serie notas de matemática y simposia

REUNION SOBRE TEORIA DE HOMOTOPIA

UNIVERSIDAD DE NORTHWESTERN
AGOSTO 1974

EDITOR DONAL DAVIS

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El propósito de esta serie es publicar, rápida e informalmente, notas de cursos y memorias de congresos y simposia sobre las diferentes áreas de la matemática.

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NUMERO 1

CONFERENCE ON HOMOTOPY THEORY

Evanston, Illinois, 1974

Editado por Donald M. Davis

SOCIEDAD MATEMATICA MEXICANA

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CONFERENCIAS

LISTA DE PARTICIPANTES

M. BARRATT
J. BECKER
J.M. BOARDMAN
L. CONLON
B. COOPER
D. DAVIS
J. EWING
S. GITLER
H. GLOVER
D. GOTTLIEB
B. GRAY
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C. HANKS
R. HILL
D.C. JOHNSON
D.S. KAHN
J. KAMINKER
P. KIRLEY
R. KNILL
S. KOCKMAN
I. KOZMA
T.Y. LIN
A. LIULEVICIUS
M. MAHOWALD
J.P. MAY
R.J. MILGRAM
H. MILLER
J. MILLER
R. MING
J. MOORE
R. PATTERSON
F. PETERSON
R. RANDALL
D. RAVENEL
C. RODRIGUEZ
S. ROSEN
C. SCHOCHET
R. SCHULTZ
P. SCHWEITZER
L. SMITH
V. SNAITH
J. STASHEFF
P. SUAREZ
M. TANGORA
H. TODA
M.L. WELLAND
G. WHITEHEAD
W.S. WILSON
R. ZAHLER

LISTA DE COLABORADORES

E.H. BROWN, Jr.
D.M. DAVIS
H.H. GLOVER
D.C. JOHNSON
D.S. KAHN
A. LIULEVICIUS
M. MAHOWALD
J.P. MAY
R.J. MILGRAM
H.R. MILLER
G. MISLIN
J.C. MOORE
E.P. PETERSON
D.C. RAVENEL
L. SMITH
V.P. SNAITH
M. TANGORA
H. TODA
W.S. WILSON
R.S. ZAHLER

CONFERENCE ON HOMOTOPY THEORY

NORTHWESTERN UNIVERSITY

August 12 - August 19, 1974

The following papers, all of which were received by December 1, 1974, were submitted by participants in the conference. Most of them are the talks presented at the conference.

On behalf of the participants, I wish to express thanks to Northwestern University for providing the facilities for the conference, to Mark Mahowald for organizing the conference, to the National Science Foundation for providing the money, and to the Mexican Mathematical Society for publishing these papers.

DONALD M. DAVIS

PROBLEMS PRESENTED BY PARTICIPANTS IN THE CONFERENCE

Donald Davis

There was no formal problem session, but rather participants were invited to submit problems in writing. The following were submitted:

TODA

1. What is a complex of the smallest number of cells in which p^{p^n} is non-trivial $(n \geq 3)$? Is the smallest number of cells $n+2$? It is known that the smallest complexes are $S^0 \cup_{\alpha_1} e^q$ for $n = 0$, $S^0 \cup_{\beta_1} e^{pq-1} \cup_p e^{pq}$ for $n = 1$,

$S^0 \cup_{\beta_1^p} e^{(p^2-1)q-1} \cup_{\alpha_1} e^{p^2q-1} \cup_p e^{p^2q}$ for $n = 2$, and probably

$S^0 \cup_{\varepsilon'(p-1)} e^{(p^3-p^2)q} \cup_{\beta_1^p} e^{(p^3-1)q-1} \cup_{\alpha_1} e^{p^3q-1} \cup_p e^{p^3q}$ for $n=3$.

2. Let $i : S^a \to K$, $f : \Sigma^b K \to K$, $\pi : K \to S^0$ be stable maps such that $f_r = \pi f^r i \in \pi^e_{rb+a}(S^0)_{(p)}$ is non-trivial for all $r \geq 1$. Is $b = 2p^n - 2$? Are $K = V(n)$ (essentially) and f a power of $\eta^{(n)}$? Also change S^a and S^0 to complexes.

3. Assume $V(n)$ exists for $n \geq 4$. Can the first element $\eta_1^{(n)}$-series be represented by a bracket of 2^{n-2} variables? In particular, is $\eta_1^{(4)} = \{\alpha_1, p, \beta_1, \gamma_{p-1}\}$ and

$\eta_1^{(5)} = \{\alpha_1, p, \beta_1, p, \alpha_1, p, \gamma_1, \eta_{p-1}^{(4)}\}$, etc.? For $n = 3$, $\gamma_1 = \eta_1^{(3)} = \alpha_1\beta_{p-1}$ and for $n = 2$, $\eta_1^{(2)} = \beta_1$ (one variable).

4. (Converse of Problem 3.) Show that $\eta_1^{(n)}$ cannot be represented by a bracket of less than 2^{n-2} variables. In particular, is $\eta_1^{(n)}$ indecomposable for $n \geq 4$?

5. If $V(n)$ exists, is $\eta_i^{(n)}$ nonzero for all i ?

6. For an odd prime p, is $\xi^{p^2-p+1} = 0$ for any $\xi \in \pi_i(S^0)_{(p)}$, $i > 0$? For example, $\beta_1^{p^2-p-1} \neq 0$ for $p > 5$ and $p = 3$, $\beta_1^{p^2-p+1} = 0$, and for $p = 3$, $\beta_1^6 = 0$.

7. Does there exist a chain complex $C(n)$ over $BP^*(BP)$, $n \geq 4$, so that we can construct $BP(n)$, $H_p(n)$, $VB(n)$, $B(\binom{n}{2})$ for large primes p ?

8. If the answers to Problems 2 and 7 are "yes", does the series $\{f_n\}$ survive in $BP(n)^{(a')}$ up to degree $p^n q-2$? ($BP(n)^{(a')}$ is a realization of the a'-skeleton $C(n)^{(a')}$ for some $a' = a \pmod q$).

9. Construct the spectrum $B(\binom{n}{2})$ for small primes p.

10. Given an integer $n \geq 4$, what is the smallest integer $k \geq 1$ such that we can construct a k'-stage generalized Postnikov system X_p, $k' \leq k$, satisfying

$$\lim_p (\text{rank} \sum_{i<p^n q-2} H_i(X_p; Z_p)) = a < \infty ?$$

11. In problem 10 for such a smallest integer k, is $a \geq 2^{n+\binom{n}{2}}$? Is this best possible? Can we take $X_p = H_p(n)$ for large p?

12. Does $\gamma_p = \{\beta_{p-1}, p, \alpha_1, \beta_1^p, \varepsilon'(p-1)\}$ or $\gamma_p = \{\beta_{p-1}, \varepsilon'(p-1), \beta_1^p, \alpha_1, p\}$? Does $\eta_p^{(n)'} = \{\eta_{p-1}^{(n-1)}, *, \ldots, *\}$, $(n+2)$-variables, and $\eta_p^{(n)} = \{\eta_j^{(n-1)}, *, \ldots, *\}$, $n+1$ variables, for $1 \leq j \leq p-1$?

<u>Remark</u>. $\beta_p = \{\alpha_{p-1}, \beta_1^p, \alpha_1, p\} = \{\alpha_{p-1}, p, \alpha_1, \beta_1^{p-1}, \beta_1\}$, and $\beta_{ip} = \{\alpha_j, p, *\}$, $1 \leq j < p-1$.

<u>JOHNSON-WILSON</u>

13. Let $k(n)$ be the connective spectra associated to Morva's extraordinary K-theories; these arise naturally from the study of Brown-Peterson homology. These spectra induce homology theories $k(n)_*(\ )$ with coefficients $k(n)_* = F_p[v_n]$, where the dimension of v_n is $2(p^n-1)$. There are natural transformations of homology theories: $g_n(X) : k(n)_*(X) \to H_*(X; F_p)$. For a finite complex X, it is true that if $g_n(X)$ is epic, then $g_{n+1}(X)$, $g_{n+2}(X), \ldots$ are also epic. (See our "BP operations and Morava's extraordinary K-theories".) We can define an invariant of the space X by taking the lowest integer n such that $g_n(X)$ is onto. (This is always finite for a finite complex.)

Can the above invariant be computed directly in terms of $BP_*(X)$? If so, how?

RAVENAL

14. Give a geometric construction of BP. Give a construction of MU which will yield more insight into the role of formal groups in complex cobordism.

LIN

15. Projective, flat and injective modules are three basic modules in homological algebra. For stable homotopy ring of spheres, the projective and flat modules are known; they are essentially the stable homotopy modules of Moore spaces of free and torsion free groups. On the other hand, there are no explicit examples of injective stable homotopy modules besides trivial ones (e.g. the rational numbers $Q = \pi_*(K(Q))$. According to Freyd's generating hypothesis, the stable homotopy modules of torsion finite CW-complexes (i.e., the identity map is of finite CW-complexes whose stable homotopy modules are injective (it is known that they are torsion spaces). For example, is the stable homotopy modules of Moore space of Z_p injective?

DAVIS-MILLER

16. Do there exist analogues for odd primes of the vanishing theorem of [Anderson-Davis, Comm. Math. Helv., 1973].

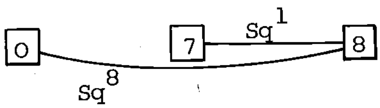
DAVIS

17. Can modified Postnikov towers for nonorientable bundles (Nussbaum-Robinson-McClendon) be used to determine whether $\mathbb{R}P^{24}$ immerses in $\mathbb{R}^{38}$.

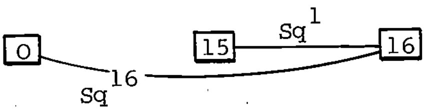
18. What is the smallest Euclidean space in which $\mathbb{R}P^n$ can be immersed for $n \equiv 3(8)$? Conjecture:

$$2n - 2\alpha(n) - \begin{cases} 2 \\ 0 \\ 1 \end{cases} \text{if} \quad \alpha(n) \equiv \begin{cases} 0(4) \\ 1(4) \\ 2,3(4) \end{cases} .$$

DAVIS-MAHOWALD

19. $H^*(bJ)$ begins 

, so that the attaching map for the 7-cell is an Arf invariant map θ_2. Can we mimic the way in which bJ is formed from bo, using the splitting of $bo \wedge bo$ [Milgram, these Proceedings], to realize higher Arf invariant classes? In particular, does $bJ \wedge bJ$ split into a wedge $bJ \vee Y_1 \vee \bigvee Y_j$, such that the fibre of $bJ \to Y_1$ begins



?

If so, can this procedure be iterated?

MAHOWALD

20. Let W(n) be the fiber of $S^{2n-1} \to \Omega^2 S^{2n+1}$. What is

the structure of $W(n)$? Conjectures: 1) $W(n)$ should be an H-space and the above fibration should be principal. 2) Consider the two maps:

a) $\Omega^2 S^{4n+1} \xrightarrow{\Omega^2(2)} \Omega^2 S^{4n+1}$ and

b) the composite

$$\Omega^2 S^{4n+1} \xrightarrow{\Omega H} \Omega^2 S^{8n+1} \xrightarrow{\Sigma^2} \Omega^4 S^{8n+3} \xrightarrow{\Omega^2 p} \Omega^2 S^{4n+1}$$

which we call f.

Then the sum $f+\Omega^2(2)$ lifts to $\phi : \Omega^2 S^{4n+1} \to S^{4n-1}$ and the fiber of φ is $BW(n)$. 3) If $W(n)$ is not an H-space then in part 2 take one more loop at every stage.

Remarks. 1) $W(1)$ is a triple loop space. Indeed if Y_4 is the fiber of $S^4 \to K(Z,4)$, $B^3W(1)$ is the fiber of $S^7 \to Y_4$ which lifts $S^7 \to S^4$. 2) The conjecture is valid at the E_2 term of the unstable spectral sequence constructed from the Λ-algebra consideration.

21. Recall that if one takes $S^3 \xrightarrow{\iota} B^3O$ be a generator and loops twice giving an H-map of $\Omega^2 S^3 \to BO$ then the Thom spectrum of ι is $K(Z_2)$. The spectrum with $\pi_0 = Z$ and only the 2-primary homology of $K(Z,O)$ in higher dimension can be realized similarly. Also there is a map of $\Omega^2 S^5 \to BF$ whose image includes all the Kervaire invariant characteristic classes. Thus the Thom spectrum is a ring spectrum whose homology algebra is a

polynomial algebra on generator

$$x_i \in H^{2^{i+1}-1}(X) \quad \text{and} \quad \varphi_i{}'{}_i U = U \cup (x_i)^*.$$

What other spectra can be realized as Thom complexes in such a fashion.

MILGRAM

22. In the bo-resolution of S^0

$$\begin{array}{ccccccc} S^0 \to bo & \longrightarrow & bo \wedge \overline{bo} & \longrightarrow & bo \wedge \overline{bo} \wedge \overline{bo} & \longrightarrow \\ & & \wr\wr & & \wr\wr & \\ & & \bigvee K(\mathbb{Z}_2) \vee \Sigma^4 bsp \vee \cdots & & \bigvee K(\mathbb{Z}_2) \vee \Sigma^8 bo^{(2)} \vee \cdots & \end{array}$$

we know the Eilenberg-MacLane spaces are acyclic for $6s > t$. What can be said about the remaining $K(\mathbb{Z}_2)$'s? In particular if $(s,t) = (3,20)$, $(2,20)$, $(3,21)$, and $(3,22)$, there seem to be nontrivial classes coming from these and representing homotopy.

SCHWEITZER

23. Compute the homotopy and cohomology of the classifying space for foliations $B\Gamma_q^r$, which classifies codimension q foliations of differentiability class C^r. It is known that the Godbillon-Vey invariant gives a surjection of $\pi_3(B\Gamma_1^r)$ onto the additive group of reals $\mathbb{R}$ provided $r \geq 2$ (Thurston). [See Lawson, Foliations, Bull. A.M.S., 1974.]

SCHULTZ

24. Which framed cobordism classes in Ω_n^{fr} are representable by exotic spheres bounding plumbings of (linear) disk bundles over S^p and S^q $(p+q = n+1)$? Presumably this would require formulas for these classes involving the bundles' classifying maps and "reasonable" homotopy operations. Hopefully such formulas should allow explicit calculation, at least when $|p-q|$ is relatively small.

Despite the geometric simplicity of the plumbing construction, this problem has proved to be difficult. Various partial results are due to D.Frank, A. Kosinski, R. Schultz, L. Smith, C.T.C. Wall, and V. Giambalvo.

25. Framed cobordism may be filtered by setting $F_{k*} = \mathrm{Ker}(\Omega_*^{fr} \longrightarrow \Omega_*^{\langle k \rangle})$, where the codomain is the bordism theory associated to $BO\langle k \rangle$. Is there a more algebraic description of F_k/F_{k+1} ? For example, $F_{k,*}$ for $* < 2k-1$ is easy to calculate, and determination of $F_{k,2k-1}$ and $F_{k,2k}$ is closely related to Problem 24. If $k \leq 4$, $F_{k,*}$ is well-known, and Giambalvo's work gives some nontrivial information about $F_{8,*}$. Wall's results on highly connected manifolds yield upper bounds for $F_{k,2k-1}$ and $F_{k,2k}$ depending on $k \bmod 8$.

Probably the most to hope for in studying $F_{k,*}$ is detection of some systematics phenomena.

26. The stabilized spaces of equivariant self-maps of

spheres F_G introduced by Becker and Schultz have the homotopy type of $Q(BG^+)$ but also admit E_∞-structures in their own right that generalize the composition structure on F. What are the Pontryagin ring structure and Dyer-Lashof operations on $H_*(F_G)$?

Results of Kochman, Madsen, May, Milgram, and Tsuchiya give partial information. Becker has suggested that the E_∞-structure may be equivalent to the smash product structure if G is abelian. Tsuchiya has suggested formulating the problem in terms of classifying spaces for finite G-sets in the spirit of Segal's Nice Congress paper on equivariant stable homotopy.

27. Let $t : S(CP^{\infty+}) \to S^0$ be the Umkehr map in stable homotopy associated to the universal S^1-bundle over CP^∞. What is the image of $t_* : \pi_*^s(S(CP^{\infty+})) \to \pi_*^s(S^0)$?

It is not onto the 2-torsion by results of Becker-Schultz and Loffler-Smith. Conjecture: It is onto the odd torsion. Even stranger. Let $X \subseteq S(CP^{\infty+})_{(p)}$ (p odd) be the subcomplex of the form $S^{q-1}_{(p)} \cup e^{2q-1}_{(p)} \cup e^{3q-1}_{(p)} \cup \ldots$, $q = 2(p-1)$. Then $t_* | \pi_*^s(X)$ is onto the p-primary component.

$H^*(MO)$ AS AN ALGEBRA OVER THE STEENROD ALGEBRA

E.H. Brown, Jr. and F.P. Peterson[(*)]

ABSTRACT. A theorem is given relating $H^*(MO)$ as a free module over the Steenrod algebra and as a module over $H^*(BO)$.

1. The Main Results

It is well known (see [4]) that $H^*(MO) = H^*(MO;Z_2)$ is a free module over A, where A is the mod 2 Steenrod algebra. $H^*(MO)$ is also a free module on one generator over $H^*(BO)$ given by the cup product. For various applications, it is important to know how these two structures are related. In this paper we give some results which partially solve the problem.

Let N be a coalgebra and let $\theta : N \to H^*(MO)$ be a monomorphism of coalgebras such that $\theta(N)$ is an A basis for $H^*(MO)$. Then θ extends to $\bar{\theta} : A \otimes N \to H^*(MO)$, an isomorphism of left A-modules and of coalgebras. The Thom isomorphism $\Phi : H^*(BO) \to H^*(MO)$ defines a right A-module structure on $H^*(BO)$ by $(u)a = \Phi^{-1}(\chi(a)(\Phi(u)))$, (see [1]. Hence, $\bar{\theta}$ defines

(*) The authors were partially supported by the N.S.F.

$\theta' : N \otimes A \to H^*(BO)$, an isomorphism of right A-modules and of coalgebras. Taking vector space duals, we obtain $F: H_*(BO) \to N^* \otimes A^*$, an isomorphism of right A^*-comodules and of algebras. The problem is to find the coalgebra structure on $N^* \otimes A^*$ which is defined by F and the coalgebra structure on $H_*(BO)$.

Theorem 1.1. One can choose N and θ in such a way that $1 \otimes \xi_k \in N^* \otimes A^*$ is primitive. Furthermore, such N and θ are unique up to a coalgebra isomorphism.

Let $\pi : N \otimes A \to A$ be projection onto the summand generated by the unit in N. Let $\{Sq^R\}$ be the Milnor basis for A. Define a different multiplication on A by $Sq^R \circ Sq^S = \binom{R+S}{R} Sq^{R+S}$, where $\binom{R+S}{R} = \prod_i \binom{r_i+s_i}{r_i}$. This multiplication is, of course, the multiplication dual to a different diagonal in A^*, namely the diagonal where ξ_k is primitive for all k. The following corollary comes immediately from the fact that

$$1 \otimes A^* \subset N^* \otimes A^* \xrightarrow{F^{-1}} H_*(BO)$$

is a map of Hopf algebras.

Corollary 1.2. With the choice of N and θ in Theorem 1.1, $\pi(\theta')^{-1} : H^*(BO) \to A$ is multiplicative with respect to the $\circ$-multiplication in A.

As an example, we recall that the Wu class $v_i = \theta'(1 \otimes Sq^i)$.

Hence, $\pi(\theta')^{-1}(v_i v_j) = \pi(\theta')^{-1}(\binom{i+j}{i} v_{i+j}) = \binom{i+j}{i} Sq^{i+j}$.

Browder [2] and Papastavrides [3], in their studies of the Kervaire invariant, have shown that the second k-invariant for the spectrum $MWu(n+1) \longrightarrow MO$ comes from the relation

$$\sum_{i=0}^{n} Sq^{n+1-i}(v_{n+1} \cdot v_i \cdot U) = 0$$

in $H^*(MO)$. One of the difficulties of using this formula is that $v_i \cdot v_{n+1} \cdot U$ is not expressed in terms of the A-basis for $H^*(MO)$. $\theta' : A \otimes N \longrightarrow M^*(MO)$ can be realized by a homotopy equivalence between MO and a wedge of $K(Z_2, i)$'s and hence $\pi(\theta')^{-1}$ can be represented by a map $g : K(Z_2, 0) \longrightarrow MO$.

<u>Corollary</u> 1.3. With the choice of N and θ' in Theorem 1.1, g^* on the above relations gives the relation

$$\sum_{i=0}^{n} Sq^{n+1-i}(\binom{n+1+i}{n+1} \chi(Sq^{n+1+i})) = 0.$$

Similar considerations apply to $H^*(MU) = H^*(MU; Z_p)$, where p is an odd prime, as a module over $'A \subset A$, the algebra of reduced powers. Theorem 1.1 and Corollary 1.2 carry over verbatim.

