A note on the Rees algebra of a bipartite graph

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Dedicated to Wolmer Vasconcelos in his 65th birthday

Abstract

Let \( R = K[x_1, \ldots, x_n] \) be a polynomial ring over a field \( K \) and let \( I \) be an ideal of \( R \) generated by a set \( x^{a_1}, \ldots, x^{a_s} \) of square-free monomials of degree two such that the graph \( G \) defined by these monomials is bipartite. We study the Rees algebra \( \mathcal{R}(I) \) of \( I \), by studying both the Rees cone \( \mathbb{R}_+ \mathcal{A} \) generated by the set \( \mathcal{A} = \{e_1, \ldots, e_n \}

\( \alpha_1, 1, \ldots, \alpha_q, 1\} \) and the matrix \( C \) whose columns are the vectors in \( \mathcal{A} \). It is shown that \( C \) is totally unimodular. We determine the irreducible representation of the Rees cone in terms of the minimal vertex covers of \( G \). Then we compute the \( \alpha \)-invariant of \( \mathcal{R}(I) \).

1 Introduction

Let \( R = K[x_1, \ldots, x_n] \) be a polynomial ring over a field \( K \) and let \( G \) be a bipartite graph with vertex set \( V = V(G) = \{v_1, \ldots, v_n\} \) and edge set \( E = E(G) \). The edge ideal of \( G \) is the square-free monomial ideal of \( R \) given by

\[
I = I(G) = (\{x_i x_j | \{v_i, v_j\} \text{ is an edge of } G\}) \subset R,
\]

and the Rees algebra of \( I \) is the \( K \)-subalgebra:

\[
\mathcal{R}(I) = K[\{x_i x_j t | v_i \text{ is adjacent to } v_j \} \cup \{x_1, \ldots, x_n\}] \subset R[t],
\]

where \( t \) is a new variable. Consider the set of vectors

\[
\mathcal{A} = \{e_i + e_j + e_{n+1} | v_i \text{ is adjacent to } v_j \} \cup \{e_1, \ldots, e_n\} \subset \mathbb{R}^{n+1},
\]

\( ^{0}\)2000 Mathematics Subject Classification. Primary 13H10; Secondary 13F20, 13A30, 52B20.

\( ^{1}\)The authors were partially supported by CONACyT grant 40201F and SNI.
where $e_i$ is the $i$th unit vector. Here we study $\mathcal{R}(I)$ by looking closely at the matrix $C$ whose columns are the vector in $\mathcal{A}$. One of our results proves that $C$ is totally unimodular, then as a consequence we derive that the presentation ideal of $\mathcal{R}(I)$ is generated by square-free binomials. As another consequence we give a simple proof of the fact that $\mathcal{R}(I)$ is a normal domain [7].

We are able to determine the irreducible representation of the polyhedral Rees cone $\mathbb{R}_+\mathcal{A}'$ generated by $\mathcal{A}'$, see Corollary 4.3. This turns out to be related to the minimal vertex covers of the graph $G$ and yields a description of the canonical module of $\mathcal{R}(I)$.

By assigning $\deg(x_i) = 1$ and $\deg(t) = -1$, the Rees algebra $\mathcal{R}(I)$ becomes a standard graded $K$-algebra, that is, it is generated as a $K$-algebra by elements of degree 1. Another of our results proves that the $a$-invariant of $\mathcal{R}(I)$, with respect to this grading, is equal to $-\beta_0 + 1$, where $\beta_0$ is the independence number of $G$.

In order to compute this invariant we use the irreducible representation of $\mathbb{R}_+\mathcal{A}'$ together with a formula of Danilov-Stanley for the canonical module of $\mathcal{R}(I)$.

2 Preliminaries

Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be a set of monomials of $R$ and let $A = (a_{ij})$ be the matrix of order $n \times q$ whose columns are the vectors $\alpha_1, \ldots, \alpha_q$. We say that the matrix $A$ is unimodular if all its nonzero $r \times r$ minors have absolute value equal to 1, where $r$ is the rank of $A$.

Recall that the monomial subring $K[F] \subset R$ is normal if $K[F] = \overline{K[F]}$, where $\overline{K[F]}$ is the integral closure of $K[F]$ in its field of fractions. The following expression for the integral closure is well known:

$$\overline{K[F]} = K[\{x^a | a \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}]\},$$

where $\mathbb{Z}\mathcal{A}$ is the subgroup spanned by $\mathcal{A}$ and $\mathbb{R}_+\mathcal{A}$ is the polyhedral cone

$$\mathbb{R}_+\mathcal{A} = \left\{ \sum_{i=1}^{q} a_i \alpha_i \mid a_i \in \mathbb{R}_+ \text{ for all } i \right\}$$

generated by $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$. Here $\mathbb{R}_+$ denotes the set of non negative real numbers. See [2, 3] and [12, Chapter 7] for a thorough discussion of the integral closure of a monomial subring and how it can be computed. For the related problem of computing the integral closure of an affine domain see [9, 10].

