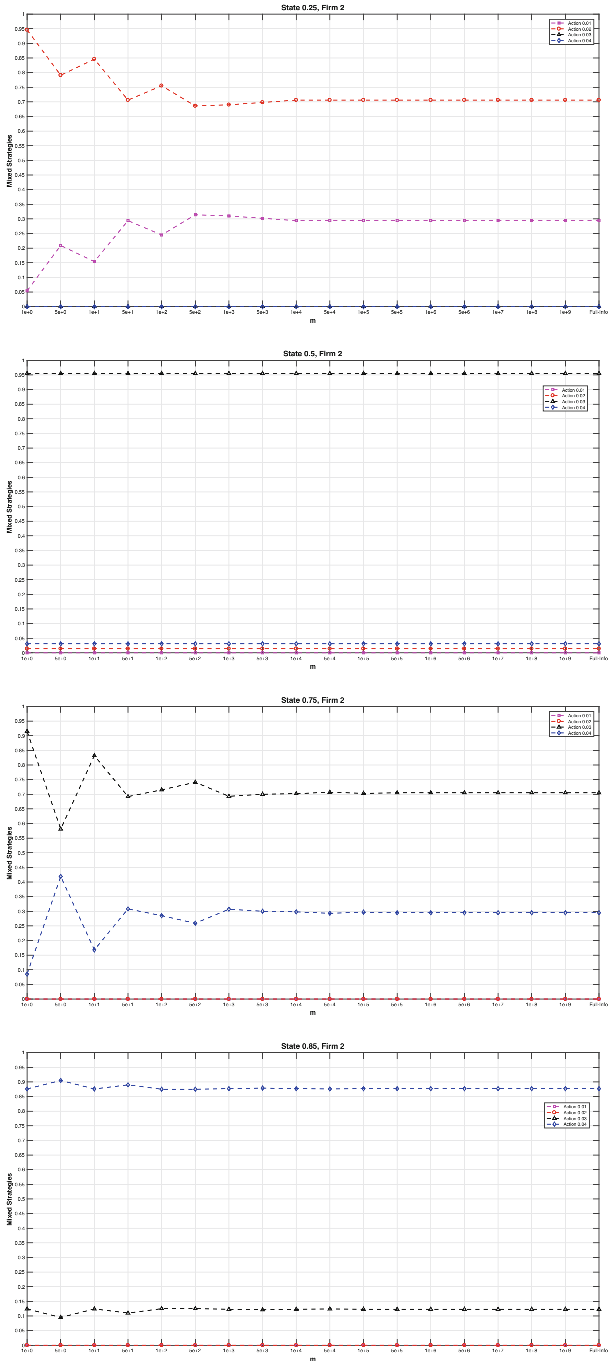

Fig. 4. Estimated equilibrium strategies of Firm 2 for different values of m at the states 0.25, 0.5, 0.75, and 0.85.

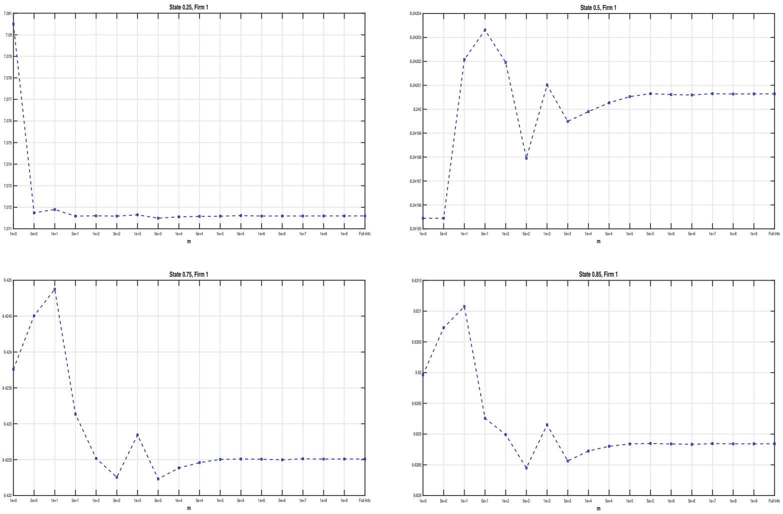

Fig. 5. Estimated equilibrium payoffs of Firm 1 for different values of m at the states 0.25, 0.5, 0.75, and 0.85.

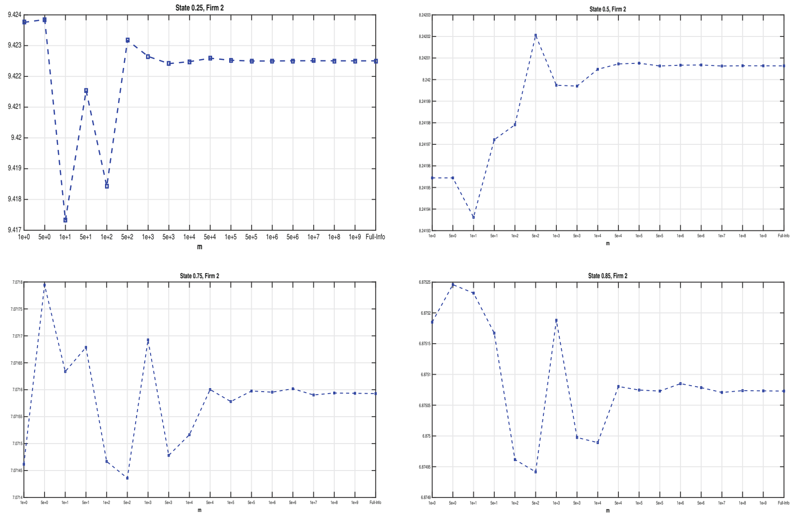

Fig. 6. Estimated equilibrium payoffs of Firm 2 for different values of m at the states 0.25, 0.5, 0.75, and 0.85.