2. <u>Proofs</u>

Let ω be a partition of $n = n(\omega)$. Then $\{s_\omega\}$ is a basis for $H^*(BO)$. Also, $\{W_\omega\}$ is a basis for $H^*(BO)$, where $W_\omega = W_{i_1} \ldots W_{i_r}$. Let $\{b_\omega\}$ be the dual basis to $\{s_\omega\}$ and $\{W^*_\omega\}$ be the dual basis to $\{W_\omega\}$. Then $H_*(BO) = Z_2[b_{(1)}, b_{(2)}, \ldots]$. Let $\Delta : H_*(BO) \to H_*(BO) \otimes A^*$ be the dual to the right A-module structure on $H^*(BO)$. The following lemma is quite easy and we leave to proof to the reader.

<u>Lemma</u> 2.1. $W^*_{(i)}$ is primitive if $i \geq 1$. $W^*_{(i)}$ is indecomposable if and only if i is odd. $W^*_{(2i)} = (W^*_{(i)})^2$.

Let $D \subset H^*(BO)$ be the decomposible elements. Let $\{Sq^R\}$ be the Milnor basis for A. The following lemma is key to our results.

<u>Lemma</u> 2.2. Let $n(R) = 2^k - 1$. Then (1) $Sq^R \in D$ if and only if $R \neq \Delta_k$, where Δ_k is the sequence with all zeros except a one in the k^{th} spot.

We procede the proof of Lemma 2.2 by some preliminary results.

<u>Lemma</u> 2.3. Let $\iota \in H^1(RP^\infty)$. Let I be an admissible Adem sequence. Then $\chi(Sq^I)(\iota) \neq 0$ if and only if $I = (2^k - I)$.

<u>Proof</u>. $\chi(Sq^{2^k-1})(i) = Sq^{2^{k-1}} Sq^{2^{k-2}} \ldots Sq^2 Sq^1(\iota) \neq 0$ is well known. For the converse, let $I = (i_1, i_2, \ldots, i_t)$. If $\chi(Sq^I)(\iota) = \chi(Sq^{i_t}) \ldots \chi(Sq^{i_1})(\iota) \neq 0$, then $i_1 = 2^j - 1$ and

$\chi(Sq^{i_1})(\iota) = \iota^{2^j}$. However, if $a \in A$, then $a(\iota^{2^j}) = 0$ if $\dim a < 2^j$. But $i_2 \le 2i_1$ and hence $i_2 < 2^j - 1$, so $\chi(Sq^I)(\iota) = 0$ if $t > 1$.

Lemma 2.4. Let $Sq^R = \Sigma a_I \chi(Sq^I)$, where Sq^R is a Milnor basis element and I is admissible. If $n(R) = 2^k - 1$, then $a_{(2^k-1)} \neq 0$ if and only if $R = \Delta_k$.

Proof. Apply the equation to $\iota \in H^1(RP^\infty)$. The excess of R is the sum of the entries. Hence $Sq^R(\iota) \neq 0$ if and only if $R = \Delta_k$. $\Sigma a_I \chi(Sq^I)(\iota) \neq 0$ if and only if $a_{(2^k-1)} \neq 0$ by 2.3.

Lemma 2.5. Let I be admissible. Then $(1)\chi(Sq^I) \notin D$ if and only if $I = (n)$.

Proof. If $I = (n)$, then $(1)\chi(Sq^n) = W_n$. If $I = (i_1, \ldots, i_t)$, then

$$(1)\chi(Sq^{i_t})\chi(Sq^{i_{t-1}}) = (W_{i_t})\chi(Sq^{i_{t-1}}) =$$

$$= \sum_{j=0}^{i_{t-1}} (1)\chi(Sq^{i_{t-1}-j}) \cdot Sq^j(W_{i_t}) \in D$$

as $i_{t-1} \ge 2i_t$.

We now prove Lemma 2.2, $(1)Sq^R \notin D$ if and only if $(1)\Sigma a_I \chi(Sq^I) \notin D$ if and only if $a_{(2^k-1)} \neq 0$ by 2.5 if and only if $R = \Delta_k$ by 2.4.

As a corollary to 2.2, we have the following result.

<u>Corollary</u> 2.6. Let $q+n(R) = 2^k-1$. Then $(W_q)Sq^R \notin D$ if and only if $R = \Delta_i$ and $q = 2^k-2^i$.

<u>Proof</u>. Since $W_q \equiv (1)Sq^S \bmod D$ for various S, we need to study when $Sq^S Sq^R = Sq^{\Delta_k} + \text{others}$. That is, we must find a Milnor matrix with a single one in it. This happens if and only if $R = \Delta_i$ and $q = 2^k-2^i$.

Using these results, we now prove our main calculational theorem.

<u>Theorem</u> 2.7. $\Delta(W^*_{(2^k-1)}) = \sum_{i=0}^{k} (W^*_{(2^{k-i}-1)})^{2^i} \otimes \xi_i$

<u>Proof</u>. Since $\{W_\omega\}$ is a basis for $H^*(BO)$ and $\{Sq^R\}$ is a basis for A, we get the following result.

$$\Delta(u) = \sum_{\omega,R} \langle (W_\omega)Rq^R, u\rangle W^*_\omega \otimes \xi_R ,$$

where $\xi_R = \xi_1^{r_1}\xi_2^{r_2}\cdots$. We apply this to $u = W^*_{(2^k-1)}$. Since $\langle (W_\omega)Sq^R, W^*_{(2^k-1)}\rangle \neq 0$ if and only if $(W_\omega)Sq^R \notin D$, we obtain

$$\Delta(W^*_{(2^k-1)}) = \sum_{q,R} \langle (W_q)Sq^R, W^*_{(2^k-1)}\rangle W^*_q \otimes \xi_R = \sum_{i=0}^{k} W^*_{(2^k-2^i)} \otimes \xi_i$$

by Corollary 2.6. Finally, $W^*_{(2^k-2^i)} = (W^*_{(2^{k-i}-1)})^{2^i}$ by 2.1.

An alternative way of constructing $F : H_*(BO) \to N^* \otimes A^*$ is the following. Let $J \subset H_*(BO)$ be an ideal with generators one indecomposible in every dimension of the form 2^q-1. Then the composition

$$F_J, H_*(BO) \xrightarrow{\Delta} H_*(BO) \otimes A^* \to H_*(BO)/J \otimes A^*$$

is an isomorphism of algebras and right A^*-comodules. That is, we take $N = (H_*(BO)/J)^*$ and construct θ by

$$N \to N \otimes 1 \subset N \otimes A \xrightarrow{F_J^*} H^*(BO) \xrightarrow{\Phi} H^*(MO).$$

Furthermore, up to an algebra isomorphism of N^*, any F can be constructed this way by taking J to be the kernel of

$$H_*(BO) \to N^* \otimes A^* \to N^*.$$

In order to prove Theorem 1.1, we must choose J such that $F_J(W^*_{(2^k-1)}) = 1 \otimes \xi_k$, since $W^*_{(2^k-1)}$ is primitive by 2.1. Let J be the ideal generated by $W^*_{(2^q-1)}$, $q \geq 1$. Then, by Theorem 2.7, $F_J(W^*_{(2^k-1)}) = 1 \otimes \xi_k$. The uniqueness of J follows from the fact that $\Delta(W^*_{(2^k-1)}) = W^*_{(2^k-1)} \otimes 1 +$ other terms.

We conclude by giving an N and $\theta' : N \to H^*(BO)$ which satisfies Theorem 1.1. Let $N = \bigotimes_{i \neq 2^q-1} C_i$, where C_i is a coalgebra with one generator ${}^i x_{(k)}$ of dimension ki, $k = 1,2,\ldots,$

and $\psi({}^i x_{(k)}) = \Sigma\, {}^i x_{(j)} \otimes {}^i x_{(k-j)}$. We define $\theta'|C_i$ and extend to N multiplicatively, i.e. $\theta'(c_i \otimes c_j) = c_i \cdot c_j$. If i is even, define $\theta'({}^i x_{(k)}) = s_\omega$, where $\omega = (i,i,\ldots,i)$, k-times. If i is odd, let $a_1, a_2, \ldots, a_j < a_{j+1}$, be the integers such that $i | 2^{a_j}-1$. Further, a_j is in the list if and only if $(2^{a_{j'}}-1) \nmid (2^{a_j}-1)$ for any $j' < j$. Let $i\ell_j = 2^{a_j}-1$. Define

$$\theta'({}^i x_{(k)}) = s_\omega + \sum_j \sum_{m=1}^{\infty} s_{\omega'(m,j)} \cdot s_{\omega''(m,j)}$$

where $\omega = (i,\ldots,i)$, k-times, $s_{\omega'(m,j)} = (i,\ldots,i)$, $k - m\ell_j$ times and $s_{\omega''(m,j)} = (2^{a_j}-1,\ldots,2^{a_j}-1)$, m-times.

Brandeis University
Massachusetts Institute of Technology

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THE IMMERSION CONJECTURE FOR $\mathbb{R}P^{8\ell+7}$

Donald M. Davis(*)

Seven years ago Gitler, Mahowald, and Milgram made a well-known conjecture about immersions of real projective space in Euclidean space. <u>Conjecture</u> [4,6]. If $n \equiv 7(8)$, the smallest Euclidean space in which $\mathbb{R}P^n$ can be immersed has dimension

$$2n - 2\alpha(n) + \begin{cases} 0 \\ 1 \\ -1 \end{cases} \text{if} \quad \alpha(n) \equiv \begin{cases} 0(4) \\ 1,2(4) \\ 3(4) \end{cases}$$

where $\alpha(n)$ denotes the number of ones in the binary expansion of n.

In joint work Mahowald and I have proved the immersion part for small $\alpha(n)$.

<u>Theorem</u> 1. If $n \equiv 7(8)$, the smallest Euclidean space in which $\mathbb{R}P^n$ can be immersed has dimension =

$$2n - 2\alpha(n) + \begin{cases} 0 \\ 1 \\ -1 \end{cases} \text{if} \quad \alpha(n) \equiv \begin{cases} 0(4) \\ 1,2(4) \\ 3(4) \end{cases}$$

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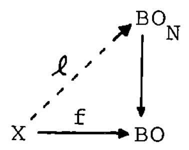
if $\alpha(n) = 4,5,6,8,$ or 9.

At the Conference and in [3] the non-immersion result was also announced. There was found a gap in the evaluation of the indeterminary of the secondary bo obstruction. That result is thus not proved.

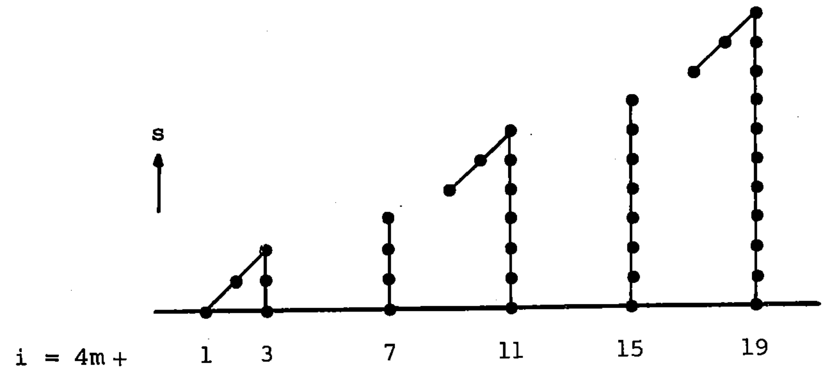
The approach is to use obstruction theory to find the geometric dimension of the stable normal bundle $(2^L-n-1)\xi_n$, where L is any sufficiently large integer and ξ_n denotes the canonical line bundle over $\mathbb{R}P^n$. Recall that the geometric dimension of a vector bundle is the smallest N such that there is a lifting ℓ of its classifying map f:



The first step is to evaluate the bo-primary obstructions for multiples pH_k of the Hopf bundle over quaternionic projective space QP^k. We show that they are effectively the symplectic Pontryagin classes e_i. To be more precise, let $\widetilde{BSp}_N$ denote the classifying space for symplectic vector bundles of real geometric dimension N. Let B_N^O denote $\widetilde{BSp}_N \wedge_{BSp} bo$, the fiberwise smash with the spectrum bo localized at 2. We show in [2] how to form the fiberwise smash for fibrations without a section.

Recall that the fibre of $\widetilde{BSp}_N \to BSp$ is $V_N = \lim_k V_{N+k,k}$

which has the same 2N-type as $P_N = \mathbb{R}P^\infty/\mathbb{R}P^{N-1}$. Thus through dimension 2N the fibre of $B_N^O \to BSp$ is $P_N \wedge bo$, whose homotopy groups are well-known [1,6] cyclic 2-groups. For example, if $N = 4m+1$, we have $\pi_i(P_{4m+1})$



where the number of dots indicates the exponent of 2 $\nu(\pi_i(P_{4m+1}))$ and the height s is Adams filtration. We prove [2]

<u>Theorem</u> 2. If $N \geq 2k$, pH_k lifts to B_N^O if and only if for all $i \leq k$, $\nu\binom{p}{i} \geq \nu(\pi_{4i-1}(P_N \wedge bo))$. Since the coefficient of $e_i(pH_k)$ is $\binom{p}{i}$, the condition of the theorem may be restated as

$$\rho(e_i(pH_k)) = 0 \in H^{4i}(QP^k; \pi_{4i-1}(P_N \wedge bo)).$$

Theorem 2 is proved inductively by writing $p = 2^i + p'$ with $0 < p' \leq 2^i$, and noting that pH_k is classified by the composite

$$QP^k \xrightarrow{\Delta} (QP^k \times QP^k)^{(4k)} = \bigcup_{\ell} QP^\ell \times QP^{k-\ell} \xrightarrow{2^i H \times p'H} BSp \times BSp \to BSp,$$

where Δ is a skeletal map homotopic to the diagonal. Let $N(p,k)$ denote the smallest N such that pH_k lifts to B^O_N. There is a pairing $B^O_N \times B^O_M \to B^O_{n+M}$ compatible with the Whitney sum pairing, so $QP^{\ell} \times QP^{k-\ell}$ lifts to $B^O_{N(2^i,\ell)+N(p',k-\ell)}$. Since $\pi_{4i}(P_N \wedge bo) = 0$ there is no obstruction to fitting lifting together, and so $N(p,k) \leq \max_{\ell}(N(2^i,\ell)+N(p',k-\ell))$. It would be nice if that were sufficient to make the induction work, but unfortunately the desired value of $N(p,k)$ is often several less than $\max_{\ell}(N(2^i,\ell)+N(p',k-\ell))$, and various technical difficulties had to be overcome to make the induction work. See [2].

The immersions of $\mathbb{R}P^{4\ell+3}$ are obtained by using the canonical map $\mathbb{R}P^{4\ell+3} \xrightarrow{h} QP^{\ell}$ and modified Postnikov towers (MPT) [3,5]. Suppose that to prove a certain immersion we are trying to show $gd((2^L-4\ell-4)\xi_{4\ell+3}) \leq N$.

Consider the diagram 1, where the spaces E_i form a $(4\ell+3)$-MPT for $\widetilde{BSp}_N \to BSp$ and E^O_i a $(4\ell+3)$-MPT for $B^O_{N+\Delta} \to BSp$. Here Δ is such that by Theorem 2, QP^{ℓ} lifts to $B^O_{N+\Delta}$; Δ will be approximately 3 or 4. By observing the tables of [7] we see that through a fixed range the Adams chart for $\pi_*(P_N)$ goes several (D) higher than the Adams chart for $\pi_*(P_{N+\Delta} \wedge bo)$. QP^{ℓ} lifts to E^O_s and we would like to show that it lifts to E_s. The obstructions to this lifting lie in $H^{4i}(QP;\pi_{4i-1}(\text{fibre}(E_s \to E^O_s)))$. Let $F_S = \text{fibre}(E_s \to BSp)$. Then $\pi_*(F_S)$ consists of those elements of $\pi_*(V_N)$ of Adams

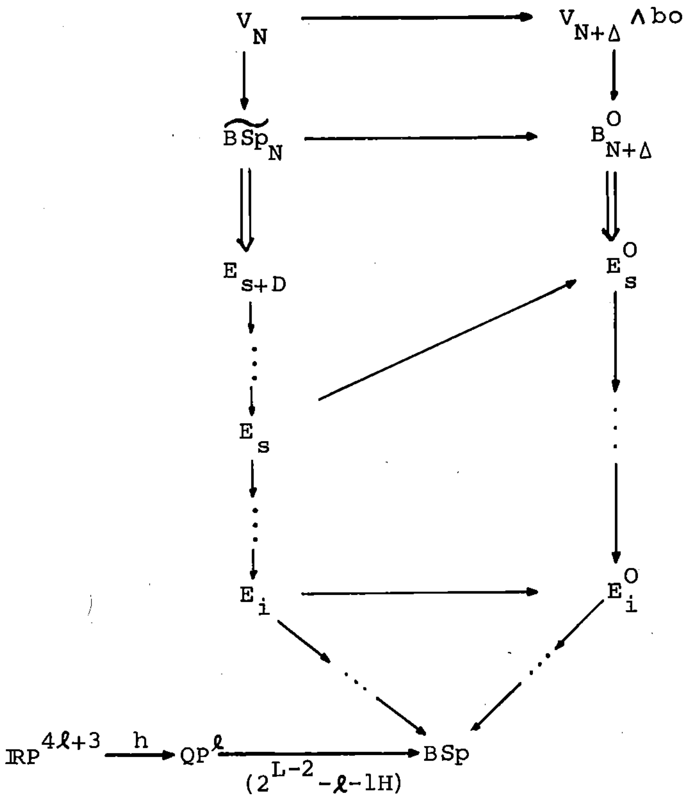


Diagram 1.

filtration $< s$, and fibre $(E_s \to E^O_s) = \text{fibre}(F_S \to V_{N+\Delta} \wedge bo)$. Thus if $\pi_{4i-1}(P_N)$ contains nothing except the tower which clearly injects into $\pi_{4i-1}(P_N \wedge bo)$, then $\pi_{4i-1}(\text{fibre}(E_s \to E^O_s)) = 0$ so QP^ℓ lifts to E_s. The tables of [7] show this is often true through $\pi_{N+18}(P_N)$; this is the reason that we can only assert the immersions for small values of $\alpha(n)$. There is a naturality argument by which we can often show that the first occurrence of a non-bo-primary element in $\pi_{4*-1}(P_N)$ cannot obstruct QP^{ℓ} [2,3], but an entirely different argument will be required to obtain immersions for large $\alpha(n)$.

Having lifted QP^{ℓ} to E_s, we lift $\mathbb{R}P^{4\ell+3}$ to $E_{s+D} = \widetilde{BSp}_N$ by indeterminacy computations [2,3].

Northwestern University

Lehigh University

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VECTOR FIELDS ON 2-EQUIVALENT MANIFOLDS

Henry H. Glover and Guido Mislin

Introduction

It is known from the work of [3], [7] and [13] that if M and N are homotopy equivalent closed differentiable manifolds of dimension n and if $k < n/2$, then M admits a k-field if and only if N does.

We generalize this result to the extent that we only assume the manifolds M and N to be equivalent at the prime 2, in a sense made precise in Section 1. The result is proved using the technique of localization of homotopy types [4], [9], [10] and relies on the main pullback theorem of localization.

We call a space p-good, if it is good with respect to the ring $\mathbb{Z}/p$ in the sense of Bousfield and Kan [4] (see also Section 1). Our main theorem is then the following.

Theorem 0.1. Let M and N be connected closed orientable differentiable manifolds of dimension n which are 2-good. Suppose that $\hat{M}_2 \simeq \hat{N}_2$ and let $k < n/2$. Assume that N admits a k-field. Then

(i) M admits a k-field if $n-k$ is odd,

(ii) M admits a $(k-1)$-field if $n-k$ is even.

A slightly different version of Theorem 0.1 could be obtained using partial $\mathbb{Z}/2$-localization, localizing all higher homotopy groups of the manifolds M and N, and using the technique of [2].

We say that M and N are 2-equivalent if $\hat{M}_2 \simeq \hat{N}_2$. Recall from [5] the definition of the semi-characteristic of a closed n-manifold M,

$$\chi^*M = \begin{cases} 1/2\ \chi M, \text{ if } n \text{ even,} \\ \left(\sum_{i=0}^{r} \text{rank } H_i(M;\mathbb{Z}/2) \right) \pmod 2, \text{ if } n = 2r+1. \end{cases}$$

(χM the Euler characteristic of M). We get from Theorem 0.1 the following corollary.

<u>Corollary</u> 0.2. Let M be a 2-good n-dimensional closed differentiable manifold which is 2-equivalent to a π-manifold. Let $\rho(n+1)$ denote the Hurwitz-Radon number (i.e. the span of S^n) and let $\chi^*(M)$ denote the semi-characteristic of M. Then one has the following.

(i) If $\dim M = 7$ then M admits a 2-field and if $\dim M = 15$ then M admits a 6-field.

(ii) If $\chi^*(M) \neq 0$ and $n \neq 7, 15$ then M admits a $2[\rho(n+1)/2]$-field.