The next result was shown in [7] if $A$ is the incidence matrix of a bipartite graph and it was shown in [8] for general $A$. The proof below, in contrast to that of [8], is direct and does not make any use of Gröbner bases techniques.

**Theorem 2.1** If $A$ is a unimodular matrix, then $K[F]$ is a normal domain.
Proof. By Eq. (1) it suffices to prove

\[ \mathcal{Z} \mathcal{A} \cap \mathbb{R}_+ \mathcal{A} = \mathbb{N} \mathcal{A}. \]

Let \( b \in \mathcal{Z} \mathcal{A} \cap \mathbb{R}_+ \mathcal{A} \). By Carathéodory’s Theorem [4, I, Theorem 2.3] there are linearly independent columns \( \alpha_{i_1}, \ldots, \alpha_{i_r} \) of \( \mathcal{A} \), where \( r = \text{rank}(\mathcal{A}) \), such that

\[ b \in \mathbb{R}_+ \alpha_{i_1} + \cdots + \mathbb{R}_+ \alpha_{i_r}. \]  \hspace{1cm} (2)

As \( \mathcal{A} \) is unimodular for each \( j \) one has

\[ \Delta_r([\alpha_{i_1} \cdots \alpha_{i_r}]) = \Delta_r([\alpha_{i_1} \cdots \alpha_{i_r} \alpha_j]) = 1, \]

where \( \Delta_r(B) \) denotes the greatest common divisor of all the nonzero \( r \times r \) minors of \( B \). Hence by a classical result of I. Heger [6, p. 51] one readily obtains

\[ \mathcal{Z} \mathcal{A} = \mathbb{Z} \alpha_{i_1} \oplus \cdots \oplus \mathbb{Z} \alpha_{i_r}. \]

Therefore

\[ b \in \mathbb{Z} \alpha_{i_1} + \cdots + \mathbb{Z} \alpha_{i_r}. \]  \hspace{1cm} (3)

Since \( \alpha_{i_1}, \ldots, \alpha_{i_r} \) are linearly independent by comparing the coefficients of \( b \) with respect to the two representations given by (2) and (3) one derives \( b \in \mathbb{N} \mathcal{A} \). Hence we have shown \( \mathcal{Z} \mathcal{A} \cap \mathbb{R}_+ \mathcal{A} \subset \mathbb{N} \mathcal{A} \). The reverse containment is clear. \hspace{1cm} \Box

3 On the defining matrix of the Rees algebra

Let \( G \) be a simple graph. The incidence matrix of \( G \) is the matrix whose columns are the vectors \( e_i + e_j \) such that \( v_i \) is adjacent to \( v_j \). A matrix \( B \) is totally unimodular if each \( i \times i \) minor of \( B \) is 0 or \( \pm 1 \) for all \( i \geq 1 \). Recall that the bipartite simple graphs are characterized as those graphs whose incidence matrix is totally unimodular [6, p. 273].

Next we present the main result of this section and two of its consequences.

Theorem 3.1 Let \( G \) be a simple bipartite graph with \( n \) vertices and \( q \) edges and let \( A = (a_{ij}) \) be its incidence matrix. If \( e_1, \ldots, e_n \) are the first \( n \) unit vectors in \( \mathbb{R}^{n+1} \) and \( C \) is the matrix

\[
C = \begin{pmatrix}
a_{11} & \cdots & a_{1q} & e_1 & \cdots & e_n \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nq} & 1 & \cdots & 1
\end{pmatrix}
\]

obtained from \( A \) by adjoining a row of 1’s and the column vectors \( e_1, \ldots, e_n \), then \( C \) is totally unimodular.
Proof. Suppose that \( \{1, \ldots, m\} \) and \( \{m+1, \ldots, n\} \) is the bipartition of the graph \( G \). Let \( C' \) be the matrix obtained by deleting the last \( n-m \) columns from \( C \). It suffices to show that \( C' \) is totally unimodular. First one successively subtracts the rows \( 1, 2, \ldots, m \) from the row \( n+1 \). Then one reverses the sign in the rows \( m+1, \ldots, n \). These elementary row operations produce a new matrix \( C'' \). The matrix \( C'' \) is the incidence matrix of a directed graph, namely, consider \( G \) as a directed graph, and add one more vertex \( n+1 \), and add the edges \( (i, n+1) \) for \( i = 1, \ldots, m \). The matrix \( C'' \), being the incidence matrix of a directed graph, is totally unimodular [6, p. 274]. As the last \( m \) column vectors of \( C'' \) are