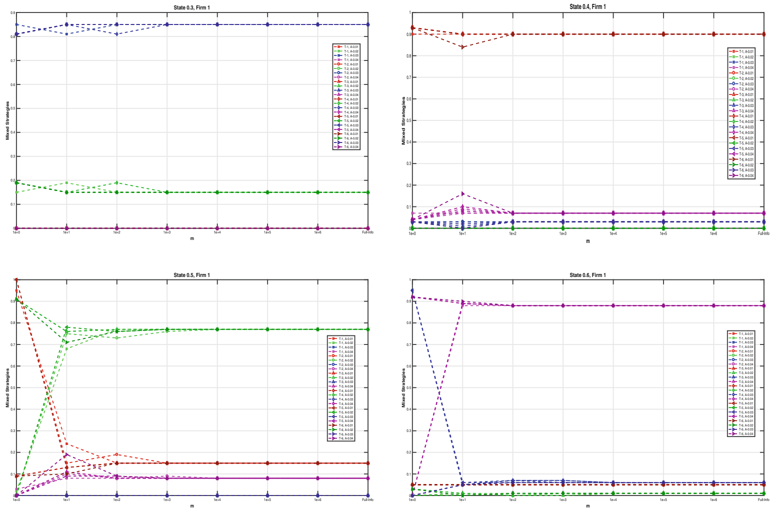

Fig. 7. Estimated equilibrium strategies of Firm 1 for six realizations of ω and different values of m at states 0.3, 0.4, 0.5, and 0.6.

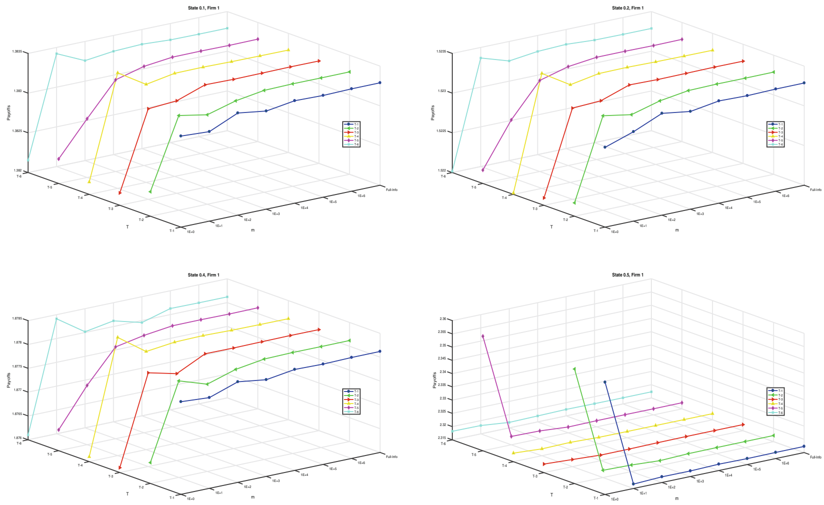

Fig. 8. Estimated equilibrium payoffs of Firm 1 for six realizations of ω and different values of m at states 0.1, 0.2, 0.4, and 0.5.

play the corresponding equilibrium strategies—of a single realization ω. If the strategies converge, then Proposition 1 asserts that, with probability 1, the estimated equilibrium strategies are close to an equilibrium of the full-information game.

For illustrative purposes, in Fig. 7, we plot the equilibrium strategies corresponding to six different realizations of ω. The game model components are given in Table 1, except for $\beta = 0.75$ and $\mathbb{X} = \{0.0, 0.1, 0.2, 0.3, \ldots, 1.0\}$. The associated payoffs are shown in Fig. 8. We plot data for some states of Firm 1 only. An interesting feature we can observe in this numerical experiment, possibly due to the uniqueness of equilibrium in the full-information game, is that the limits of the estimated equilibrium strategies and the estimated payoffs are independent of ω.

4 Conclusions

We have shown how to estimate equilibrium strategies in a stochastic advertising game with unknown distributions of the response to advertising efforts. From the numerical results, it is worth remarking some features of our model. First, since we deal with a finite game, the equilibrium strategies are mixed instead of pure strategies—obtained in most of the deterministic differential games of advertising—because the corresponding action spaces in those models are convex. Second, the qualitative behavior of the equilibrium strategies we found corresponds to that in the existing literature, namely, for higher market shares the advertising efforts are also higher. Third, we assume that at the $m-$th decision epoch, firms have recorded m values of the advertising responses; hence firms have good estimators (θ_n, ϑ_n) only when m is large enough. However, firms can improve the estimators by using information of previous advertising campaigns as well as information acquired between decision epochs. With such improved estimators, the conclusion of Proposition 1 does not change. Finally, the problem of multiple equilibria and/or the non convergence of the estimated equilibrium strategies can be overcame by passing to a subsequence as is shown in [19].

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