(iii) If $\chi^*(M) = 0$ then M admits a $(n-2)/2$-field if dim $M \equiv 0(4)$, $(n-1)/2$-field if dim $M \equiv 1(4)$, $(n-4)/2$-field if dim $M \equiv 2(4)$, $(n-3)/2$-field if dim $M \equiv 3(4)$.

In case $n = 7$, (ii) of Corollary 0.2 does not hold. Sjerve gave an example of a lens space which is not parallelizable, but which is 2-equivalent to the 7-sphere [12]. However, in case M is any spherical space form with fundamental group of odd order, then (ii) of the above corollary is sharp if $\rho(n+1)$ is even; if $\rho(n+1)$ is odd such an M admits a $\rho(n+1)$-field as was proved by Yoshida [14]. A complete solution of the vector field problem for generalized spherical space forms was given by Becker [1].

The plan of the paper follows. In section 1 we give the facts about localization and spherical fibrations needed.

In Section 2 we prove a lifting theorem and generalize the results of Sutherland [13] and Dupont [7] to our situation.

In Section 3 we prove Theorem 0.1 and Corollary 0.2.

1. Localization of Spherical Fibrations.

All spaces we consider are supposed to be of the homotopy type of well-pointed connected CW complexes.

Let $\mathbb{Z}_p$ denote the integers localized at (p) and $\mathbb{Z}/p$ the field with p elements. According to [4] a space X has a

$\mathbb{Z}_p$-localization and a $\mathbb{Z}/p$-localization, which we denote by X_p and $\hat{X}_p$ respectively. For the following definition compare with [4].

Definition 1.1. X is called p-good if the canonical map $X \to \hat{X}_p$ induces

$$H_*(X;\mathbb{Z}/p) \xrightarrow{\cong} H_*(\hat{X}_p;\mathbb{Z}/p).$$

For instance, if X is nilpotent or if $H_1(X;\mathbb{Z}/p) = 0$, then X is p-good.

Let $\xi : F \to E \xrightarrow{pr} X$ denote a fibration over X. Then we call the mapping cone of pr the Thom space of ξ, and denote it by $T\xi$. In case the fibration in question is of a specific type, classified by some map $X \to B$, we will often use the same letter ξ to denote the classifying map.

Denote by $SF(n)$ the monoid of degree one pointed maps of S^n and let $SF = USF(n)$, with classifying spaces $BSF(n)$ and BSF. Recall that $BSF(n)$ classifies oriented S^n-fibrations with cross section. The obvious maps $SO(n) \to SF(n)$ and $SO \to SF$ induce maps of classifying spaces, which we will denote by the same letter $J : BSO(n) \to BSF(n)$, $J : BSO \to BSF$ respectively. The composite of a map $f : X \to BSO(n)$, $(g : Y \to BSF(n))$ with the canonical map $BSO(n) \to BSO$, $(BSF(n) \to BSF)$ will be denoted by $f_{st}(g_{st})$. Since BSF is 1-connected and since all homotopy groups of BSF are finite, one has $(BSF)_p \cong (BSF)^{\wedge}_p$

and therefore, for a p-good space X, the canonical map $X \longrightarrow \hat{X}_p$ induces a bijection.

$$[\hat{X}_p, BSF_p] \cong [X, BSF_p]$$

by the universal property of $\mathbb{Z}/p$-localization for p-good spaces.

All manifolds we consider are supposed to be orientable, closed, connected and differentiable.

<u>Lemma</u> 1.2. Let $S^k \longrightarrow E \xrightarrow{pr} M$ denote an oriented sphere fibration with cross section over some manifold M and let $\xi : M \longrightarrow BSF(k)$ be the classifying map. Suppose that M is p-good. Then

(i) $(pr)^\wedge_p : \hat{E}_p \longrightarrow \hat{M}_p$ has fiber $\hat{S}^k_p$,

(ii) E is p-good,

(iii) The fibration $\hat{S}^k_p \longrightarrow \hat{E}_p \longrightarrow \hat{M}_p$ is fiberhomotopy equivalent to the fibration induced by $\hat{\xi}_p : \hat{M}_p \longrightarrow BSF(k)^\wedge_p$ from the universal $\hat{S}^k_p$-fibration over $BSF(k)^\wedge_p$,

(iv) $T(\xi)^\wedge_p \simeq (T(\hat{\xi}_p))^\wedge_p$.

<u>Proof</u>. As $S^k \longrightarrow E \xrightarrow{pr} M$ is induced by the universal fibration $S^k \longrightarrow ESF(k) \longrightarrow BSF(k)$ and since $\pi_1 BSF(k) = 0$, we conclude that $\pi_1 M$ operates trivially on the homotopy groups $\pi_i S^k$ of the fiber of pr. Hence (i) follows by [4;II, 4.8]. To see (ii), we consider the induced morphism of Serre spectral sequences, associated with the map of fibrations

$$\begin{array}{ccccc} S^k & \longrightarrow & E & \longrightarrow & M \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \hat{S}^k_p & \longrightarrow & \hat{E}_p & \longrightarrow & \hat{M}_p \end{array}$$

where α, β and γ denote the canonical maps. As α and β induce isomorphisms in mod p homology and as $\pi_1 M$ operates trivially on $H_* S^k$, it follows that β induces an isomorphism in mod p homology. Hence E is p-good. Since $\pi_1 BSF(k) = 0$, we obtain by $\mathbb{Z}/p$-localization a fibration

$$\hat{S}^k_p \longrightarrow ESF(k)^{\wedge}_p \longrightarrow BSF(k)^{\wedge}_p$$

and using the universal property of $\mathbb{Z}/p$-localization of p-good spaces it is immediate that there is a canonical map of fibrations of degree 1 on the fiber

$$\begin{array}{ccc} \hat{S}^k_p & \longrightarrow & \hat{S}^k_p \\ \downarrow & & \downarrow \\ \hat{E}_p & \longrightarrow & X \\ & {\scriptstyle\hat{p}r_p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \hat{M}_p & \end{array}$$

where q denoted the fibration over $\hat{M}_p$ induced by $\hat{\xi}_p$. This proves (iii). For (iv) we consider the diagram

$$\begin{array}{ccc}
E & \longrightarrow & \hat{E}_p \\
{\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \hat{pr}_p} \\
M & \xrightarrow{\ g\ } & \hat{M}_p \\
\downarrow & & \downarrow \\
T(\xi) & \xrightarrow{\ h\ } & T(\hat{\xi}_p)
\end{array}$$

with f, g and h the obvious maps. As f and g induce isomorphisms in mod p homology the same is true for h. Therefore, h induces a homotopy equivalence $T(\xi)^{\wedge}_{p} \longrightarrow (T(\hat{\xi}_p))^{\wedge}_{p}$ [4;I, 5.5] and the proof of the lemma is completed.

This allows us to reformulate Proposition 4.1 of [8] to get by the same kind of argument the following result.

<u>Lemma</u> 1.3. Let M and N be p-good oriented manifolds and $\lambda : \hat{M}_p \longrightarrow \hat{N}_p$ a homotopy equivalence. Then

$$(J\nu(N)_{st})^{\wedge}_{p} \circ \lambda = (J\nu(M)_{st})^{\wedge}_{p} \quad \text{in} \quad [\hat{M}_p, BSF^{\wedge}_{p}] = [M, BSF_p].$$

where $J\nu(\ )_{st}$ denotes the classifying map for the oriented stable normal fibration.

2. <u>The</u> <u>Lifting</u> <u>Theorem</u>.

The following theorem is an unstable analogue of [8, Prop.

3.5].

Theorem 2.1. Let M be an n-dimensional manifold and let ξ denote an oriented $\mathbb{R}^n$-bundle over M, classified by $\xi : M \to BSO(n)$. Let $s > n/2$ be an integer. Then the following are equivalent.

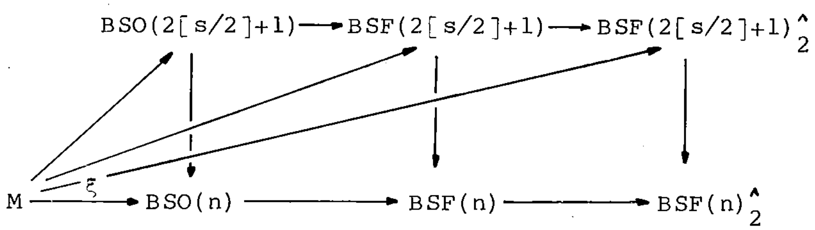
(i) ξ lifts to $BSO(2[s/2]+1)$.

(ii) $\hat{e}_2 \circ J\xi : M \to BSF(n)^\wedge_2$ lifts to $BSF(2[s/2]+1)^\wedge_2$

($\hat{e}_2 : BSF(n) \to BSF(n)^\wedge_2$ denotes the canonical map.)

Proof. Of course one has only to prove that (ii) implies (i). Consider the diagram

(2.2)



Case 1: n odd. If n is odd, one has isomorphisms

$$can : \pi_i(SF(n), SF(2[s/2]+1)) \to \pi_i(SF(n)^\wedge_2, SF(2[s/2]+1)^\wedge_2)$$

for $i \leq 4[s/2]+2$, (see for instance [8, Lemma 2.2]) taking into account that all the homotopy groups of $SF(k)$ are finite in case k is odd. Since $4[s/2]+2 \geq 4[n/4]+2 \geq n-1$ and $\dim M = n$, it follows by an obstruction argument first observed by Sutherland [13] (c.f. Lemma 2.3 of [8]) that the existence of the lift h. By a result of James [11]

$$\mathrm{can} : \pi_i(SO(n), SO(2[s/2]+1)) \longrightarrow \pi_i(SF(n), SF(2[s/2]+1))$$

is an isomorphism for $i \leq 4[s/2]+2$. Hence, again by obstruction theory, the lift k exists if one has the lift h.

Case 2: n even. We will first show that if $\hat{e}_2 \circ J\xi$ lifts to $BSF(n-1)$ then ξ lifts to $BSO(n-1)$. For this consider the diagram

$$\begin{array}{ccc}
BSO(n-1) & \longrightarrow & BSF(n-1)^{\wedge}_2 \\
\downarrow & & \downarrow \\
BSO(n) & \longrightarrow & BSF(n)^{\wedge}_2 \\
E \downarrow & & \downarrow \bar{E}^{\wedge}_2 \\
K(\mathbb{Z}, n) & \xrightarrow{\ \mathrm{loc}\ } & K(\hat{\mathbb{Z}}_2, n)
\end{array}$$

with $\bar{E}^{\wedge}_2$ the first Postnikov invariant of $BSF(n-1)^{\wedge}_2 \to BSF(n)^{\wedge}_2$. Since the coefficient homomorphism

$$\theta : H^n(M;\mathbb{Z}) = \mathbb{Z} \longrightarrow H^n(M;\hat{\mathbb{Z}}_2) = \hat{\mathbb{Z}}_2$$

is injective, we see that if $\hat{e}_2 \circ J\xi$ lifts to $BSF(n-1)^{\wedge}_2$ then $\bar{E}^{\wedge}_2(\hat{e}_2 J\xi) = \theta(E(\xi)) = 0$ and therefore $E(\xi) = 0$, which implies that ξ lifts to $BSO(n-1)$. As $n-1$ is odd, the proof is then completed by proceeding in the same way as Case 1.

The next lemma is a special case of a theorem of Sutherland [13].

Lemma 2.4. Let M be an oriented even dimensional manifold with oriented tangent bundle τ. Suppose ξ is an oriented n-spherical fibration over M with cross section such that

(i) $\xi_{st} = J(\tau)_{st} : M \to BSF$,

(ii) $\bar{E}(\xi) = E(\tau)$, $E : BSO(n) \to K(\mathbb{Z},n)$ and $\bar{E} : BSF(n) \to K(\mathbb{Z},n)$ the Euler classes. Then $\xi = J(\tau) : M \to BSF(n)$.

An analogous lemma in case $\dim M$ is odd may be formulated by using an invariant introduced by Dupont [7] in place of the Euler class. Let M denote an n-manifold with n odd and let $\xi, \eta : M \to BSF(n)$. If $\xi_{st} = \eta_{st} : M \to BSF$ then either $\xi = \eta$ or $\xi = \lambda * \eta$ where $\lambda \in \pi_n BSF(n)$ is the characteristic element associated with the tangent bundle of S^n and

$$* : \pi_n BSF(n) \times [M, BSF(n)] \to [M, BSF(n)]$$

is the operation induced by the usual cooperation

$$M \to M \vee S^n$$

coming from the attachment of the top cell of M (see [6]). Dupont defines then [7] an invariant $b(\xi) \in \mathbb{Z}/2$ for $\xi : M \to BSF(n)$, which is well defined in case there is an $\tilde{\eta} : M \to BSO(n)$ with $\xi \neq J(\tilde{\eta})$ but $\xi_{st} = J(\tilde{\eta})_{st}$. Further, $b(J\tilde{\eta})$ is well defined too and in this situation $b(\xi) \neq b(J\tilde{\eta})$. We extend his definition in the following way: Let $\xi : M \to BSF(n)$.

Then

$$D(\xi) = \begin{cases} b(\xi), & \text{if } b(\xi) \text{ is well defined} \\ 0, & \text{if } b(\xi) \text{ is not defined.} \end{cases}$$

It follows then that the invariant D has the following property.

<u>Lemma</u> 2.5. Let M be an oriented n-manifold, n odd, and let ξ be an oriented n-spherical fibration over M with cross section such that

(i) $\xi_{st} = J(\tau)_{st} : M \longrightarrow BSF$

(ii) $D(\xi) = DJ(\tau)$.

Then $\xi = J(\tau) : M \longrightarrow BSF(n)$.

There is a local version of Lemma 2.5. Namely, for n odd $BSF(n) \simeq \Pi BSF(n)_p$ and as the element $\lambda \in \pi_n BSF(n)$ associated with the tangent bundle of S^n has order at most 2, it follows that for $\xi, \eta : M \longrightarrow BSF(n)$ with $\xi_{st} = \eta_{st}$ one has $e_p \circ \xi = e_p \circ \eta$ for all odd primes, and hence

$$\xi \simeq \eta \text{ if and only if } e_2 \circ \xi \simeq e_2 \circ \eta$$

($e_p : BSF(n) \longrightarrow BSF(n)_p$ the canonical map). Also, it makes perfect sense to define D for $\delta : M \longrightarrow BSF(n)_2$, n odd, since $[M, BSF(n)] \longrightarrow [M, BSF(n)_2]$ is surjective, by defining $D(\delta) = D(\bar{\delta})$, where $\bar{\delta}$ is any lift of δ to $BSF(n)$.

The following lemma follows then immediately from the above remarks.

Lemma 2.6. Let n be odd and let M be an oriented 2-good n-manifold. Let $\xi : M \to BSF(n)$, $\tau(M) : M \to BSO(n)$ be such that

(i) $\hat{\xi}_2 \simeq J\tau(M)^{\wedge}_2$

(ii) $\xi_{st} \simeq J\tau(M)_{st}$.

Then $\xi \simeq J\tau(M)$.

Lemma 2.7. Let n be odd and let M and N be oriented n-manifolds. Let $\lambda : \hat{M}_2 \simeq \hat{N}_2$. Suppose M (and hence N) is 2-good and suppose given $\xi : M \to BSF(n)$, such that

(i) $\xi_{st} \simeq J\tau(M)_{st}$

(ii) $\xi^{\wedge}_2 \simeq J\tau(N)^{\wedge}_2 \circ \lambda$.

Then $D(\xi) = D(J\tau(M))$ and hence $\xi = J\tau(M) : M \to BSF(n)$.

The proof of Lemma 2.7 follows from the proof of Theorem 5.1 of [7] using naturality and the good behavior of the Thom spaces of spherical fibrations under localization (c.f. Lemma 2.2).

We can now prove the following local version of [7, Lemma 5.3 and Theorem 5.7].

Theorem 2.8. Let M and N be closed oriented differentiable manifolds of odd dimension which are p-good. Suppose $\lambda : \hat{M}_p \simeq \hat{N}_p$. Then

$$J\tau(M)^{\wedge}_p = J\tau(N)^{\wedge}_p \circ \lambda \in [M^{\wedge}_p, BSF(n)^{\wedge}_p].$$

Proof. For p odd, the theorem follows from Lemma 1.3 since $[M,BSF(n)_p]\longrightarrow[M,BSF_p]$ is then a bijection. In case $p = 2$, choose $\xi : M \longrightarrow BSF(n)$ by putting $\hat{e}_p\circ\xi = \hat{e}_p\circ J\tau(M)$ for p odd and $\hat{e}_2\circ\xi = J\tau(N)^{\wedge}_2\circ\lambda\circ\hat{e}_2$. Then, in view of Lemma 1.3, one has

$$\xi_{st} = J\tau(M)_{st}$$

and by construction

$$\hat{\xi}_2 = J\tau(N)^{\wedge}_2\circ\lambda.$$

Hence $\xi \simeq J\tau(M)$ by Lemma 2.7 and therefore $\hat{\xi}_2 = J\tau(M)^{\wedge}_2 =$ $= J\tau(N)^{\wedge}_2\circ\lambda$.

An analogous theorem for even dimensional manifolds would follow from Lemma 2.4 in case one assumes for instance that $\chi(M) = 0$.

3. The Proof of Theorem 0.1 and Corollary 0.2.

Notice that for 2-good manifolds M and N, $\hat{M}_2 \simeq \hat{N}_2$ implies that the Euler characteristics $\chi(M)$ and $\chi(N)$ agree, since they are determined by the mod 2 homology of the spaces involved. To prove Theorem 0.1, assume that $\lambda : \hat{M}_2 \longrightarrow \hat{N}_2$ is a homotopy equivalence. We distinguish two cases.

Case 1: n odd. By Theorem 2.8, we have $J\tau(N)^{\wedge}_2\circ\lambda = J\tau(M)^{\wedge}_2$.

As N has a k-field $\tau(N)$ lifts to $BSO(n-k)$. Hence $J\tau(N)^{\wedge}_2 \circ \lambda = J\tau(M)^{\wedge}_2$ lifts to $BSF(n-k)_2$ and since $n-k > n/2$, we conclude by Theorem 2.1 that $\tau(M)$ lifts to $BSO(2[(n-k)/2]+1)$. Hence M admits a $n-2[(n-k)/2]-1$ field, which completes the proof of Theorem 0.1 in case n is odd.

Case 2: n even. If $\chi(N) \neq 0$, there is nothing to prove since then $k = 0$. Hence we may assume that $\chi(M) = \chi(N) = 0$. This implies that $J\tau(N)^{\wedge}_2 \circ \lambda \hat{e}_2$ lifts to a map $\alpha : M \longrightarrow \overline{BSF(n)}_2$, where $\overline{BSF(n)}$ denotes the fiber of $BSF(n) \xrightarrow{\bar{E}} K(\mathbb{Z},n)$. Notice that $\overline{BSF(n)} = \Pi \overline{BSF}(n)_p$ since all homotopy groups of $\overline{BSF}(n)$ are finite and $\Pi_1 \overline{BSF}(n) = 0$.

Because by assumption $E(\tau(M)) = 0$ we can lift $J\tau(M)$ to $\beta : M \longrightarrow \overline{BSF(n)}$. Now choose

$$\bar{\omega} : M \longrightarrow \overline{BSF(n)} = \Pi \overline{BSF(n)}_p$$

such that for odd primes $e_p \circ \bar{\omega} = e_p \circ \beta$ and for $p = 2$, $e_2 \circ \bar{\omega} = \alpha$, and define

$$\omega : M \xrightarrow{\bar{\omega}} \overline{BSF(n)} \xrightarrow{\text{can}} BSF(n).$$

Then ω will be a map such that

$$\omega_{st} = J\tau(M)_{st} \quad \text{(using Lemma 1.3)}$$

and $E(\omega) = E(\tau(M)) = 0$.

Hence $\omega \simeq J\tau(M)$ be Lemma 2.4. But, by construction, $\hat{e}_2 \circ \omega$

lifts to $BSF(n-k)^{\wedge}_{2}$, since N admits a k-field, $k < n/2$. Hence $\tau(M)$ lifts to $BSO(2[(n-k)/2]+1)$ from which Theorem 0.1 follows in case n is even.

To prove Corollary 0.2, let N be a π-manifold with $\hat{M}_2 \simeq \hat{N}_2$. Notice that the semi-characteristic of M and N must agree:

$$\chi^*(M) = \chi^*(N).$$

Suppose now that $\chi^*(M) \neq 0$. Then N admits a $\rho(n+1)$ field by [5]. For n even there is nothing to prove; for n odd, $n \neq 1,3,7,15$ one has $\rho(n+1) < n/2$. Hence (ii) follows from Theorem 0.1 in case $n \neq 1,3,7,15$; if $n = 1,3$ the conclusion of (i) is trivial. If $\chi^*(M) = 0$, then N is parallelizable. Hence we can choose $k = [(n-1)/2]$ in Theorem 0.1, from which we get (iii) of the corollary; (i) is a special case of (iii) in case $\chi^*(M) = 0$. If $\chi^*(M) \neq 0$ and $\dim M = 7(15)$ then N has 7(8) fields by [5]. Hence we can use Theorem 0.1 with $k = 2(6)$ to get the result.