\[
e_1 - e_{n+1}, \ldots, e_m - e_{n+1},
\]

one can successively pivot on the first nonzero entry of \( e_i - e_{n+1} \) for \( i = 1, \ldots, m \) and reverse the sign in the rows \( m+1, \ldots, n \) to obtain back the matrix \( C' \). Here a pivot on the entry \( c'_{st} \) means transforming column \( t \) of \( C'' \) into the \( st \)th unit vector by elementary row operations. Since the operation of pivoting preserves total unimodularity [5, Lemma 2.2.20] one derive that \( C' \) is totally unimodular, and hence so is \( C \). This proof is due to Bernd Sturmfels, it is simpler than our original proof. \( \Box \)

Let \( \alpha_1, \ldots, \alpha_q \) be the columns of the incidence matrix of a graph \( G \) and let \( I \) be its edge ideal. There is an epimorphism of graded algebras

\[
\varphi: B = K[x_1, \ldots, x_n, t_1, \ldots, t_q] \longrightarrow \mathcal{R}(I) \quad (x_i \not\sim x_j), \quad (t_i \not\sim tx^\alpha),
\]

where \( B \) is a polynomial ring with the standard grading.

**Corollary 3.2** If \( G \) is a bipartite graph, then the toric ideal \( J = \ker(\varphi) \) has a universal Gröbner basis consisting of square-free binomials.

**Proof.** It follows from Theorem 3.1 and [8, Proposition 8.11]. \( \Box \)

**Corollary 3.3** ([7]) If \( G \) is a bipartite graph and \( I \) is its edge ideal, then the Rees algebra \( \mathcal{R}(I) \) is normal.

**Proof.** It follows from Theorem 3.1 and Theorem 2.1. \( \Box \)

### 4 The irreducible representation of the Rees cone

If \( 0 \neq a \in \mathbb{R}^n \), then the set \( H_a \) will denote the hyperplane of \( \mathbb{R}^n \) through the origin with normal vector \( a \), that is,

\[
H_a = \{ x \in \mathbb{R}^n | \langle x, a \rangle = 0 \}.
\]
This hyperplane determines two closed half-spaces

\[ H^+_a = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle \geq 0 \} \quad \text{and} \quad H^-_a = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle \leq 0 \}. \]

A subset \( F \subset \mathbb{R}^n \) is a proper face of a polyhedral cone \( \mathbb{R}_+A \) if there is a supporting hyperplane \( H_a \) such that

(i) \( F = \mathbb{R}_+A \cap H_a \neq \emptyset \),

(ii) \( \mathbb{R}_+A \subset H^+_a \) and \( \mathbb{R}_+A \not\subset H^-_a \).

A proper face \( F \) of \( \mathbb{R}_+A \) is a facet if \( \dim(F) = \dim(\mathbb{R}_+A) - 1 \).

The facets of the Rees cone

In the sequel \( G \) is a connected bipartite graph with edge set \( E(G) \), vertex set \( V = V(G) = \{ v_1, \ldots, v_n \} \), and height of \( I(G) \) greater or equal than 2.

The polyhedral cone of \( \mathbb{R}^{n+1} \) generated by the set of vectors

\[ A' = \{ e_i + e_j + e_{n+1} \mid v_i \text{ is adjacent to } v_j \} \cup \{ e_i \mid 1 \leq i \leq n \} \subset \mathbb{R}^{n+1} \]

is called the Rees cone of \( G \) and it will be denoted by \( \mathbb{R}_+A' \). Note that the Rees cone has dimension \( n+1 \).

A subset \( C \subset V \) is called a minimal vertex cover of \( G \) if the face ideal \( p = (\{ x_i | v_i \in C \}) \) is a minimal prime of \( I(G) \) and a subset \( A \subset V \) is called an independent set of \( G \) if any two vertices in \( A \) are non adjacent. Thus \( A \) is a maximal independent set if and only if \( V \setminus A \) is a minimal vertex cover.

In order to describe the facets of the Rees cone we need to introduce another graph theoretical notion. The cone \( C(G) \) of \( G \) is the graph obtained by adding a new vertex \( v_{n+1} \) to \( G \) and joining every vertex of \( G \) to \( v_{n+1} \). If \( A \) is an independent set of \( C(G) \) we define

\[ \alpha_A = \sum_{v_i \in A} e_i - \sum_{v_i \in N(A)} e_i, \]

where \( N(A) \) is the neighbor set of \( A \) in \( C(G) \) consisting of all vertices of \( C(G) \) that are adjacent to some vertex of \( A \).