Ohio State University, Columbus

Eidgenössische Technische Hochschule, Zürich

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BOUNDARY HOMOMORPHISMS IN THE GENERALIZED ADAMS SPECTRAL SEQUENCE AND THE NONTRIVIALITY OF INFINITELY MANY γ_t IN STABLE HOMOTOPY

D.C. Hohnson, H.R. Miller, W.S. Wilson, and R.S. Zahler

We apply the computation announced in [8] to prove the following result on the nontriviality of an infinite subset of the family $\{\gamma_t : t > 0\}$ in the stable homotopy of the sphere.

<u>Theorem</u>. $\gamma_t \in \pi^S_{2(p^3-1)t-2(p^2-1)-2(p-1)-3}(S^0)$ is essential if $t = rp^s$, $r = 2,\dots,p-1$, $s > 0$.

The elements γ_t have been detected for $t = ap+b$, $0 \leq a < b \leq p-1$, by E. Thomas and R.S. Zahler [12,13]. Several programs for detecting the whole gamma family are currently under way, but as far as we know, none has yet succeeded.

Our approach is to reduce the theorem to an algebraic question in E_2 of the Adams spectral sequence for BP homology and then appeal to [8] and arithmetic to deduce the result. The arithmetic actually shows $\gamma_t \neq 0$ for other values of t in a set of density zero.

Our methods allow a systematic detection of elements in infinite families in stable homotopy. We illustrate this by proving that all the elements in the alpha and beta families are nontrivial, assuming only the existence of the self-maps required for their construction. The same technique could be used to detect the known members of the epsilon family, again assuming their construction.

The link between algebra and homotopy theory is provided in the first section by a folk theorem relating algebraic and geometric connecting homomorphisms. The second section defines the stable homotopy elements of interest to us and uses [8] to detect many of them.

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1. Algebraic Boundaries and Geometric Boundaries.

Our goal in this section is to prove a general result (1.7) relating connecting homomorphisms in E_2 and in the abutment of generalized homology Adams spectral sequences.

We recall the construction [3, III, §15] of an Adams spectral sequence based on a homotopy-associative ring spectrum E with unit $\eta : S^0 \longrightarrow E$. Let $F^0 = S^0$ and for $s \geq 0$ let F^{s+1} complete the cofibration sequence

$$(1.1) \qquad F^s \xrightarrow{j_s} E \wedge F^s \xrightarrow{k_s} F^{s+1} \xrightarrow{i_s} F^s$$

in which k_s has degree -1 and

$$j_s = \eta \wedge F^s : F^s \simeq S^0 \wedge F^s \longrightarrow E \wedge F^s.$$

The sequences (1.1) splice together to form an Adams resolution for S^0. When smashed with a connective spectrum X, they form an Adams resolution for X. Note that

$$(1.2) \qquad E_*(i_s \wedge X) = 0.$$

If we apply $\pi_*^S(\ )$, we obtain an exact couple whose associated spectral sequence is the E-homology Adams spectral sequence $E_r^{**}(X)$.

Define a filtration of $\pi_*^S(X)$ by

$$(1.3) \qquad F^s \pi_*^S(X) = \text{image}\left\{\pi_*^S(F^s \wedge X) \longrightarrow \pi_*^S(X)\right\}.$$

Lift $x \in F^s \pi_*^S(X)$ to $y \in \pi_*^S(F^s \wedge X)$. Then $(j_s \wedge X)y \in \pi_*^S(E \wedge F^s \wedge X) =$ $= E_1^{s,*}(X)$ is a permanent cycle and projects to an element of $E_\infty^{s,*}(X)$ which depends only on x modulo $F^{s+1}\pi_*^S(X)$. Thus we have a homomorphism of bigraded modules

$$(1.4) \qquad E_0^* \pi_*^S(X) \longrightarrow E_\infty^{**}(X).$$

Now suppose $E_* = E_*(S^0)$ is commutative and $E_*(E)$ is flat over E_*. The left unit $\eta_L : E_* \longrightarrow E_*(E)$ is split by the multiplication map so the cokernel, $E_*(F^1)$ by (1.2), is also flat. Then $E_*(F^1) \otimes_{E_*} E_*(-)$ gives a homology theory naturally equivalent to $E_*(F^1 \wedge -)$. Using the observation that $F^t \simeq F^1 \wedge F^{t-1}$, we prove inductively that:

$$E_*(F^t \wedge X) \cong E_*(F^t) \otimes_{E_*} E_*(X) \tag{1.5}$$

for any connective spectrum X. Then [2,3]

$$E_2^{**}(X) \cong \mathrm{Ext}^{**}_{E_*(E)}(E_*, E_*(X)).$$

This Ext is an Ext of comodules over the "coalgebra" $E_*(E)$; it is computed using extended $E_*(E)$ comodules as injectives.

<u>Definition</u> 1.6. The class $\bar{x} \in \mathrm{Ext}^{t,*}_{E_*(E)}(E_*, E_*(X))$ is said to converge to $x \in \pi_*^S(X)$ provided that

(i) $\bar{x}$ is a permanent cycle representing the class $\{\bar{x}\} \in E_\infty^{t,*}(X)$;

(ii) $x \in F^t\pi_*^S(X)$; and

(iii) The homomorphism (1.4) sends the coset $x + F^{t+1}\pi_*^S(X)$ to $\{\bar{x}\}$.

We define a map $f : X \longrightarrow Y$ to be E-<u>proper</u> provided that $E_*(f) = 0$. (This terminology was suggested by Larry Smith.) If in the cofibration sequence

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} SW$$

the map h is E-proper, we obtain a short exact sequence

$$0 \longrightarrow E_*(W) \xrightarrow{E_*(f)} E_*(X) \xrightarrow{E_*(G)} E_*(Y) \longrightarrow 0.$$

In turn, this induces a long exact sequence

$$\begin{array}{ccc} \mathrm{Ext}^{**}_{E_*(E)}(E_*, E_*(W)) & \longrightarrow & \mathrm{Ext}^{**}_{E_*E}(E_*, E_*(X)) \\ & {\scriptstyle\delta}\nwarrow \quad \swarrow & \\ & \mathrm{Ext}^{**}_{E_*(E)}(E_*, E_*(Y)) & \end{array}$$

where the connecting homomorphism δ is as in [4, p. 55] and has bidegree (1,0).

<u>Theorem</u> 1.7. (<u>Geometric Boundary Theorem</u>) Let E be a homotopy associative ring spectrum with unit such that E_* is commutative and $E_*(E)$ is flat over E_* is commutative and $E_*(E)$ is flat over E_*. Let $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} SW$ be a cofibre sequence of finite spectra with h an E-proper map. If $\bar{x} \in \mathrm{Ext}^{t,*}_{E_*(E)}(E_*, E_*(Y))$ converges to $x \in \pi^S_*(Y)$, then $\delta(\bar{x})$ converges to $h_*(x) \in \pi^S_*(W)$.

<u>Proof</u>. Smash the Adams resolution for the sphere with the cofibration sequence $W \longrightarrow X \longrightarrow Y$. Part of the resulting diagram

is displayed in (1.8). Let $y \in \pi_*^S(F^t \wedge Y)$ be such that $(j_t \wedge Y)y$ represents $\bar{x}$ in $E_2^{t,*}(Y)$ and

$$(1.9) \qquad (i_0 \ldots i_{t-1} \wedge Y)y = x.$$

By (1.5), the $E \wedge F^t \wedge -$ row in (1.8) is short in homotopy, so $0 = (E \wedge F^t \wedge h)(j_t \wedge Y)y = (j_t \wedge W)(F^t \wedge h)y$ and there exists $y_1 \in \pi_*^S(F^{t+1} \wedge W)$ such that

$$(1.10) \qquad (i_t \wedge W)y_1 = (F^t \wedge h)y.$$

We come now to the main geometric step.

<u>Claim</u> 1.11. There is an element $y_2 \in \pi_*^S(E \wedge F^t \wedge X)$ such that

$$(j_t \wedge Y)y = (E \wedge F^t \wedge g)y_2,$$

$$(k_t \wedge X)y_2 = (F^{t+1} \wedge f)y_1.$$

To see this, pass to the Spanier-Whitehead dual cofibration sequence $DW \leftarrow DX \leftarrow DY$. Take maps $y^\#$ and $y_1^\#$ dual to y and y_1. We have

$$\begin{array}{ccccc} DX & \longleftarrow & DY & \longleftarrow & S^{-1}DW \\ {\scriptstyle y_2^\#}\downarrow & & \downarrow{\scriptstyle y^\#} & & \downarrow{\scriptstyle S^{-1}y_1^\#} \\ E \wedge F^t & \longleftarrow & F^t & \longleftarrow & F^{t+1}. \end{array}$$

Let $y_2^\#$ complete the map of cofibrations (see [16], p. 170).

Then the map $y_2 \in \pi_*^S(E \wedge F^t \wedge Y)$ dual to $y_2^\#$ satisfies the conditions of the claim.

Now it is easy, using the definition of the connecting homomorphism δ, to chase (1.8) and see that

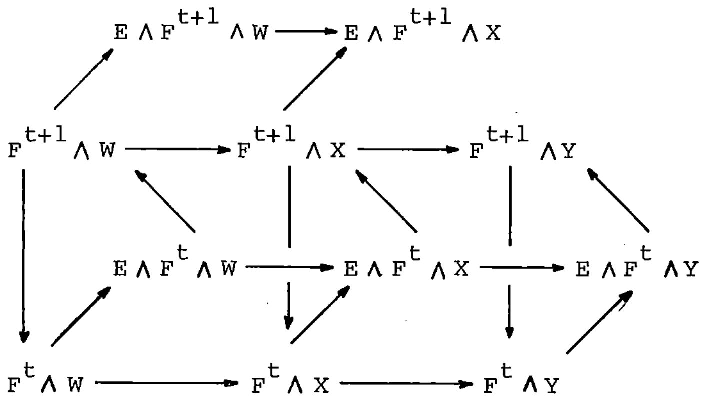
$$(j_{t+1} \wedge W) y_1 \in \pi_*^S(E \wedge F^{t+1} \wedge W)$$

represents $\delta(\bar{x})$. Because it factors through $F^{t+1} \wedge W$ it is a permanent cycle. Since $(i_s \wedge W)(F^{s+1} \wedge h) = (F^s \wedge h)(i_s \wedge Y)$ for all s, we have by (1.9) and (1.10)

$$(i_0 \cdots i_t \wedge W) y_1 = (i_0 \cdots i_{t-1} \wedge W)(F^t \wedge h) y = h \cdot (i_0 \cdots i_{t-1} \wedge Y) y = h(x).$$

That is, $\delta(\bar{x})$ represents $h(x)$.

(1.8)



2. Detecting Stable Homotopy Families.

In this section we show how the Miller-Wilson results may be combined with the geometric boundary theorem to detect stable homotopy. First we recover known results on homotopy elements of BP filtration 1 and 2; then we prove our main theorem on the gamma family.

Recall that $BP_*(\ )$ is the Brown-Peterson homology theory associated with the prime p; it has coefficient ring $BP_* \cong Z_{(p)}[v_1, v_2, \ldots]$ with $|v_n| = 2(p^n-1)$. Define $I_n = (p, v_1, \ldots, v_{n-1})$ with the convention that $v_0 = p$ and $I_0 = (0)$. All of the results of Section 1 hold for BP. In fact, there is an Adams spectral sequence

$$\mathrm{Ext}^{**}_{BP_*(BP)}(BP_*, BP_*(X)) \Longrightarrow \pi_*^S(X)$$

converging to $\pi_*^S(X) \otimes Z_{(p)}$. Henceforth we shall delete "$BP_*(BP)$" from our Ext notation.

Let $V(-1) = S^0$. For $n = 0,1,2,$ or 3, and $p > 2n$, there is a cofibre sequence

$$(2.1) \qquad S^{2p^n-2}V(n-1) \xrightarrow{\phi_n} V(n-1) \xrightarrow{a_n} V(n) \xrightarrow{h_n} S^{2p^n-1}V(n-1),$$

in which h_n is BP-proper, inducing the short exact sequence

$$0 \to BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0.$$

($n=1$: [1]; $n=2$: [10]; $n=3$: [15].)

On the E_2 level of the Adams spectral sequence, these short exact sequences induce exact triangles

(2.2)
$$\begin{array}{ccc} \mathrm{Ext}^{**}(BP_*, BP_*/I_n) & \xrightarrow{v_n} & \mathrm{Ext}^{**}(BP_*, BP_*/I_n) \\ {\scriptstyle \delta_n} \nwarrow & & \swarrow {\scriptstyle \rho_n} \\ & \mathrm{Ext}^{**}(BP_*, BP_*/I_{n+1}) & \end{array}$$

where δ_n has bidegree $(1, 2-2p^n)$.

Now we need to quote two theorems.

Theorem 2.3. (Landweber [7], or see [5]) Let $n > 0$; then

$$\mathrm{Ext}^{0,*}(BP_*, BP_*/I_n) \cong \mathbb{F}_p[v_n].$$

Thus $\mathrm{Ext}^{**}(BP_*, BP_*/I_n)$ is a module over $\mathbb{F}_p[v_n]$. When $n = 0$, $\mathrm{Ext}^{0,*}(BP_*, BP_*) \cong Z_{(p)}$, concentrated in degree zero.

Theorem 2.4. (Miller-Wilson [8] passim) Let $n = 0, 1, 2$, or 3. If $0 \neq \rho_n(x) \in \mathrm{Ext}^{1,*}(BP_*, BP_*/I_{n+1})$, then $v_{n+1}\rho_n(x) \neq 0$.

Lemma 2.5. Let $n = 0, 1, 2$, or 3, and $p > 2n$. If $g_n \in \pi_s^S(V(n))$, $s > 0$, is such that

$$0 \neq BP_*(g_n) \in \mathrm{Hom}_{BP_*BP}(BP_*, BP_*/I_{n+1}) \cong \mathbb{F}_p[v_{n+1}]$$

then $0 \neq h_n g_n \in \pi_*^S(V(n-1))$. Furthermore $0 \neq h_{n-1} h_n g_n \in \pi_*^S(V(n-2))$ for $n \neq 0$.

Proof. By (2.3) the exact sequence induced by (2.2) begins, for $n = 0$,

$$(2.6) \qquad 0 \longrightarrow Z_{(p)} \xrightarrow{p} Z_{(p)} \xrightarrow{\rho_0} \mathbb{F}_p[v_1] \xrightarrow{\delta_0} \mathrm{Ext}^{1,*}(BP_*, BP_*)$$

and for $n > 0$;

$$(2.7) \qquad \begin{aligned} &0 \longrightarrow \mathbb{F}_p[v_n] \xrightarrow{v_n} \mathbb{F}_p[v_n] \xrightarrow{\rho_n} \mathbb{F}_p[v_{n+1}] \xrightarrow{\delta_n} \\ &\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n) \xrightarrow{v_n} \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n). \end{aligned}$$

In either case, $\delta_n(BP_*(g_n)) \neq 0$ since g_n has positive degree. By Theorem 1.7, $\delta_n(BP_*(g_n))$ is a permanent cycle converging to $h_n g_n$; see (1.6). Differentials increase homological degree by at least two, so $h_n g_n$ survives nontrivially.

Now by (2.7), $v_n \delta_n(BP_*(g_n)) = 0$. Thus by (2.4), $\delta_n(BP_*(g_n))$ cannot be in the image of ρ_{n-1} (in (2.2)). Hence, by exactness of (2.2),

$$0 \neq \delta_{n-1}\delta_n(BP_*(g_n)) \in \mathrm{Ext}^{2,*}(BP_*, BP_*/I_{n-1}) \cong E_2^{2,*}(V(n-2)).$$

By Theorem 1.7, $\delta_{n-1}\delta_n(BP_*(g_n))$ is a permanent cycle which converges to $h_{n-1}h_n g_n$; see (1.6). A glance at (2.3) shows that $\delta_{n-1}\delta_n(BP_*(g_n))$ cannot be hit by any differential so $\delta_{n-1}\delta_n(BP_*(g_n))$ survives nontrivially to E_∞ and $h_{n-1}h_n g_n \neq 0$.

Definition 2.8. $\phi_n^1 = \phi_n$ and $\phi_n^t = \phi_n \phi_n^{t-1}$, $t > 1$ (using the same symbol for a stable map and its suspension).

We now consider some examples. For $t > 0$ and $p > 2$, $\alpha_t \in \pi^S_{2(p-1)t-1}(S^0)$ is defined as the composition

$$S^{2(p-1)t} \xrightarrow{a_0} S^{2(p-1)t}V(0) \xrightarrow{\phi_1^t} V(0) \xrightarrow{h_0} S^1.$$

Notice that

$$BP_*(\phi_1^t a_0) = v_1^t \in \mathbb{F}_p[v_1] \cong \mathrm{Ext}^{0,*}(BP_*, BP_*/(p)).$$

By Lemma 2.5 we have:

Corollary 2.9 (Toda [14]). $\alpha_t \neq 0$ for all $t > 0$.

For $t > 0$ and $p > 3$, $\beta_t \in \pi^S_{2(p^2-1)t-2(p-1)-2}(S^0)$ is defined as the composition

$$S^{2(p^2-1)t} \xrightarrow{a_1 a_0} S^{2(p^2-1)t}V(1) \xrightarrow{\phi_2^t} V(1) \xrightarrow{h_1} S^{2p-1}V(0) \xrightarrow{h_0} S^{2p}.$$

Notice that

$$BP_*(\phi_2^t a_1 a_0) = v_2^t \in \mathbb{F}_p[v_2] \cong \mathrm{Ext}^{0,*}(BP_*, BP_*/(p, v_1)).$$

By Lemma 2.5 we have

Corollary 2.10 (Smith [10]). $\beta_t \neq 0$ for all $t > 0$.

Remark 2.11. In [8], $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ and the maps ρ_n are described completely for $n = 0,1,2,$ and 3. We have stated in (2.4) only the minimal result necessary to study the beta and gamma families. Using the more complete information availble in [8] one can use techniques similar to these to detect all of the epsilons of Oka [9], Smith [11], and Zahler [17] assuming only the spaces and self maps used in their definition.

Let $p > 5$ and $t > 0$. There are elements

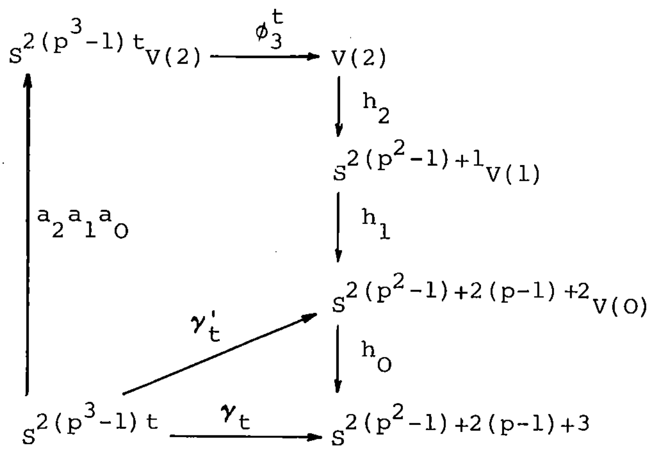
$$\gamma'_t \in \pi^S_{2(p^3-1)t-2(p^2-1)-2(p-1)-2}(V(0))$$

and

$$\gamma_t \in \pi^S_{2(p^3-1)t-2(p^2-1)-2(p-1)-3}(S^0)$$

defined by the following diagram.

(2.12)



Observe that

$$BP_*(\phi_3^t a_2 a_1 a_0) = v_3^t \in \mathbb{F}_p[v_3] \cong \mathrm{Ext}^{0,*}(BP_*, BP_*/(p, v_1, v_2)).$$

By Lemma 2.5 we have the following folk result.

Corollary 2.13. $\gamma_t' \neq 0$ for all $t > 0$.

In this case, the proof of Lemma 2.5 showed that

$$0 \neq \delta_1 \delta_2 (v_3^t) \in \mathrm{Ext}^{2,w(t)}(BP_*, BP_*/(p)),$$

where $w(t) = 2(p^3-1)t-2(p^2-1)-2(p-1)$.

Lemma 2.14. Suppose for some $t > 0$ that

$$\mathrm{Ext}^{2,w(t)}(BP_*, BP_*) = 0.$$

Then $\gamma_t \neq 0$.

Proof. With $k = w(t)$ we have

$$0 \neq \delta_1 \delta_2 (v_3^t) \in \mathrm{Ext}^{2,k}(BP_*, BP_*/(p))$$

and the exact sequence

$$0 = \mathrm{Ext}^{2,k}(BP_*, BP_*) \xrightarrow{\rho_0} \mathrm{Ext}^{2,k}(BP_*, BP_*/(p)) \xrightarrow{\delta_0} \mathrm{Ext}^{3,k}(BP_*, BP_*).$$

Thus $\delta_0 \delta_1 \delta_2 (v_3^t) \neq 0$. By Theorem 1.7, this is a permanent cycle. Since $\mathrm{Ext}^{*,i}(BP_*, BP_*) = 0$ for $i \not\equiv 0$ modulo $2(p-1)$, no non-zero differential (bidegree $= (r, r-1)$) has range $\mathrm{Ext}^{3,w(t)}(BP_*, BP_*)$.

So $\delta_0\delta_1\delta_2(v_3^t)$ survives nontrivially to E_∞ and by (1.7) converges to γ_t.

We are not so lucky as to have $\mathrm{Ext}^{2,w(t)}(BP_*,BP_*) = 0$ for all t. In [6] it was observed that $\mathrm{Ext}^{2,w(1)}(BP_*,BP_*) = 0$, giving a confirmation of the theorem of Thomas and Zahler [12] that $\gamma_1 \neq 0$. From Theorem B of [8] and the discussion following it we have

<u>Theorem</u> 2.15 (Miller-Wilson). Let $p > 2$. $\mathrm{Ext}^{2,n}(BP_*,BP_*)$ is the direct sum of j nontrivial cyclic $Z_{(p)}$-modules where j is the number of times n appears in the following list.

(i) $[p^s(p+1)-i]q$ $\quad s \geq 0,\ 0 < i \leq p^s$

(ii) $[a(p+1)-1]q$

(iii) $[ap(p+1)-i]q$ $\quad 0 < i \leq p$

(iv) $[ap^s(p+1)-i]q$ $\quad s > 1,\ 0 < i \leq p^s+p^{s-1}-1$

where $1 < a$, $(a,p) = 1$, $q = 2(p-1)$.

<u>Proposition</u> 2.16. For $r = 2,3,\ldots,p-1$, $s > 0$, and $k = w(rp^s) = 2(p^3-1)rp^s-2(p^2-1)-2(p-1)$, $\mathrm{Ext}^{2,k}(BP_*,BP_*) = 0$.

<u>Proof</u>. It is easy to check that $k = w(rp^s)$ is not in the above list. For example: $k/q \equiv rp^s-1$ modulo $(p+1)$; $a(p+1)-1 \equiv -1$ modulo $(p+1)$; $rp^s \not\equiv 0$ modulo $(p+1)$ since $2 \leq r \leq p-1$; and thus k cannot be of form (ii). The other cases require equally elementary and entertaining arguments.

This gives us our main result.

Corollary 2.17. $\gamma_{rp^s} \neq 0$ for $r = 2,3,\ldots,p-1$ and $s > 0$.

Proof. (2.14) and (2.16).

University of Kentucky

Princeton University

Princeton University and Institute for Advanced Study

Rutgers University (Douglass College)

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HOMOLOGY OF THE BARRATT - ECCLES DECOMPOSITION MAPS

Daniel S. Kahn(*)

Section 1.

In this paper, we propose an approach to the calculation of the homology homomorphism induced by the Barratt-Eccles maps which decompose $\Sigma\Gamma^{+}X$ ([2]). The interest of these calculations is suggested by two applications given here as well as in some recent work in progress of Barratt and Kirley. The first application gives a new proof of the Kahn-Priddy theorem [6] along lines suggested by an argument of G. Segal [14] except that the details are carried out by means of homology calculations. We conclude by sketching a result of the present author and L. Finkelstein which generalizes the first result.

Section 2.

For a finite group G, let WG denote a contractible countable CW complex on which G acts freely and cellularly.

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$BG = WG/G$ is a classifying space for G. In this paper, all actions of G are on the left; actions of G on products of G-spaces will be the diagonal action. We will denote the quotient space of the diagonal action on $A \times B$ by $A \times_G B$. In the special case where $G \subset \Sigma_n$, the symmetric group on n objects, and $B = A^n$ with the evident G-action, we denote the quotient space $WG \times_G A^n$ by $WG \times_\Sigma A^n$.

Consider the infinite loop space $QA = \varinjlim \Omega^n S^n A$ [4], where Ω^n is the n-fold loop space functor and $S^n A = A \wedge S^n$ for a space A with base point. For each integer m, QS^0 has a path component $Q_m S^0$ which is represented by a map $S^n \longrightarrow S^n$ of degree m. We use the symbol $[m]$ to denote both a point of $Q_m S^0$ and the singular homology class $[m] \in H_0(QS^0;R)$ represent by that point. (R is any ring with unit.) Letting $x*y$ denote the Pontryagin product of x and y derived from loop sum on QS^0, we have $[m]*[n] = [m+n]$. We denote by $*[m] : QS^0 \longrightarrow QS^0$ the map given by adding to each loop a fixed loop of degree m. Thus $*[m](Q_n S^0) \subset Q_{m+n} S^0$.

For any infinite loop space A, there exist the Dyer-Lashof maps $\theta_n : W\Sigma_n \times_\Sigma A^n \longrightarrow A$ [4]. (Σ_n = symmetric group on n symbols, or to be more specific we will take Σ_n to be the automorphism group of the set $\underline{n} = \{1,2,\ldots,n\}$.) Consider the composite $\rho_n : B\Sigma_n \longrightarrow Q_n S^0$

$$B\Sigma_n = W\Sigma_n \times_\Sigma [1]^n \longrightarrow W\Sigma_n \times_\Sigma (Q_1 S^0)^n \longrightarrow Q_n S^0$$

where the first map is induced by the inclusion of [1] in Q_1S^0 and the second map is the restriction of $\theta_n : W\Sigma_n \times_{\Sigma} (QS^0)^n \to QS^0$. We define $\bar{\rho}_n : B\Sigma_n \to Q_0S^0$ to be the composite

$$B\Sigma_n \xrightarrow{\rho_n} Q_nS^0 \xrightarrow{*[-n]} Q_0S^0.$$

Following Adams [1], for a prime $p > 2$, we set $L_p = {}_p(B\Sigma_p \cup_j CB\Sigma_2)$ where $j : B\Sigma_2 \to B\Sigma_p$ is induced by the inclusion $\underline{2} \to \underline{p}$, and ${}_p(A)$ is the p-primary factor of A. For $p = 2$, we set $L_2 = B\Sigma_2$. The inclusion $j : B\Sigma_p \to L_p$ is the identity for $p = 2$ and induces a Z_p-homology isomorphism if $p > 2$. Also for $p > 2$, L_p is simply-connected. This follows from the van Kampen theorem. It follows by obstruction theory that the composite

$$B\Sigma_p \xrightarrow[\bar{\rho}_p]{} Q_0S^0 \xrightarrow[\gamma_p]{} {}_p(Q_0S^0)$$

extends to a map $\hat{\rho}_p : L_p \to {}_p(Q_0S^0)$. (Here γ_p is the projection of Q_0S^0 on its p-primary factor ${}_p(Q_0S^0)$.)

Since ${}_p(Q_0S^0)$ is an infinite loop space, the map $\hat{\rho}_p : L_p \to {}_p(Q_0S^0)$ extends to a map $\omega : QL_p \to {}_p(Q_0S^0)$ [7].

We wish to prove the following

<u>Theorem</u> 2.1. There exists a map $\alpha : {}_p(Q_0S^0) \to QL_p$ such that $\omega \circ \alpha$ is homotopic to the indentity map of ${}_p(Q_0S^0)$.

(2.1) is equivalent to the Kahn-Priddy theorem. See [1] for a discussion of the different forms of this theorem.

Proof. We construct the homorphism $\varphi_{n,p} : \Sigma_n \to \Sigma_N \int \Sigma_p$ used by Barratt and Eccles in [2], where $N = \binom{n}{p}$ and $\Sigma_N \int \Sigma_p$ is the wreath product [5]. (In our notation $\Sigma_N \int \Sigma_p$ is a semi-direct product of Σ_N with $(\Sigma_p)^N$.)

Let $\mathcal{P}(n)$ be the set of subsets of $\underline{n}$ of cardinality p. The cardinality of $\mathcal{P}(n)$ is N. For notational convenience, we shall think of $\mathcal{P}(n)$ as the set of monotone functions $f : \underline{p} \to \underline{n}$. Enumerate these functions: $f_1, f_2, \ldots, f_N$. Each $\sigma \in \Sigma_n$ permutes the elements of $\mathcal{P}(n)$. Thus we obtain $\bar{\sigma} \in \Sigma_N$ defined by $\sigma \circ f_i = f_{\sigma(i)} \circ \sigma$ also defines elements $\tau_i \in \Sigma_p$, $i = 1, 2, \ldots, p$ by demanding that $\tau_i^{-1} \circ f_{\sigma(i)}^{-1} \circ \sigma \circ f_i$ be monotone increasing. (τ_i may be thought of as the internal permutation on $f_i(\underline{p})$ induced by σ.) Finally we define $\varphi_{n,p}(\sigma) = \bar{\sigma} \cdot (\tau_1 \times \tau_2 \times \cdots \times \tau_N)$. It is easy to see that a change in the enumeration of $\mathcal{P}(n)$ only changes $\varphi_{n,p}$ by an inner automorphism of $\Sigma_N \int \Sigma_p$. Thus the homotopy class of

$$B\varphi_{n,p} : B\Sigma_n \to B(\Sigma_N \int \Sigma_p) = W\Sigma_N \times_{\Sigma} (B\Sigma_p)^N$$

is well-defined.

Consider the composite μ_n

$$B\Sigma_n \xrightarrow{B\varphi_{n,p}} W\Sigma_N \times_\Sigma (B\Sigma_p)^N \xrightarrow{\theta'_N} QB\Sigma_p \xrightarrow{Qj} QL_p \xrightarrow{\omega} {}_p(Q_0S^0)$$

where θ'_N is the restriction of the Dyer-Lashof map

$$\theta_N : W\Sigma_N \times_\Sigma (QB\Sigma_p)^N \longrightarrow QB\Sigma_p .$$

We also set

$$\zeta_n = Qj\theta'_N B\varphi_{n,p} : B\Sigma_n \longrightarrow QL_p .$$

The remaining sections of this paper are devoted to proving the following two results.

Lemma 2.2. The diagrams

$$\begin{array}{ccc} B\Sigma_n & \xrightarrow{\zeta_n} & \\ {\scriptstyle Bi_n}\downarrow & & QL_p \\ B\Sigma_{n+1} & \xrightarrow{\zeta_{n+1}} & \end{array}$$

are homotopy-commutative, where $i_n : \Sigma_n \longrightarrow \Sigma_{n+1}$ is induced by the inclusion $\underline{n} \subset \underline{n+1}$.

Using the homotopy extension theorem, we obtain from (2.2) a map $\zeta : B\Sigma_\infty \longrightarrow QL_p$ such that $\zeta | B\Sigma_n \simeq \zeta_n$.

Proposition 2.3. $(\omega \circ \zeta)_* : H_i(B\Sigma_\infty ; Z_p) \longrightarrow H_i({}_p(Q_0S^0) ; Z_p)$ is an isomorphism for $i \geq 0$.

An immediate consequence is that the composite

$$ {}_p(QB\Sigma_\infty) \xrightarrow{{}_pQ\zeta} QQL_p \xrightarrow{Q\omega} Q({}_p(Q_0S^0)) $$

is a homotopy equivalence. Let $\beta : Q({}_p(Q_0S^0)) \longrightarrow {}_p(QB\Sigma_\infty)$ be a homotopy inverse. Let $h_1 : QQL_p \longrightarrow QL_p$ and $h_2 : Q({}_p(Q_0S^0)) \longrightarrow {}_p(Q_0S^0)$ be the canonical maps derived from the infinite loop structures of QL_p and ${}_p(Q_0S^0)$, respectively [7].

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
QB\Sigma_\infty & \xrightarrow{\quad Q\zeta \quad} & QQL_p & \xrightarrow{\quad Q\omega \quad} & Q({}_p(Q_0S^0)) \\
\uparrow \beta & & \downarrow h_1 & & \downarrow h_2 \\
Q({}_pQ_0S^0) & & QL_p & \xrightarrow{\quad \omega \quad} & {}_p(Q_0S^0)
\end{array}
$$

Let $\iota : {}_p(Q_0S^0) \longrightarrow Q({}_p(Q_0S^0))$ be the standard inclusion [7]. Then $Id = h_2 \circ \iota \simeq h_2 \circ Q\omega \circ Q\zeta \circ \beta \circ \iota = \omega \circ h_1 \circ Q\zeta \circ \beta \circ \iota$. Thus, by setting $\alpha = h_1 \circ Q\zeta \circ \beta \circ \iota$, we have completed the proof of (2.1).

Section 3.

In this section we recall the results of Nakaoka [12] on $H_*(B\Sigma_\infty)$ and of Dyer and Lashof [4] on $H_*(QA)$ for a prime $p > 1$. For the remainder of this paper, p shall be a fixed prime > 1 and $H_*(X)$ shall mean homology with coefficients in Z_p, the integers moduls p.

Let $\pi \subset \Sigma_p$ be the subgroup generated by the cyclic permutation $T = (12\ldots p)$. For notational convenience, we denote by $\Gamma_p A$ the space $W\pi \times_\pi A^p$. Then, for example, if $\Sigma(p^r,p)$ is a p-Sylow subgroup of Σ_p, we may take

$$B\Sigma(p^r,p) = \bar{\Gamma}_p^{r-1}\overline{\Gamma_p\ldots\Gamma_p}B\pi$$

and

$$W\Sigma(p^r,p) \times_\Sigma A^{p^r} = \bar{\Gamma}_p^r\overline{\Gamma_p\ldots\Gamma_p}A.$$

The standard model for $W\pi$ has cells $e_i, Te_i, T^2e_i, \ldots, T^{p-1}e_i$ in dimension i [8]. If $x, x_1, \ldots, x_p$ are cycles of $C_*(A)$, then $e_i \int x = e_i \otimes x^p$ is a cycle of $C_*(\Gamma_p A)$ as is $e_0 \otimes (x_1 \otimes \ldots \otimes x_p)$. To simplify notation we shall frequently use the same symbol to denote a cycle as well as its homology class.

Let us recall the description of $H_*(\Gamma_p A)$ [8]. Let $x_1, x_2, \ldots$ be a basis for $H_*(A)$. Let $\mathcal{O}$ be a set of orbit representatives of the action of π on sequences $\{(j_1,\ldots,j_p) \mid j_i \geq 1\}$. Then the set

$$\{e_i \int x_j \mid i \geq 1, j \geq 1\} \cup \{e_0 \otimes (x_{j_1} \otimes \ldots \otimes x_{j_p}) \mid (j_1,\ldots,j_p) \in \mathcal{O}\}$$

is a basis for $H_*(\Gamma_p A)$.

If $I = \{i_1,\ldots,i_k\}$ is a sequence of non-negative integers, we denote by e_I the element of $H_*(B\Sigma(p^r,p))$ represented by

$e_{i_1} \int e_{i_2} \int \cdots \int e_{i_k}$ and we denote by $\bar{e}_I$ its image in $H_*(B\Sigma_{p^r})$ or $H_*(B\Sigma_\infty)$. (Recall that $H_*(B\Sigma_n) \longrightarrow H_*(B\Sigma_\infty)$ is a monomorphism [3, 11].) We use $*$ to denote the juxtaposition product in $H_*(B\Sigma_n)$.

We may now state Nakaoka's theorem on $H_*(B\Sigma_\infty)$ [12].

Theorem 3.1 (Nakaoka). $H_*(B\Sigma_\infty)$ is the free commutative graded algebra over Z_p on $\{\bar{e}_I \mid I \in \mathcal{A}\}$.

(See [12] to see exactly what $\mathcal{A}$ is.)

We now recall the results of Dyer and Lashof on $H_*(QA)$. Let A be a countable connected CW complex and let $x_1, x_2, \ldots$ be a basis for $H_*(A)$. If $I = \{i_1, \ldots, i_k\}$ is a sequence of non-negative integers and $x \in H_*(A)$, $Q_I x$ shall denote the image of $e_{i_1} \int \cdots \int e_{i_k} \int x$ under the evident map

$$\overline{\Gamma_p^k \cdots \Gamma_p} A \longrightarrow W\Sigma_{p^k} \times_{\Sigma} A^{p^k} \xrightarrow{\ \theta\ } QA.$$

(If I is empty, we set $Q_I x = x$.)

We can now state

Theorem 3.2 (Dyer and Lashof [4]). Let A be a connected countable CW complex and let $x_1, x_2, \ldots$ be a basis for $\tilde{H}_*(A)$. Then $H_*(QA)$ is the free commutative graded algebra over Z_p on $\{Q_I x_j \mid Q_I x_j \in \mathcal{A}'\}$.

(See [4] to see exactly what $\mathcal{A}'$ is.)

We next state the results of Dyer and Lashof on $H_*(Q_0S^0)$ as modified by Milgram [9].

Theorem 3.3 (Dyer and Lashof [4]). $H_*(Q_0S^0)$ is the free commutative graded algebra over Z_p on $\{(Q_I[1]*[-p^{|I|}] \mid I \in \mathscr{A}\}$, where $|I| = k$ if $I = \{i_1,\ldots,i_k\}$.

It is important to know that the $\mathscr{A}$ of (3.3) is the same as the $\mathscr{A}$ of (3.1). For our purposes we do not need to know exactly what $\mathscr{A}$ is.

Section 4.

Recall from Section 2 the map $\zeta_n : B\Sigma_n \longrightarrow QL_p$. We wish to examine the behavior of ζ_n with respect to the juxtaposition product $B\Sigma_m \times B\Sigma_n \longrightarrow B\Sigma_{m+n}$.

The inclusion $i_{m,n} : \Sigma_m \times \Sigma_n \longrightarrow \Sigma_{m+n}$ induces an action of $\Sigma_m \times \Sigma_n$ on $\mathscr{P}(m+n)$ which has orbit representatives

$$\{1,2,\ldots,k,m+1,\ldots,m+p-k\},\ 0 \le k \le m \text{ and } 0 \le p-k \le n.$$

It follows that with an appropriate ordering of $\mathscr{P}(m+n)$, the map

$$B\varphi_{m+n,p} \circ Bi_{m,n} : B(\Sigma_m \times \Sigma_n) \longrightarrow W\Sigma_N \times_\Sigma (B\Sigma_p)^N, N = \binom{m+n}{p}$$

may be written as a composition

$$B(\Sigma_m \times \Sigma_n) \simeq B\Sigma_m \times B\Sigma_n \xrightarrow{\text{diag}} (B\Sigma_m \times B\Sigma_n)^t$$

$$\xrightarrow[\Pi\mu_k]{} \Pi B(\Sigma_{\binom{m}{k}\binom{n}{p-k}} \int (\Sigma_k \times \Sigma_{p-k}))$$

$$\xrightarrow[\Pi\nu_k]{} \Pi B(\Sigma_{\binom{m}{k}\binom{n}{p-k}} \int \Sigma_p)$$

$$\xrightarrow[Bh]{} B((\Pi\Sigma_{\binom{m}{k}\binom{n}{p-k}}) \int \Sigma_p)$$

$$\xrightarrow[B(i\int 1)]{} B(\Sigma_N \int \Sigma_p) = W\Sigma_N \times_\Sigma (B\Sigma_p)^N,$$

where $\mu_p = \zeta_m$ and $\mu_0 = \zeta_n$. Here, the product ranges over $0 \le k \le p$ and $0 \le p-k \le n$ and t is the number of such k; ν_k is induced by the inclusion $\Sigma_k \times \Sigma_{p-k} \subset \Sigma_p$, h is the evident isomorphism and i is the inclusion $\Pi\Sigma_{\binom{m}{k}\binom{n}{p-k}} \subset \Sigma_N$.

The map

$$\zeta_{m+n} \circ Bi_{m,n} : B\Sigma_m \times B\Sigma_n \longrightarrow QL_p$$

is obtained by composing $\theta_N \circ B\varphi_{m+n,p} \circ B\iota_{m,n}$ with $Qj : QB\Sigma_p \longrightarrow QL_p$. Using the fact that any map $B(\Sigma_k \times \Sigma_{p-k}) \longrightarrow QL_p$ is null-homotopic if $0 < k < p$ (since $\tilde{H}^*(B\Sigma_k \times B\Sigma_{p-k}) = 0$ and $\pi_i(QL_p)$ is of order p^s for some s), we obtain

<u>Proposition</u> 4.1. The following diagrams are homotopy-

commutative:

(i)

$$\begin{array}{ccc} B\Sigma_m - B\Sigma_m \times B\Sigma_1 & & \\ \downarrow{\scriptstyle Bi_{m,1}} & \searrow^{\zeta_m} & QL_p \\ B\Sigma_{m+1} & \nearrow_{\zeta_{m+1}} & \end{array}$$

(ii) for m _ p, n _ p

$$\begin{array}{ccc} B\Sigma_m \times B\Sigma_n & \xrightarrow{Bi_{m,n}} & B\Sigma_{m+n} \\ \downarrow{\scriptstyle \zeta_m \times \zeta_n} & & \downarrow{\scriptstyle \zeta_{m+n}} \\ QL_p \times QL_p & \xrightarrow{*} & QL_p \end{array}$$

Note that (4.1(i)) is just (2.2).

Corollary 4.2. $\zeta : B\Sigma_\infty \longrightarrow QL_p$ and $B\Sigma_\infty \longrightarrow {}_p(Q_0S^0)$ induce ring homomorphisms in homology.

The proof of (2.3) will be completed by an examination of the behavior of μ_* on the ring generators of $H_*(B\Sigma_\infty)$. This is the subject of the next section.