**Lemma 4.1** Let \( \mathbb{R}_+B \) be the polyhedral cone in \( \mathbb{R}^{n+1} \) generated by the set

\[ B = \{ e_i + e_j \mid \{ v_i, v_j \} \in E(G) \} \cup \{ e_i + e_{n+1} \mid 1 \leq i \leq n \}. \]

Then \( F \) is a facet of \( \mathbb{R}_+B \) if and only if

(i) \( F = \mathbb{R}_+B \cap H_{e_i}, \) for some \( 1 \leq i \leq n \), or

(ii) \( F = \mathbb{R}_+B \cap H_{\alpha_A}, \) where \( A \) is a maximal independent set of \( C(G) \).
Proof. \(\Rightarrow\) Applying [11, Theorem 3.2] to the graph \(C(G)\) it follows that we can write \(F\) as in (i) or we can write \(F = \mathbb{R}_+ B \cap H_{A}^\perp\) for some independent set \(A\) of \(C(G)\) such that the induced subgraph \(\langle V \cup \{v_{n+1}\} \setminus (A \cup N(A)) \rangle\) has non-bipartite connected components. Since this induced subgraph is bipartite one has \(V \cup \{v_{n+1}\} = A \cup N(A)\), that is, \(A\) is a maximal independent set of \(C(G)\).

\(\Leftarrow\) If \(F\) is as in (i), note \(G \setminus \{v_i\}\) is connected and non-bipartite. Hence \(F\) is a facet. Assume \(F\) is as in (ii). First note \(V \cup \{v_{n+1}\} = A \cup N(A)\) because \(A\) is a maximal independent set of \(C(G)\). Consider the subgraph \(L_1\) of \(C(G)\) with vertex set \(A \cup N(A)\) and edge set \(E(L_1) = \{z \in E(C(G)) | z \cap A \neq \emptyset\}\). One can rapidly verify (by considering a bipartition of \(G\) and showing that \(L_1\) has only even cycles) that \(L_1\) is a connected bipartite graph. Therefore \(F\) is a facet by [11, Theorem 3.2]. \(\square\)

**Theorem 4.2** \(F\) is a facet of the Rees cone \(\mathbb{R}_+ A^\perp\) if and only if

(a) \(F = \mathbb{R}_+ A^\perp \cap H_{e_i}\) for some \(1 \leq i \leq n + 1\), or

(b) \(F = \mathbb{R}_+ A^\perp \cap \{x \in \mathbb{R}^{n+1} | -x_{n+1} + \sum_{v \in C} x_i = 0\}\) for some minimal vertex cover \(C\) of \(G\).

**Proof.** \(\Rightarrow\) Since the Rees cone is of dimension \(n + 1\), there is a unique \(a \in \mathbb{Z}^{n+1}\) with relatively prime entries such that \(F = \mathbb{R}_+ A^\perp \cap H_{a}\) and \(\mathbb{R}_+ A^\perp \subset H_{a}^+\). Hence the entries of \(a\) must satisfy \(a_i \geq 0\) for \(1 \leq i \leq n\). Consider the vector

\[
b = (b_i) = (2a_1 + a_{n+1}, \ldots, 2a_n + a_{n+1}, -a_{n+1}).
\]

Using the equalities

\[
2(e_i + e_j + e_{n+1}, a) = \langle e_i + e_j, b \rangle \quad \text{if} \quad \{v_i, v_j\} \in E(G),
\]

\[
2(e_i, a) = \langle e_i + e_{n+1}, b \rangle \quad \text{if} \quad 1 \leq i \leq n,
\]

we obtain that \(F' = \mathbb{R}_+ B \cap H_b\) is a facet of \(\mathbb{R}_+ B\) with \(\mathbb{R}_+ B \subset H_b^+\). Thus from Lemma (4.1) we can write \(b\) in one of the following three forms:

\[
b = \begin{cases} 
\lambda e_i & 1 \leq i \leq n, \\
\lambda (1, \ldots, 1, -1), & \\
\lambda e_{n+1} & \text{for some maximal independent set } A \text{ of } G,
\end{cases}
\]

for some integer \(\lambda \neq 0\). In the first and second case we get \(a = e_i\) with \(1 \leq i \leq n\) and \(a = e_{n+1}\) respectively. Now consider the case \(b = \lambda e_{n+1}\) with \(A \subset V\) a maximal independent set of \(G\). Note \(A \cup N(A) = V \cup \{v_{n+1}\}\) and \(v_{n+1} \notin A\). Hence the entries of \(b\) satisfy

\[
b_i = \begin{cases} 
-\lambda & \text{if } v_i \in V \setminus A, \\
\lambda & \text{if } v_i \in A, \\
-\lambda & \text{if } i = n + 1.
\end{cases}
\]

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Thus $a_i = 0$ if $v_i \in A$. If $v_i \in V \setminus A$, then $a_i = -a_{n+1}$. It follows that $a_{n+1} = -1$. Therefore setting $C = V \setminus A$ we fall into case (b).