Section 5.

This section is devoted to the proof of

Proposition 5.1. Under the homomorphism

$$\mu_* : H_*(B\Sigma_\infty) \longrightarrow H_*({}_p(Q_0 S^0)), \text{ for } I \in \mathcal{A}, \mu_* e_I = tQ_I[I]*[-p^n]$$

modulo decomposable elements, for some t prime to p.

Remark. Combining (5.1) with (4.3), (3.1) and (3.3) we obtain (2.3) as an immediate corollary.

We define subgroups Δ_n of Σ_{p^r} inductively. First set $\Delta_1 = \pi \subsetneq \Sigma_p$. Assume we have already defined the inclusion $\alpha_{n-1} : \Delta_{n-1} \longrightarrow \Sigma_{p^{n-1}}$ is defined. We take Δ_n as the image of the composite

$$\pi \times \Delta_{n-1} \xrightarrow{1 \times \mathrm{diag}} \pi \int \Delta_{n-1} \xrightarrow{1 \int \alpha_{n-1}} \pi \int \Sigma_{p^{n-1}} \subset \Sigma_{p^n}.$$

Note that Δ_n is transitive on $\underline{p}^n$ and hence that α_n is (up to conjugacy) the regular representation of Δ_n (since Δ_n has p^n elements).

Proposition 5.2. If $I \in \mathcal{A}$, then $\bar{e}_I$ is in the image of

$$(B\alpha_n)_* : H_*(\Delta_n) \longrightarrow H_*(\Sigma_{p^n})$$

modulo the subgroup generated by elements of the form

$$\bar{e}_{0,j_2,\ldots,j_n}.$$

This is an easy consequence of Quillen's result on detecting cohomology classes and certain formulas of Adem [8, (9.1)] and Nishida [8, (9.4)].

Now suppose $I \in \mathcal{A}$ and $x_I \in H_*(B\Delta_n)$ is an element such that $(B\alpha_n)_*(x_I) = \bar{e}_I$ modulo a linear combination of the In view of (4.3), we only need prove that $(B\alpha_n)_*(x_I) = Q_I[1]*[-p^n]$ modulo decomposables in order to complete the proof of (5.1). (Note that $e_{0,j_2,\ldots,j_n}$ is decomposable in $H_*(B\Sigma_\infty)$.)

Consider the composite homology homomorphisms

$$(5.3)\quad H_*(W\Sigma_N \times_\Sigma (B\Sigma_p)^N) \xrightarrow{\theta_*} H_*(QB\Sigma_p) \xrightarrow{(Qj)_*} H_*(QL_p) \xrightarrow{\omega_*} H_*({}_p(Q_0S^0))$$

and

$$(5.4)\quad H_*(W\Sigma_N \times_\Sigma (B\Sigma_p)^N) \xrightarrow{\sigma_*} H_*(B\Sigma_{pN}) \xrightarrow{\theta_*} H_*({}_p(Q_{pN}S^0)) \xrightarrow{*[-pN]} H_*({}_p(Q_0S^0))$$

where σ is induced by the inclusion $\Sigma_N \int \Sigma_p \subset \Sigma_{pN}$ and both θ's refer to appropriate Dyer-Lashof maps. By a well-known argument [9], (5.3) and (5.4) agree modulo decomposables. Thus, the proof of (5.1) will be complete after we have shown that under the composite

$$H_*(B\Delta_n) \xrightarrow{(B\alpha_n)_*} H_*(B\Sigma_{p^n}) \xrightarrow{(B\omega_{p^n,p})_*} H_*(B(\Sigma_N \int \Sigma_p)) \xrightarrow{\sigma_*} H_*(B\Sigma_{pN}) \xrightarrow{\theta_*'} H_*(Q_{pN}S^0),$$

x_I goes to $tQ_I[1]$ modulo decomposables, for some number t prime to p.

In order to prove this, we first study the composite

$$\Delta_n \xrightarrow[\alpha_n]{} \Sigma_{p^n} \xrightarrow[\varphi_{p^n,p}]{} \Sigma_N \int \Sigma_p .$$

Consider the action of Δ_n on $\mathscr{P}(\underline{p}^n)$. Since the cardinality of Δ_n is p^n, the transitivity of Δ_n on $\underline{p}^n$ demands that each orbit must contain either p^{n-1} or p^n elements of $\mathscr{P}(p^n)$. An easy counting argument shows that the number t of orbits containing exactly p^{n-1} elements is prime to p. Let $\{\mathcal{O}_1, \ldots, \mathcal{O}_t\}$ be the set of those orbits and let $\{\mathcal{O}'_1, \ldots, \mathcal{O}'_s\}$ be the remaining orbits. The inclusion $\sigma : \Sigma_N \int \Sigma_p \longrightarrow \Sigma_{pN}$ may be viewed as letting $\Sigma_N \int \Sigma_p$ act on $\underline{N} \times \underline{p}$ [5]. Since

$$\underline{N} = \mathcal{O}_1 \coprod \cdots \coprod \mathcal{O}_t \coprod \mathcal{O}'_1 \coprod \cdots \coprod \mathcal{O}'_s ,$$

we can write

$$\underline{N} \times \underline{p} = (\mathcal{O}_1 \times \underline{p}) \coprod \cdots .$$

The proofs of the following two lemmas are easy if one recalls that Δ_n is transitive on $\underline{p}^n$ and that Δ_n has p^n elements.

Lemma 5.5. The action of Δ_n under $\sigma \circ \varphi_{p^n,p}$ on $\mathcal{O}_i \times \underline{p}$, $i = 1, \ldots, t$ is transitive and hence conjugate in $\mathrm{Aut}(\mathcal{O}_i \times \underline{p})$ to the regular representation.

Lemma 5.6. Up to conjugacy in $\Sigma_{p^{n+1}} = \mathrm{Aut}(\mathcal{O}_j' \times \underline{p})$, the action of Δ_n on $\mathcal{O}_j' \times \underline{p}$ factors as

$$\Delta_n \xrightarrow{\alpha_n} \Sigma_{p^n} \xrightarrow{\delta} \Sigma_{p^n} \int \Sigma_p \subset \Sigma_{p^{n+1}}$$

where δ is the inclusion of Σ_{p^n} as a semi-direct factor of $\Sigma_{p^n} \int \Sigma_p$.

We summarize the above discussion in the following

Proposition 5.7. The following diagram is commutative up to conjugacy in Σ_{pN}:

$$\begin{array}{ccccccc}
\Delta_n & \xrightarrow{\alpha_n} & \Sigma_{p^n} & & \xrightarrow{\varphi_{p^n,p}} & & \Sigma_N \int \Sigma_p \\
\downarrow \text{diag} & & & & & & \\
(\Delta_n)^t \times (\Delta_n)^s & & & & & & \downarrow \sigma \\
\downarrow (\alpha_n)^{t+s} & & & & & & \\
(\Sigma_{p^n})^t \times (\Sigma_{p^n})^s & \xrightarrow{1 \times \delta^s} & (\Sigma_{p^n})^t \times (\Sigma_{p^n} \int \Sigma_p)^s & \longrightarrow & (\Sigma_{p^n})^t \times (\Sigma_{p^{n+1}})^s & \longrightarrow & \Sigma_{pN}
\end{array}$$

Using the fact that the image of the composite

$$\tilde{H}_*(\Sigma_{p^n}) \xrightarrow{\delta_*} \tilde{H}_*(\Sigma_{p^n} \int \Sigma_p) \longrightarrow \tilde{H}_*(\Sigma_{p^{n+1}})$$

is generated by decomposable elements, it is now easy to complete the proof of (5.1).

Section 6.

In this section we outline some joint work with L. Finkelstein. We now use the notation of [2]. In particular, the $\Gamma_n^+ A$ shall mean the image of $W\Sigma_n \times_\Sigma A^n$ in $\Gamma^+ A$, the latter having the homotopy type of QA if A is a connected CW-complex.

Let A be a connected infinite loop space of finite type. Consider the composite

$$(6.1)\quad \Gamma^+ A \xrightarrow[f_1]{} (\Gamma^+ D_2 A) \times A \xrightarrow[f_2]{} (\Gamma^+ \Gamma_2^+ A) \times A \xrightarrow[f_3]{} \Gamma^+ A \times \Gamma^+ A \xrightarrow[f_4]{} \Gamma^+ A.$$

In (6.1), the first component of f_1 is the map h_2^A of [2: p. 201] and the second component is the natural map $\mathcal{L}$ for infinite loop spaces [7]. The second component of f_2 is the identity map of A while the first component is the evident map derived from the homotopy equivalence $\Gamma^+\Gamma_2^+ A \simeq (\Gamma^+ D_2 A) \times (\Gamma^+ A)$ proved in [2; Lemma 3.1]. The second component of f_3 is the natural inclusion while the first component is $\Gamma^+ \mathcal{L} | \Gamma^+ \Gamma_2^+ A$. Finally f_4 is the map representing the loop sum.

<u>Theorem</u> 6.2. The composite (6.1) is a mod 2 homology equivalence if A is an infinite loop space of finite type.

The proof of (6.2) uses the general methods of Sections 4 and 5 although it is rather more complicated. One presumes that a similar result will hold with 2 replaced by an odd prime, but the details have not been worked out yet for odd primes. Details will appear elsewhere.

We note that (6.2) says that, after localization at 2, the stable homotopy of A is a direct summand of the direct sum of the stable homotopy of D_2A and the unstable homotopy of A. In the special case $A = S^1$, we obtain as a corollary the Kahn-Priddy Theorem since $D_2S^1 = SRP^\infty$. (D_2A is also known as the quadratic construction on A.)

Northwestern University

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ON THE BIRTH AND DEATH OF ELEMENTS IN COMPLEX COBORDISM

Arunas Liulevicius(*)

In this paper we investigate a filtration on the ring of cobordism classes of stably almost complex manifolds Ω_*^U and the structure of the associated graded algebra $E_{*,*}$. The filtration is defined in terms of normal characteristic numbers: an element is said to have filtration s if all of its normal Chern numbers involving c_i with $i > s$ vanish. It is surprising that the structure of $E_{*,*}$ is quite different from the analogous algebra associated to unoriented cobordism and investigated in [6]. We also investigate a related question -given a cohomology class u in $H^*(BU;\mathbb{Z})$, what is the smallest positive integer $d(u)$ such that it is a normal u-number of a stably almost complex manifold.

The paper is organized as follows: in the first section we state and comment on the results, in the second we give a sketch of the proofs, in the third we give tables of characteristic numbers, incidence relations, and numerical invariants which express a given element in Ω_*^U in terms of our basis for real dimensions ≤ 10.

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1. A filtration on Ω_*^U and the structure of the ass associated graded algebra.

We shall use the complex dimension rather than the real dimension as the grading of Ω_*^U, that is Ω_n^U is identified with $\pi_{2n}(MU)$, where MU is the Thom spectrum of the unitary group. Let M^{2n} be a representative for $x \in \Omega_n^U$ with normal structure $\nu : M \longrightarrow BU$. The symbol $I = (i_1, i_2, \ldots)$ denotes a finitely non-zero sequence of natural numbers and $c^I = c_1^{i_1} c_2^{i_2} \ldots$ the corresponding monomial in the Chern classes.

Definition. We say $x \in F_s\Omega_n^U$ if $\langle \nu^* c^I, [M] \rangle = 0$ for all monomials c^I with $i_k \neq 0$ for some $k > s$.

Remark 1. In terms of the Hurewicz homomorphism $h : \pi_{2n}(MU) \longrightarrow H_{2n}(MU;Z)$, $x \in F_s\Omega_n^U$ if $h(x)$ comes from $H_{2n+2s}(MU(s);Z)$ —we say x is born homologically on $MU(s)$.

We let $E = \{E_{s,t}\}$ be the associated graded object with $E_{s,t} = F_s\Omega_{s+t}^U / F_{s-1}\Omega_{s+t}^U$. Since the filtration is well behaved with respect to the product in Ω_*^U, E is a bigraded algebra.

Theorem 1. E is free abelian and an integral domain. Indeed $E \otimes Q$ is a polynomial algebra on generators in $E_{1,*} \otimes Q$.

Remark 2. E is of finite type with rank $E_{s,t} =$ rank $H^{2t}(BU(s);Z)$. In particular $E_{1,t} \cong Z$ for $t \geq 0$.

Remark 3. E is definitely not a polynomial algebra.

Let $\bar{E} = \{x \in E_{s,t} | s+t > 0\}$, $QE = \bar{E}/\bar{E} \cdot \bar{E}$ the group of indecomposable elements of E.

Theorem 2. The group $QE_{s,t}$ is a finite torsion group if $s \neq 1$ and $QE_{1,n} = E_{1,n} \cong Z$ if $n \geq 0$.

Remark 4. The second half of the theorem is immediate since $E_{0,t} = 0$ if $t \neq 0$.

Remark 5. For $0 < s$, $0 < s+t \leq 5$, $QE_{s,t} \neq 0$ (see the next table) so there are many more indecomposables in E than in $H_*(MU;Z)$.

In the following table the elements $\alpha_{i,j}$ and $\beta^r_{s,t}$ will be specified more precisely later.

TABLE 1

Indecomposables in E in degrees ≤ 5

$s+t = 1$:

$$QE_{1,0} = E_{1,0} = Z \text{ on } \alpha_{1,0}$$

$s+t = 2$:

$$QE_{2,0} = Z_2 \text{ on cls } \alpha_{2,0}$$

$QE_{1,1} = E_{1,1} = Z$ on $\alpha_{1,1}$

$E_{0,2} = 0$

$s+t = 3$:

$QE_{3,0} = Z_2$ on cls $\alpha_{3,0}$

$QE_{2,1} = Z_6$ on cls $\alpha_{2,1}$

$QE_{1,2} = E_{1,2} = Z$ on $\alpha_{1,2}$

$E_{0,3} = 0$

$s+t = 4$:

$QE_{4,0} = Z_2$ on cls $\alpha_{4,0}$

$QE_{3,1} = Z_2$ on cls $\alpha_{3,1}$

$QE_{2,2} = Z_{12}+Z_2$ on cls $\beta^1_{2,2}$, cls $\beta^2_{2,2}$, respectively

$QE_{1,3} = E_{1,3}$ on $\alpha_{1,3}$

$E_{0,4} = 0$

$s+t = 5$:

$QE_{5,0} = Z_2$ on cls $\alpha_{5,0}$

$QE_{4,1} = Z_2$ on cls $\alpha_{4,1}$

$QE_{3,2} = Z_6$ on cls $\beta^1_{3,2}$, and cls $\beta^2_{3,2} = 3$ cls $\beta^1_{3,2}$

$QE_{2,3} = Z_{120}+Z_6$ on cls $\beta^1_{2,3}$, -4 cls $\beta^1_{2,3}$ + cls $\beta^2_{2,3}$

$QE_{1,4} = E_{1,4} = Z$ on $\alpha_{1,4}$

$E_{0,5} = 0$.

Remark 6. It is not true that $QE_{n,0} = Z_2$ for $n \geq 1$. Indeed $QE_{6,0} = 0$. This is related to the following:

Theorem 3.

(i) All polynomial generators $g_n \in \Omega^U_n$ for $n \leq 5$ have filtration n,

(ii) there exists a polynomial generator $g_6 \in \Omega_6^U$ of filtration 5 and no polynomial generator of Ω_6^U has smaller filtration.

<u>Definition</u>. We shall say that an element of Ω_n^U is young (or born today) if its filtration is precisely n; if its filtration is less than n, we'll say that it is old (or born yesterday).

<u>Theorem</u> 4. An infinite number of the Hazewinkel generators for Ω_*^U weren't born yesterday.

<u>Remark</u> 7. It is probably true that the preceding theorem can be improved to say that all Hazewinkel generators are young (not just those in gradings one less than a power of a prime).

<u>Theorem</u> 5. If g_n is a polynomial generator for Ω_*^U and $n > 1$ then g_n cannot be very old (that is, filtration $g_n > 1$).

Let us present the complete multiplicative structure of E in gradings ≤ 5. Let P_n be the cobordism class of the complex projective n-space with the usual complex structure: $\tau + 1 = (n+1)\eta$, where η is the canonical complex line bundle over CP^n (the conjugate of the Hopf line bundle). Define polynomial generators for Ω_*^U by

$$g_1 = P_1,$$
$$g_2 = P_2 - P_1^2,$$
$$g_3 = \tfrac{1}{2}P_3 - P_1P_2 + \tfrac{1}{2}P_1^3,$$
$$g_4 = -P_4 + 9P_1^2P_2 - 9P_1^4,$$
$$g_5 = \frac{1}{6}P_5 + 5P_1P_4 - \tfrac{1}{2}P_1P_2^2 - \frac{134}{3}P_1^3P_2 + 45P_1^5$$

and specify a basis for Ω_*^U (here $\gamma_{s,t}$ indicates a basis element of filtration precisely s) by letting

$$\alpha_{1,0} = g_1$$
$$\alpha_{2,0} = g_2$$
$$\alpha_{1,1} = -2g_2 + g_1^2$$
$$\alpha_{3,0} = g_3$$
$$\alpha_{2,1} = -2g_3 + g_1g_2$$
$$\alpha_{1,2} = 12g_3 - 8g_1g_2 + g_1^3$$
$$\alpha_{4,0} = g_4$$
$$\alpha_{3,1} = -2g_4 + g_1g_3$$
$$\beta^1_{2,2} = 604g_4 + 4527g_1g_3 + 1007g_2^2 - 4176g_1^2g_2 + 629g_1^4$$
$$\beta^2_{2,2} = -1380g_4 - 10344g_1g_3 - 2300g_2^2 + 9541g_1^2g_2 - 1437g_1^4$$
$$\alpha_{1,3} = -24g_4 - 180g_1g_3 - 40g_2^2 + 166g_1^2g_2 - 25g_1^4$$
$$\alpha_{5,0} = g_5$$
$$\alpha_{4,1} = 2g_5 + 5g_1g_4 - 6g_2g_3$$
$$\beta^1_{3,2} = 39584g_5 + 200587g_1g_4 - 56007g_2g_3 - 90538g_1^2g_3 + 2088g_1^5$$
$$\beta^2_{3,2} = 172668g_5 + 874974g_1g_4 - 244306g_2g_3 - 394933g_1^2g_3 + 9108g_1^5$$

$$\beta^1_{2,3} = 62676g_5+357253g_1g_4-73121g_2g_3+146242g_1^2g_3+55711g_1g_2^2$$
$$-267764g_1^3g_2+43960g_1^5$$

$$\beta^2_{2,3} = 250584g_5+1428328g_1g_4-292348g_2g_3+584690g_1^2g_3$$
$$+222740g_1g_2^2-1070545g_1^3g_2+175756g_1^5$$

$$\alpha_{1,4} = -720g_5-4104g_1g_4+840g_2g_3-1680g_1^2g_3-640g_1g_2^2+3076g_1^3g_2$$
$$-505g_1^5$$

Remark 8. The generators g_n were chosen so that their projections into E yield basic indecomposables.

Remark 9. The choice of $\alpha_{1,n}$ is unique up to sign.

Remark 10. The $\beta^r_{s,t}$ are really second choices —they were chosen to make the multiplication table of E look as simple as possible. Here it is:

TABLE 2

Multiplication table of E

$$\alpha^2_{1,1} = 2\alpha_{2,0}$$
$$\alpha_{1,0}\alpha_{2,0} = 2\alpha_{3,0}$$
$$\alpha_{1,0}\alpha_{1,1} = 6\alpha_{2,1}$$
$$\alpha_{1,0}\alpha_{3,0} = 2\alpha_{4,0}$$
$$\alpha_{1,0}\alpha_{2,1} = -2\alpha_{3,1}$$
$$\alpha_{1,0}\alpha_{1,2} = 2\beta^2_{2,2}$$

$$\alpha_{1,1}\alpha_{2,0} = -6\alpha_{3,1}$$
$$\alpha_{1,1}^2 = 12\beta^1_{2,2} + 6\beta^2_{2,2}$$
$$\alpha_{2,0}^2 = 4\alpha_{4,0}$$
$$\alpha_{1,0}\alpha_{4,0} = -2\alpha_{5,0}$$
$$\alpha_{1,0}\alpha_{3,1} = -2\alpha_{4,1}$$
$$\alpha_{1,0}\beta^1_{2,2} = 3\beta^1_{3,2} + \beta^2_{3,2}$$
$$\alpha_{1,0}\beta^2_{2,2} = -2\beta^2_{3,2}$$
$$\alpha_{1,0}\alpha_{1,3} = 30\beta^2_{2,3}$$
$$\alpha_{2,0}\alpha_{3,0} = 2\alpha_{5,0}$$
$$\alpha_{2,0}\alpha_{2,1} = 2\alpha_{4,1}$$
$$\alpha_{2,0}\alpha_{1,2} = -2\beta^2_{3,2}$$
$$\alpha_{1,1}\alpha_{3,0} = 6\alpha_{4,1}$$
$$\alpha_{1,1}\alpha_{2,1} = 6\beta^1_{3,2}$$
$$\alpha_{1,1}\alpha_{1,2} = -24\beta^1_{2,3} + 6\beta^2_{2,3}$$

<u>Remark</u> 11. No, it is not true that $\alpha_{1,0}\alpha_{n,0} = \pm 2\alpha_{n+1,0}$. For example, $\alpha_{1,0}\alpha_{5,0} = \alpha_{6,0}$. We shall return to this later.

<u>Remark</u> 12. The first really interesting multiplicative relation in E is $2\alpha_{1,0}\beta^1_{2,2} + \alpha_{1,1}\beta^2_{2,2} - \alpha_{1,1}\alpha_{2,1} = 0$. It is represented by $-172g_1g_4 - 4g_2g_3 - 1288g_1^2g_3 - 284g_1g_2^2 + 1188g_1^3g_2 - 179g_1^5$ which projects to $-4\beta^1_{2,3} + 216\beta^2_{2,3}$.

<u>Theorem</u> 6. Let $A(n)$ be the smallest natural number sat-

isfying $A(n) \mid \langle \nu_M^* c_n, [M] \rangle$ for all stably almost complex manifolds with normal structure $\nu_M : M \longrightarrow BU$, then $A(n)$ is always an even integer, moreover for low values of n we have

n	1	2	3	4	5	6	7	8	9	10
A(n)	2	2	2	2	2	4	2	2	2	4

<u>Remark</u> 13. It is easy to see that $A(n)$ is always a power of 2, since if we let $x = -P_1$, then the normal number of x determined by c_1 is 2, so the normal number of x^n determined by c_n is 2^n, so $A(n)$ divides 2^n. This was pointed out to me by P.S. Landweber.

<u>Remark</u> 14. Elmer Rees and P. Emery Thomas have determined [11] $A(n)$ for all n. F. Peterson told me on the flight to Vancouver that the power of 2 in $A(n)$ as determined by Rees and Thomas is given as the minimum integer k such that $\alpha(n+k) \leq 2k$, $k \geq 1$, where $\alpha(s)$ is as usual the number of ones in the dyadic expansion of s. This gives in particular that $A(n) = 8$ for $n = 29$ for the first time.

<u>Remark</u> 15. In the table of Theorem 6, $A(n) = 4$ if and only if there is a polynomial generator g_n of Ω_n^U which <u>was</u> born yesterday (and not any earlier). The preceding remark indicates that there is a polynomial generator g_{29} having filtration 27 and no polynomial generator can be chosen to have

smaller filtration in that dimension.

Definition. For $x \in H^{2n}(BU;Z)$ we let $d(x)$ be the greatest common divisor of the integers $\langle \nu_M^*(x), [M^{2n}] \rangle$ where $\nu_M : M \longrightarrow BU$ is a stably almost complex structure on M^{2n}.

TABLE 3

Values of $d(c^E)$

c_1	c_2	c_1^2	c_3	c_1c_2	c_1^3	c_4	c_1c_3	c_2^2	$c_1^2c_2$	c_1^4
2	2	1	2	2	2	2	1	1	3	1

c_5	c_1c_4	c_2c_3	$c_1^2c_3$	$c_1c_2^2$	$c_1^3c_2$	c_1^5
2	2	1	2	2	4	2

c_6	c_1c_5	c_2c_4	c_3^2	$c_1c_2c_3$	$c_1^3c_3$	c_2^3	$c_1^2c_2^2$	$c_1^4c_2$	c_1^6	$c_1^2c_4$
4	2	1	1	1	2	1	1	4	1	2

2. Methods of proof.

The main technique involves a careful scrutiny of characteristic numbers. The first step in the process is the familiar reduction to homotopy and homology. Under the Thom-Pontrjagin isomorphism $\Omega_n^U = \pi_{2n}(MU)$ the subgroup $F_s\Omega_n^U$ is specified by the pullback diagram

$$\begin{array}{ccc} F_s\Omega^U_n & \hookrightarrow & \pi_{2n}(MU) \\ \big\downarrow & & \big\downarrow h \\ \tilde{H}_{2s+2n}(MU(s);Z) & \xrightarrow{\lambda_s} & H_{2n}(MU;Z) \end{array}$$

where λ_s is the direct limit map and h is the Hurewicz homomorphism. Let $\varphi_* : H_*(MU;Z) \longrightarrow H_*(BU;Z)$ be the Thom isomorphism. We specify polynomial generators b_n for $H_*(MU;Z)$ by the condition

$$\langle c^E, \varphi_* b_n \rangle = \begin{cases} 1 & \text{if } E = (n,0,0,\ldots) \\ 0 & \text{otherwise.} \end{cases}$$

We should not be tied down to the presentation $H_*(MU;Z) = Z[b_1,\ldots,b_n,\ldots]$, for although this is convenient for the study of $\operatorname{Im} \lambda_s$ (indeed, $\operatorname{Im} \lambda_s$ is precisely the subgroup spanned by b^E with algebraic degree $b^E \leq s$ see [7], for example), it is not convenient for the study of $\operatorname{Im} h$ —for example polynomial generators of $\pi_*(MU)$ tend to have complicated and unreasonable expressions in terms of the monomials b^E. There is fortunately a set of generators much better suited for the study of $\operatorname{Im} h$: define m_n by $h(P_n) = (n+1)m_n$, or alternatively consider the formal power series $\omega = t+b_1t^2+\ldots+b_nt^{n+1}+\ldots$, and write $t = \omega+m_1\omega^2+m_2\omega^3+\ldots$. Of course, we must have an interpretation of $\operatorname{Im} \lambda_s$ in terms of these new generators if they are to be of use.

Let $\Delta_i : H_{2n}(MU;Z) \longrightarrow H_{2n-2i}(MU;Z)$ be defined as the dual homomorphism to taking the cup product with c_i (under the identification $\varphi_* : H_*(MU;Z) \longrightarrow H_*(BU;Z)$). It is immediate that $\Delta_i b_n = 0$ if $i \neq 0,1$, $\Delta_1 b_n = b_{n-1}$. Since $\Delta_i(xy) = \sum_{j+k=i} \Delta_j(x)\Delta_k(y)$ it is convenient to consider the total Δ operation

$$\Delta : H_*(MU;Z) \longrightarrow H_*(MU;Z)[s]$$

which preserves grading if we let grade $s = 2$. Now Δ is a homomorphism of algebras over Z, moreover it turns out that $x \in \operatorname{Im} \lambda_n$ if and only if $\Delta(x)$ has degree $\leq n$ as a polynomial in s.

It is not difficult to derive formulae for Δm_n from $\Delta b_t = b_t + b_{t-1}s$.

TABLE 4

Value of Δ on m_n

$$\Delta m_1 = m_1 - s$$

$$m_2 = m_2 - 3m_1 s + 2s^2$$

$$m_3 = m_3 - (4m_2 + 2m_1^2)s + 10m_1 s^2 - 5s^3$$

$$m_4 = m_4 - (5m_3 + 5m_1 m_2)s + (15m_2 + 15m_1^2)s^2 - 35m_1 s^3 + 14s^4$$

$$m_5 = m_5 - (6m_4 + 6m_1 m_3 + 3m_2^2)s + (21m_3 + 42m_1 m_2 + 7m_1^3)s^2$$

$$-(56m_2+84m_1^2)s^3+126m_1s^4-42s^5$$

$$m_6 = m_6-(7m_5+7m_1m_4+7m_2m_3)s+(29m_4+56m_1m_3+28m_2^2+28m_1^2m_2)s^2$$
$$-(84m_3+252m_1m_2+84m_1^3)s^3+(210m_2+420m_1^2)s^4-462m_1s^5+132s^6$$

The sequence $1,2,5,14,42,132,\ldots$ is a famous sequence of natural numbers, the so-called Catalan numbers [13] which go back to von Segner [12] and Euler. Indeed, $\Delta_n m_n = (-1)^n C_n$, where the n-th Catalan number is specified by

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and satisfy the recursion relation $C_1 = 1$, $\frac{C_{n+1}}{C_n} = 4 - \frac{6}{n+2}$, which shows that the Catalan numbers grow at a good rate (for example $C_{19} = 1{,}767{,}263{,}190$) a fact useful in the proof of Theorem 4.

A few words about the proof of the other theorems: The first part of Theorem 1 uses the fact that the filtration F_s is a direct summand of F_{s+1} since they are specified in terms of kernels of the maps Δ_i. The second part is a consequence of the theorem of Milnor [10] that $h\otimes 1 : \pi_*(MU)\otimes Q \longrightarrow H_*(MU;Q)$ is an isomorphism and a result of Newton on the algebra $H_*(BU;Q)$ (see [7], for example). Theorem 2 follows without difficulty from Theorem 1. Theorem 3 is a consequence of techniques for investigating characteristic numbers which we described above. Theorem 5 uses the theorems of Stong [14] and Hattori [3] in the formulation of Adams and Liulevicius [2] —to check that an element in the homology of MU is in the image of h it is necessary and sufficient that it satisfy the Riemann-Roch integrality relations (see [4] for a formulation of the integrality relations which are specially convenient for the proof of Theorem 5).

3. Incidence matrices and characteristic numbers.

The aim of this section is two-fold. First, we wish to present the incidence matrices for the basis of $H_*(MU;Z)$ in terms of monomials in the polynomial generators m_i and the basis of monomials in the Chern classes for $H^*(MU;Z)$ (which we identify with $H^*(BU;Z)$ under the Thom isomorphism). Second, we wish to enable the reader to read off the class of an element in E in terms of our basis provided he knows the normal Chern numbers of the class. Our tables will be presented in the following format: for a given grading we shall first present the incidence matrix (for each monomial m^E the row indexed by it gives its Chern numbers), and right below it the normal Chern numbers of the appropriate $\alpha_{i,j}$, $\beta^r_{s,t}$. The matrix following is the inverse of the incidence matrix —it allows one to read off the expression of h(x) in terms of the monomial basis m^E provided one knows the normal Chern numbers of x. Our tables go up to complex dimension 6 for the incidence matrices. For incidence numbers for the b^E and c^F bases the reader is reffered to VandeVelde's thesis [15].

GRADE 1

	c_1
m_1	-1
$\alpha_{1,0}$	2

	m_1
c_1	-1

GRADE 2

	c_2	c_1^2
m_2	2	3
m_1^2	1	2
$\alpha_{2,0}$	2	1
$\alpha_{1,1}$	0	6

	m_2	m_1^2
c_2	2	-3
c_1^2	-1	2

GRADE 3

	c_3	c_1c_2	c_1^3
m_3	-5	-10	-16
m_1m_2	-2	-5	-9
m_1^3	-1	-3	-6
$\alpha_{3,0}$	-2	-2	-2
$\alpha_{2,0}$	0	-2	-2
$\alpha_{1,2}$	0	0	-24

	m_3	m_1m_2	m_1^3
c_3	-3	12	-10
c_1c_2	3	-14	13
c_1^3	-1	5	-5

GRADE 4

	c_4	c_1c_3	c_2^2	$c_1^2c_2$	c_1^4
m_4	14	35	45	75	125
m_1m_3	5	15	20	36	64
m_2^2	4	12	17	30	54
$m_1^2m_2$	2	7	10	19	36
m_1^4	1	4	6	12	24
$\alpha_{4,0}$	2	5	-9	-51	-193
$\alpha_{3,1}$	0	-2	26	114	402
$\beta_{2,2}^1$	0	0	3	-6	2990
$\beta_{2,2}^2$	0	0	0	24	-6804
$\alpha_{1,3}$	0	0	0	0	-120

	m_4	m_1m_3	m_2^2	$m_1^2m_2$	m_1^4
c_4	4	-20	-10	60	-35
c_1c_3	-4	23	10	-72	45
c_2^2	-2	10	7	-36	22
$c_1^2c_2$	4	-23	-12	76	-51
c_1^4	-1	6	3	-21	14

GRADE 5

	c_5	c_1c_4	c_2c_3	$c_1^2c_3$	$c_1c_2^2$	$c_1^3c_2$	c_1^5
m_5	-42	-126	-196	-336	-441	-756	-1296
m_1m_4	-14	-49	-80	-145	-195	-350	-625
m_2m_3	-10	-35	-60	-107	-148	-266	-480
$m_1^2m_3$	-5	-20	-35	-66	-92	-172	-320
$m_1m_2^2$	-4	-16	-29	-54	-77	-144	-270
$m_1^3m_2$	-2	-9	-17	-33	-48	-93	-180
m_1^5	-1	-5	-10	-20	-30	-60	-120
$\alpha_{5,0}$	-2	16	-111	-524	-1242	-3664	-9956
$\alpha_{4,1}$	0	-2	-122	-578	-1290	-3772	-10142
$\beta_{3,2}^1$	0	0	-2	-4	-58964	73532	622906
$\beta_{3,2}^2$	0	0	0	24	-257112	321024	2717892
$\beta_{2,3}^1$	0	0	0	0	6	20	-62606
$\beta_{2,3}^2$	0	0	0	0	0	8	-250544
$\alpha_{1,4}$	0	0	0	0	0	0	720

	m_5	m_1m_4	m_2m_3	$m_1^2m_3$	$m_1m_2^2$	$m_1^3m_2$	m_1^5
c_5	-5	30	30	-105	-105	280	-126
c_1c_4	5	-34	-30	125	115	-340	161
c_2c_3	5	-30	-36	114	129	-336	156
$c_1^2c_3$	-5	34	33	-131	-127	374	-181
$c_1c_2^2$	-5	32	36	-124	-136	372	-178
$c_1^3c_2$	5	-34	-35	134	137	-399	196
c_1^5	-1	7	7	-28	-28	84	-42

GRADE 6

	c_6	c_1c_5	c_2c_4	$c_1^2c_4$	c_3^2	$c_1c_2c_3$	$c_1^3c_3$	c_2^3	$c_1^2c_2^2$	$c_1^4c_2$	c_1^6
m_6	132	462	840	1470	1008	2352	4116	3136	5488	9604	16807
m_1m_5	42	168	322	588	392	973	1764	1323	2394	4320	7776
m_2m_4	28	112	223	402	275	695	1240	945	1710	3100	5625
m_3^2	25	100	200	360	250	620	1120	856	1552	2816	5120
$m_1^2m_4$	14	63	129	243	160	420	785	585	1090	2025	3750
$m_1m_2m_3$	10	45	95	177	120	315	587	444	828	1544	2880
m_2^3	8	36	78	144	99	261	486	372	693	1296	2430
$m_1^3m_3$	5	25	55	106	70	193	370	276	528	1008	1920
$m_1^2m_2^2$	4	20	45	86	58	160	306	231	442	846	1620
$m_1^4m_2$	2	11	26	51	34	98	192	144	282	552	1080
m_1^6	1	6	15	30	20	60	120	90	180	360	720

	m_6	m_1m_4	m_2m_4	m_3^2	$m_1^2m_4$	$m_1m_2m_3$	m_2^3	$m_1^3m_3$	$m_1^2m_2^2$	$m_1^4m_2$	m_1^6
c_6	6	-42	-42	-21	168	336	56	-504	-756	1260	-462
c_1c_5	-6	47	42	21	-198	-366	-56	609	861	-1540	588
c_2c_4	-6	42	50	21	-180	-376	-76	564	906	-1510	567
$c_1^2c_4$	6	-47	-46	-21	206	386	66	-649	-941	1695	-658
c_3^2	-3	21	21	15	-84	-204	-28	282	450	-750	281
$c_1c_2c_3$	12	-89	-92	-51	378	820	132	-1242	-1935	3346	-1285
$c_1^3c_3$	-6	47	46	24	-206	-413	-66	674	1001	-1805	708
c_2^3	2	-14	-18	-7	62	132	30	-198	-333	554	-211
$c_1^2c_2^2$	-9	68	73	36	-298	-626	-111	980	1535	-2690	1048
$c_1^4c_2$	6	-47	-48	-24	209	425	72	-692	-1052	1896	-750
c_1^6	-1	8	8	4	-36	-72	-12	120	180	-330	132

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ON THE UNSTABLE J-HOMOMORPHISM

Mark Mahowald(*)

We are interested in the following diagram of sequences:

$$\begin{array}{ccccccc}
\pi_j(SO(m)) & \xrightarrow{i_1} & \pi_j(SO) & \xrightarrow{p_1} & \pi_j(V_m) & \xrightarrow{\partial_1} & \cdots \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle j_2} & & \\
\pi_j(\Omega^m S^m) & \xrightarrow{i_2} & \pi_j(\Omega^\infty S^\infty) & \xrightarrow{H_n} & \pi_j(\Omega^\infty S^\infty, \Omega^m S^m) & \xrightarrow{\partial_2} & \cdots .
\end{array}$$

The maps J and j_1, induced by the inclusion of $SO \subset \Omega^\infty S^\infty$ and $SO(n) \subset \Omega^m S^m$, are the stable and unstable J homomorphism respectively. The map J is completely determined now. If $2m-1 > j \geq 16$ then ∂_1 is a monomorphism onto a direct summand [2]. It is also known that j_2 is on isomorphism if $j < 2m-1$ [6]. It is well established that $j_1 \partial_1$ is not a monomorphism but all the elements known to be in the kernel occur for values of j larger than or equal to but near a power of 2 except for the Kervaire invariant family in dim 30 and 62. For example if $j = 16$, 32, or 64 and $m \leq j-3$, there is an element in the ker $j_1 \partial_1$

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[5]. Also if $j = 30$ there is an element in the kernel for $m \leq 22$. In this case the element α in $\pi_{30}(\Omega^\infty S^\infty)$ such that $p_2\alpha$ generates $\ker \partial_2$ is the Kervaire invariant element in dimension 30. Our goal here is to prove that this phenomenon is typical for the behaviour of j_1.

First we need some notation. If $n = 4a+b$, $0 \leq b \leq 3$ let $\varphi(n) = 8a+2^b$. Let $\rho(n)$ be defined by $n \equiv 2^{\rho(n)} \bmod 2^{\rho(n)+1}$. We will prove the following result.

<u>Theorem</u> 1. For each integer n the map $j_1\partial_1$ is a monomorphism for $j < n-1$ and $m \geq n-\varphi(\rho(n))$.

In [1] Adams introduces a spectral sequence based on the periodicity of the spectra $\Sigma^{2^{\varphi(n-k)}} P_k^n \simeq P_{k+2^{\varphi(n-k)}}^{n+2^{\varphi(n-k)}}$

$$E^2_{p,q} = Z_2 \otimes \pi_q(S)$$
$$= \mathrm{Tor}_1^Z(Z_2, \pi_q^S(S^0)).$$

This spectral sequence is defined for all $p \in Z$. The proof of Theorem 1 also proves the following result.

<u>Theorem</u> 2. $E^\infty_{p,q} = 0$ for $q+p < -1$.

The argument can also be applied to show the following:

<u>Theorem</u> 3. $\mathrm{Ext}_A^{s,t}(Z_2[x,x^{-1}], Z_2) = 0$ if $t < 0$ and

$s-t \neq -1$ or if $3s > t+4$.

In particular this shows that the edge of $\mathrm{Ext}_A^{*,*}(Z_2(x,x^{-1}),Z_2)$ is the same as $\mathrm{Ext}_A^{**}(\tilde{H}^*(S^{-1}),Z_2)$. These two results give some evidence for the conjecture in [1].

2. Some general results.

The connection between Theorems 1 and 2 is implied by the following result which is "well known" but I am not sure where a proof or a statement of it occurs. First some notation:

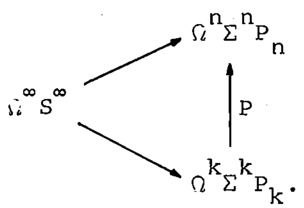
Recall that H_n is the map

$$H_n : \pi_j(\Omega^\infty S^\infty) \longrightarrow \pi_j(\Omega^\infty S^\infty, \Omega^n S^n).$$

By a theorem of Toda [6], $\pi_j(\Omega^\infty S^\infty, \Omega^n S^n) \simeq \pi_j(\Omega^n \Sigma^n P_n)$ for $j < 3n-2$.

Theorem 2.1. If $\alpha \in \pi_j(\Omega^\infty S^\infty)$ has $H_n(\alpha) \neq 0$ for $n > \frac{j+2}{3}$ then $H_n(\alpha)$ is in the image of $\pi_j(\Omega^k \Sigma^k P_k) \longrightarrow \pi_j(\Omega^n \Sigma^n P_n)$ where $k = \left\lfloor \frac{j+2}{3} \right\rfloor + 1$.

Proof. Consider the following diagram:



Clearly if $H_n(\alpha) \neq 0$ then $PH_k(\alpha) \neq 0$ and this is the theorem.

We will use this to prove Theorem 1. The key step will be the following result which also needs some additional terminology. Let $n \equiv 0 \bmod 2^i$. Let $\lambda : P_n^{n+2^i-2} \to S^n$ be the cofiber map of the projection $P_{n-1}^{n+2^i-1} \to P_n^{n+2^i-1}$. Clearly λ desuspends $n-2^i+1$ times. Let $\varphi(i)$ be the vector field number, i.e., if $i = 4a+b$, $0 \leq b \leq 3$ then $\varphi(i) = 8a+2^b$.