$\Leftarrow$ It follows using the same type of arguments as above. $\square$

As a consequence we get the irreducible representation of the Rees cone, which is the main result of this section on polyhedral geometry:

**Corollary 4.3** $\mathbb{R}_+\mathcal{A}'$ is the intersection of the closed halfspaces given by the linear inequalities

$$x_i \geq 0 \quad i = 1, \ldots, n + 1,$$

$$-x_{n+1} + \sum_{v_i \in C} x_i \geq 0 \quad C \text{ is a minimal vertex cover of } G,$$

and none of those halfspaces can be omitted from the intersection.

**Remark 4.4** Below we will give applications of Corollary 4.3. One noteworthy consequence is that we can use this result to compute the minimal vertex covers of $G$ using linear programming. Normaliz [2] can in practice be used to determine the facets of the Rees cone.

**The canonical module and the a-invariant**

Let $I = I(G)$ be the edge ideal of $G$. Since the Rees algebra $\mathcal{R}(I)$ is a normal domain and a standard graded $K$-algebra, according to a formula of Danilov-Stanley [1, Theorem 6.3.5] its canonical module is the ideal of $\mathcal{R}(I)$ given by

$$\omega_{\mathcal{R}(I)} = \{ x_1^{a_1} \cdots x_n^{a_n} t^{n+1} | a = (a_i) \in (\mathbb{R}_+\mathcal{A}')^0 \cap \mathbb{Z}^{n+1} \},$$

where $(\mathbb{R}_+\mathcal{A}')^0$ is the topological interior of the Rees cone. Thus Corollary 4.3 yields a description of the canonical module of $\mathcal{R}(I)$ in terms of halfspaces.

For use below $\beta_0$ will denote the maximal size of an independent set of $G$ and $\alpha_0$ will denote the height of $I(G)$. Thus $n = \alpha_0 + \beta_0$. The integer $\beta_0$ is called the independence number of $G$. In algebraic terms $\beta_0$ is the Krull dimension of the edge ring $R/I(G)$.

**Proposition 4.5** If $a(\mathcal{R}(I))$ is the a-invariant of $\mathcal{R}(I)$ with respect to the grading induced by $\deg(x_i) = 1$ and $\deg(t) = -1$, then

$$a(\mathcal{R}(I)) = - (\beta_0 + 1).$$

**Proof.** The $a$-invariant of $\mathcal{R}(I)$ can be expressed as

$$a(\mathcal{R}(I)) = - \min \{ i | (\omega_{\mathcal{R}(I)})_i \neq 0 \},$$
see [1]. Let $a = (a_i)$ be an arbitrary vector in $(\mathbb{R}_+ \mathcal{A})^0 \cap \mathbb{Z}^{n+1}$. By Corollary 4.3 $a$ satisfies $a_i \geq 1$ for $1 \leq i \leq n + 1$ and

$$-a_{n+1} + \sum_{v \in C} a_i \geq 1$$

for any minimal vertex cover $C$ of $G$. Let $C$ be a vertex cover of $G$ with $\alpha_0$ elements and let $A = V \setminus C$. Note $\beta_0 = |A|$. Hence if $m = x_1^{a_1} \cdots x_n^{a_n} t^{a_{n+1}}$, then

$$\deg(m) = a_1 + \cdots + a_n - a_{n+1}$$
$$= \sum_{v \in A} a_i + \sum_{v \in C} a_i - a_{n+1} \geq \beta_0 + 1.$$  

This proves $a(\mathcal{R}(I)) \leq -(\beta_0 + 1)$. On the other hand using Corollary 4.3 and the assumption $\alpha_0 \geq 2$ we get that the monomial $m = x_1 \cdots x_n t^{\alpha_0 - 1}$ is in $\omega_\mathcal{R}(I)$ and has degree $\beta_0 + 1$. Thus $a(\mathcal{R}(I)) \geq -(\beta_0 + 1)$. \hfill \Box

References