Theorem 2.2. Suppose $j \leq n+2^i-2$. If $\beta \in \pi_j^S(P_n^{n+2^i-2})$ projects non zero into $\pi_j^S(P_{n+2^i-\varphi(i)}^{n+2^i-2})$ then $\lambda\beta \neq 0$ and hence β is not in the image of $\pi_j^S(P_{n-1}^{n+2^i-2}) \to \pi_j(P_n^{n+2^i-2})$.

Combining 2.1 and 2.2 we get a proof of Theorem 1. Theorem 2 follows from 2.2 by choosing n correctly. In particular if $n = 2^{2^{2^i}}-2^i$ then $P_n^{n+2^i-2}$ is stably equivalent to $P_{-2^i}^{-2}$ and thus all classes which have filtration between $-\varphi(i)$ and -2 are either not cycles or are boundaries in $P_{-2^i-1}^{-2}$.

Now we will prove 2.2. First observe that $P_n^{n+2^i-2}$ is the Thom complex of $n\xi_{2^i-2}$ where ξ_ℓ is the Hopf bundle over RP^ℓ. Thus by [3] and the vector field problem we see that $P_n^{n+2^i-2}$ desuspends at least $n-2^i+\varphi(i)$ times. We have the following diagram

$$\begin{array}{ccccc}
\Sigma^{-(n-2^i+\varphi(i))}P_n^{n+2^i-\varphi(i)-1} & \longrightarrow & \Sigma^{-(n-2^i+\varphi(i))}P_n^{n+2^i-2} & \longrightarrow & \Sigma^{-(n-2^i+\varphi(i))}P_{n+2^i-\varphi(i)}^{n+2^i-2} \\
\Big\downarrow \bar{\lambda} & & \Big\downarrow \bar{\lambda} & & \Big\downarrow h \\
S^{2^i-\varphi(i)} & \longrightarrow & \Omega^{\varphi(i)-1}S^{2^i-1} & \longrightarrow & \Sigma^{2^i-\varphi(i)}P_{2^i-\varphi(i)}^{2^i-2}
\end{array}$$

<u>Lemma</u> 2.3. The map h is a homotopy equivalence.

<u>Proof</u>. The map λ carries a functional Sq^j for all $j \le 2^i-1$ and thus the map h has positive degree in dimension between $2\cdot 2^i-2\varphi(i)$ and $2\cdot 2^i-\varphi(i)-2$. Thus h induces an isomorphism in homology and so is a homotopy equivalence.

To complete the proof of the theorem we need the following diagram and another lemma.

$$\begin{array}{ccccc}
\Sigma^{-(n-2^i+\varphi(i))}P_n^{n-\varphi(i)+2^i-1} & \longrightarrow & \Sigma^{-(n-2^i+\varphi(i))}P_n^{n-2+2^i} & \longrightarrow & \Sigma^{-(n-2^i+\varphi(i))}P_{n-\varphi(i)+2^i}^{n-2+2^i} \\
\Big\downarrow & \searrow & \Big\downarrow \lambda & & \Big\downarrow h' \\
\Big\downarrow & & \Omega^{n-2^{i+1}+\varphi(i)}P_{n-2^i}^{n-2} & \longrightarrow & G \\
\Big\downarrow & & \Big\downarrow \lambda'' & & \Big\downarrow h'' \\
S^{2^i-\varphi(i)} & \longrightarrow & \Omega^{\varphi(i)}S^{2^i-1} & \longrightarrow & \Sigma^{2^i-\varphi(i)}P_{2^i-\varphi(i)}^{2^i-2}
\end{array}$$

Lemma 2.4. If $\alpha \in \pi_j(\Sigma^{-(n-2^i+\varphi(i))} P^{n-2+2^i}_{\varphi(i)+2^i})$ for $j \leq 2^{i+1}-2-\varphi(i)$ is in the image of $\bar{\varphi}$ then $0 \notin \lambda''(\bar{\varphi}^{-1}(\alpha))$. Thus α is not in the image of the map induced by the projection $P^{n-2+2^i}_{n-1} \longrightarrow P^{n-2+2^i}_{n-\varphi(i)+2^i}$.

Proof. The first part follows immediately since $h''h'$ is a homotopy equivalence. The second statement follows from the fact that the fiber of λ'' is $\Omega^{[n-2^{i+1}+\varphi(i)]} P^{n-2+2^i}_{n-1}$ through the range of groups we are considering.

This completes the proof of 2.2.

Northwestern University

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PROBLEMS IN INFINITE LOOP SPACE THEORY

J.P. May

This will be a rambling discussion of the status of infinite loop space theory, with the emphasis on unsolved problems in geometric topology, algebraic K-theory, cobordism, homology operations, and the general abstract theory. Many of the problems will involve E_∞ ring spaces and spectra. This is not solely a reflection of my personal interest. Implicitly or explicitly, these structures are central to most of the applications. The basic reference is [9], which will contain the material of the preprints "Coordinate-free spectra", "$\mathcal{J}$-functors and orientation theory", "E_∞ ring spectra" (with Frank Quinn and Nigel Ray), and "On kO-oriented bundle theories", as well as the material on E_∞ ring spaces about which I talked at the conference and other material developed since (some of which is sketched below).

We now have a coherent theory of infinite loop spaces and spectra, and it has gradually become apparent what the theory can and cannot do. I know of very few significant examples of H-spaces which are suspected but not known to be infinite loop spaces and there are no outstanding theoretical problems concerning the recognition of infinite loop spaces. The only possible exception might be Quillen's old conjecture that the representing space of a functor with an appropriate transfer is an infinite loop space. It is by now well understood that this conjecture is wholly implausible, but that any counter-example would be sufficiently ugly as to be wholly uninteresting (see Lada [4]).

The situation as regards the recognition of E_∞ ring spectra which are suspected but not known to be E_∞ ring spectra. The only Thom spectrum not yet accounted for is MPL (and Quinn is working on this). A very provocative example is BP.

Problem 1. Does the Brown-Peterson spectrum admit a model as an E_∞ ring spectrum?

The point here is that the notion of an E_∞ ring spectrum seems not to be a purely homotopical one; good concrete geometric models are required, and no such model is known for BP. For much the same reason, we have the awkward state of affairs revealed by the following question.

Problem 2. Are localizations and completions of E_∞ ring spectra again E_∞ ring spectra?

Of course, we can use global E_∞ ring spectra to obtain infinite loop space information and can then localize or complete (as must be done even to make sense of some of the geometric problems below).

One way to handle the local version of Problem 2 would be to carry out the following program, which is certainly well within reach and should have other useful applications.

Problem 3. Develop recognition principles for passage from "p local" E_∞ spaces and E_∞ ring spaces to p local spectra and E_∞ ring spectra.

The idea here is to reconstruct the entire machine of [7, 8, 9] with the full symmetric groups replaced by suitably compatible p-Sylow subgroups.

The situation as regards the recognition of infinite loop maps is still less satisfactory. The machine does produce lots of infinite loop maps and does show that once certain key maps are proven to be infinite loop maps then it will follow that various other maps are so as well. Unfortunately, many of the key maps have yet to be proven to be infinite loop maps. I suspect that this is in the general nature of things and that improvements in the abstract theory will be of little help with the remaining problems. However, a solution to Problem 3 would be of some use since the machine has a marked aversion to virtual representations and the localization at p of a global map defined in terms of virtual representations can sometimes be defined in terms of honest representations.

Of course, we need not rely only on naturality properties of the machine to construct infinite loop maps, since all of the familiar techniques of stable homotopy theory are also available.

There is as yet no analog of this statement for maps of E_∞ ring spectra: the only technique currently available is the construction of models acceptable to the machine.

Given this picture of the abstract situation, let's turn to concrete applications. We would like to have a complete understanding of Adams' analysis of the groups $JO(X)$ and of Sullivan's analysis of $KTop(X)$ away from 2 on the level of cohomology theories or, more or less equivalently, on the level of infinite loop space structures on the classifying spaces of various fibration and bundle theoretic functors. Explicitly, we would like to decompose the localizations or completions of all relevant infinite loop spaces into products (or, at 2, fibrations) of specific atomic pieces, namely BCoker J and pieces obtained solely from BO (such as B IM J, the various factors of BO at odd primes, BSpin, etc.). The following three conjectures would supply the key infinite loop maps necessary for this purpose.

<u>Conjecture</u> 1. The complex Adams conjecture holds on the infinite loop space level.

This means that, for each r, the composite

$$BU \xrightarrow{\psi^r - 1} BU \xrightarrow{Bj} BSF$$

is trivial as an infinite loop map when localized away from r. The real analog is wildly false. With $r = 3$ and U replaced by O, Madsen [5] showed that not even the first delooping is null homotopic. The splitting of BSF as an infinite loop space at odd primes (Tornehave [16] or [9]) shows that the conjecture is at least plausible, and several people seem to be working on it.

To state the next two conjectures, we require some preliminaries from [9]. Let G be a bundle theory (O,U,Spin,Top,F, etc.). Let $B(G;E)$ be the classifying space for E-oriented stable G-bundles and let $q : B(G;E) \longrightarrow BG$ correspond to neglect of orientation. An E-orientation of G is an H-map $g : BG \longrightarrow B(G;E)$ such that $q \circ g \simeq 1$. Thus g specifies natural E-orientations, with product formula, of stable G-bundles. When E is an E_∞ ring spectrum, $B(G;E)$ is an infinite loop space. Say that g is perfect if it is an infinite loop map and if $q \circ g \simeq 1$ as infinite loop maps.

<u>Conjecture</u> 2. The Atiyah-Bott-Shapiro orientation $g : BSpin \longrightarrow B(Spin;kO)$ is perfect.

<u>Conjecture</u> 3. The Sullivan orientation $\bar{g} : BSTop \longrightarrow B(STop;kO[1/2])$ is perfect.

Quinn is working on the second of these.

To relate these conjectures to BCoker J, we compose g and $\bar{g}$ with the natural infinite loop maps from their ranges to $B(SF;kO)$ (suitably localized) and then apply the universal cannibalistic class $c(\psi^r) : B(SF;kO) \longrightarrow BSpin_\otimes$, where $BSpin_\otimes$ is the 2-connective cover of the special unit infinite loop space $BO_\otimes = SF\, kO$ (see [9]). At p, the fibre of $c(\psi^{r(p)})$ is BCoker J, where $r(2) = 3$ and, for $p > 2$, $r(p)$ is a power of a prime $q \neq p$ such that $r(p)$ reduces mod p^2 to a generator of the group of units of Z_{p^2}. An affirmative answer to the folwing question would imply that $c(\psi^r)$ is an infinite loop map.

Problem 4. Is $\psi^r : kO[1/r] \longrightarrow kO[1/r]$ a map of E_∞ ring spectra?

As our abstract discussion makes clear, we are not yet close to an answer. To avoid this question, we turn to discrete models and algebraic K-theory. Recall from [9] that "bipermutative categories" naturally give rise to E_∞ ring spaces and thus to E_∞ ring spectra. For a commutative topological ring A, we have bipermutative categories $\mathcal{O}A$ and $\mathcal{GL}A$ of orthogonal and general linear groups. The E_∞ ring spectrum kO used above is obtained from $\mathcal{O}\mathbb{R}$. Let kO^δ denote the completion away from q (as above, when thinking of p) of the E_∞ ring spectrum obtained from $\mathcal{O}\bar{k}_q$, where $\bar{k}_q$ is an algebraic closure of the field of q elements. Brauer lifting yields an equivalence $\hat{\lambda} : kO^\delta \longrightarrow \hat{k}O[1/q]$ of ring spectra. A local version of this result (in the complex case) was proven by Tornehave [15]; on the completed level (which is the one of topological interest), the result is in fact extremely simple [9].

Problem 5. Is $\hat{\lambda} : kO^{\delta} \longrightarrow k\hat{O}[1/q]$ a map of E_{∞} ring spectra?

Again, we are not yet close to an answer. The Frobenius automorphism $\phi^r : \mathcal{O}\bar{k}_q \longrightarrow \mathcal{O}\bar{k}_q$ $(r = q^a)$ is a morphism of bipermutative categories, and the induced map (again denoted ϕ^r) on kO^{δ} is transported to ψ^r via $\hat{\lambda}$. $c(\phi^r) : kO^{\delta} \longrightarrow BSpin^{\delta}_{\otimes}$ (the 2-connective cover of $BO^{\delta}_{\otimes} = SF\,kO^{\delta}$) is an infinite loop map. Its fibre at p, with $r = r(p)$, is BCoker J endowed with an infinite loop space structure. This fibre admits a more conceptual description as $B(SF;jO^{\delta})$, where jO^{δ} is a certain E_{∞} ring spectrum which is a discrete model for the fibre jO of $\psi^{r(p)} - 1 : kO \longrightarrow kSpin$ at p (see [9]). In view of Problems 4 and 5, it should be apparent that, at present, Conjectures 2 and 3 would be much more useful if proven with kO replaced by kO^{δ}. So reformulated, they and Conjecture 1 would complete the desired analysis of Adams' work and of Sullivan's work away from 2. The sketch above is philosophically sound because kO^{δ} and $k\hat{O}[1/q]$ are indistinguishable on the motivating level of multiplicative cohomology theories and because $BO^{\delta}_{\otimes}$ and $B\hat{O}_{\otimes}[1/q]$ are equivalent as infinite loop spaces at $p \neq q$.

Here the last clause follows from a recent result of Adams and Priddy which shows that, up to equivalence, there is only one p-local or p-complete connective spectrum with zeroth space equivalent to BSO localized or completed at p. Again, maps are more difficult.

Problem 6. When is an H-map between two infinite loop spaces, both equivalent to BSO localized or completed at p, an infinite loop map?

Adams's map $\rho^3 : BSO \to BSO_{\otimes}$ is a good test case. The cannibalistic classes $\rho^r : BSpin \to BSpin_{\otimes}$ will be infinite loop maps when completed away from r if Conjecture 2 holds (in its original form) and (a weakened form of) either Problem 4 or Problem 5 has an affirmative answer. The Adams-Priddy result shows that F/Top and $BO_{\otimes}$ are equivalent as infinite loop spaces at each $p > 2$. If Conjecture 3 holds (in its original form), then the Sullivan L-genus equivalence $F/Top \to BO_{\otimes}$ away from 2 will be an infinite loop map.

We are still very far from understanding F/Top at $p = 2$, and the following problem is probably beyond reach at present.

Problem 7. Describe F/Top and F/PL as infinite loop spaces at $p = 2$; the describe BTop and BPL.

Madsen [5] showed that $B^3(F/Top)$ does not split at 2 as a product of Eilenberg-MacLane spaces; Madsen and Milgram [6] have shown that $B^2(F/Top)$ does so split.

We return for a moment to the categories $\mathcal{UL}A$ and $\mathcal{O}A$ for a discrete commutative ring A. Let kA and kOA denote the resulting E_∞ ring spectra. Quillen's algebraic K-groups are $K_iA = \pi_i kA$ for $i \geq 1$ (and the E_∞ ring structure gives K_*A a ring structure). Quillen's work suggests that it may be reasonable to define $KO_iA = \pi_i kOA$ for $i \geq 1$ and to regard the natural map $KO_*A \longrightarrow K_*A$ as analogous to complexification.

Problem 8. What are the images of the stable stems in KO_*Z and K_*Z under the induced maps on homotopy groups of the units e of the spectra kOZ and kZ?

Quillen showed that, in degrees 4s-1, the image of J maps monomorphically to K_*Z (onto a direct summand except possibly for 2-torsion when s is odd), and the image of J maps monomorphically onto a direct summand of KO_*Z in all degrees [9]. Beyond the obvious stable families of order 2, which map monomorphically onto direct summands of both KO_*Z and K_*Z, nothing is known about the behavior of the cokernel of J. While one really wants to know all of K_*Z, such a calculation seems unlikely to come out of infinite loop space techniques.

Problem 9. What is the precise relationship between the machine-built spectrum kA and the Gersten-Wagoner spectrum KA [2, 18]?

One would hope that, modulo adjustment necessitated by K_0A, kA is the connective spectrum associated to KA. This problem, and the category A, are closely related to Hermitian K-theory and algebraic L-theory, an area which abounds in bipermutative categories whose associated E_∞ ring spectra have yet to be studied. A good solution of Problem 9 should also yield the conjecture of Karoubi [3, p. 397 and 392] in the topological form stated by Wall [19, p. 292]. Incidentally, the theory of E_∞ ring spectra should answer the question about products raised in [19, p. 292] although here, and in various other applications, a solution of the following problem may eventually be required in order to obtain a really complete picture.

Problem 10. Develop a theory of E_∞ pairings of E_∞ module spectra over an E_∞ ring spectrum.

We turn next to homological calculations in geometric topology. On the infinite classifying space level, the calculations are quite complete (and will be summarized in [11]), although we still know relatively little about how to interpret them.

Problem 11. Find fibration-theoretic interpretations of characteristic classes for spherical fibrations.

Ravenel [13] and others have given such interpretations of certain classes, but not enough to generate $H^*(BSF)$ (under all structure in sight, including the duals of the homology operations).

We can read off $H^*(BSF(n);Z_p)$ from $H^*(BSF;Z_p)$ when $p = 2$ or n is odd, and similarly for $G(n+1)$. The following problem should not be too difficult.

Problem 12. Compute $H_*(BSF(n);Z_p)$ and $H_*(BSG(n+1);Z_p)$ for p odd and n even.

Provided that $PL(n)$ and $Top(n)$ are interpreted in the (comparatively uninteresting) block bundle sense, $G(n)/PL(n) \simeq G/PL$ for $n \geq 3$ and $Top(n)/PL(n) \simeq Top/PL$ for $n \geq 5$. Thus the following problem should be solvable by comparison with the infinite case.

<u>Problem</u> 13. Compute $H^*(BPL(n);Z_p)$ and $H^*(BTop(n);Z_p)$ for all primes p.

The analog for the usual $PL(n)$ and $Top(n)$ is still beyond reach (and iterated loop space techniques probably have little relevance).

On the Thom spectrum level, there is still much to be done. At the prime 2, Brumfiel, Madsen, and Milgram [1] in the unoriented case and Madsen and Milgram [unpublished] in the oriented case have obtained essentially complete information about Top and PL cobordism. At odd primes, Tsuchiya [17] showed that the kernel of the natural map $A \longrightarrow H^*(MSTop)$ is the left ideal generated by the Milnor elements Q_0 and Q_1. Unfortunately, that now seems to be the easiest step in the following program.

<u>Problem</u> 14. Compute $H^*(MSTop;Z_p)$ as a module over the Steenrod algebra A; then compute π_*MSTop by use of the Adams spectral sequence; find representative manifolds for the resulting cobordism classes.

See Peterson [12] for a possible approach to the first step. The following easier problem is also still open.

Problem 15. Compute $\pi_* MSF$ explicitly; use the Levitt exact sequence to read off the oriented Poincaré duality cobordism groups and find representative Poincaré complexes for the resulting cobordism classes.

Since MSF splits as a product of Eilenberg-MacLane spectra (see Peterson [12]), the first step (on the additive level) requires only a counting argument from the known structure of $H^*(BSF)$.

The previous problems deal primarily with the cohomology of spectra MG for stable bundle theories G. Since the MG are E_∞ ring spectra, their zeroth spaces M_0G are E_∞ ring spaces and their unit spaces $FMG \subset M_0G$ are infinite loop spaces. The infinite loop map $Be : BF \longrightarrow BFMG$ is the universal obstruction to the MG-orientability of stable spherical fibrations [9]. Even when MG is just a product of Eilenberg-MacLane spectra, the spectrum determined by FMG may well be complicated. The following would be a first step towards understanding these spectra. Define an AR-Hopf bialgebra (with χ) to be an A-coalgebra together with two structures of R-algebra (and a conjugation for the additive structure) subject to all requisite commutation formulas between the various pieces of structure [10, 11]. (The less appropriate term "Hopf ring" has been used by other authors.)

Problem 16. Compute $H_*(M_0G;Z_p)$ as an AR-Hopf bialgebra; then compute $H_*(BFMG;Z_p)$.

When $G = \{e\}$, MG is the sphere spectrum and $H_*(M_0G;Z_p)$ is the free AR-Hopf bialgebra with χ generated by H_*S^0 because the "mixed Cartan formula" and "mixed Adem relations" completely determine the multiplicative homology operations in terms of the additive homology operations [11]. Similarly, the free AR-Hopf bialgebra (without χ) is realized by $H_*(\coprod B\Sigma_n;Z_p)$.

Ravenel and Wilson [14] have computed $H_*(M_0U;Z_p)$ as a Hopf bialgebra (without homology operations).

While the theory of homology operations on E_∞ ring spaces is well understood, there should also be a related theory of homotopy operations.

Problem 17. Analyze the homotopy operations implicit in the definition of E_∞ ring spaces.

Kahn's $\cup_i$-products on the stable stems are consequences of the E_∞ ring structure on QS^0, but their definition uses only a very small part of the total structure available. I suspect that this problem is intimately related to the Arf invariant question in the (2^s-2)-stems.

The notion of E_∞ ring space is clearly essential to infinite loop space theory. There are those who feel that the abstract theory cannot be regarded as complete until the structures used are shown to be homotopy invariant.

Problem 18. Is a space of the homotopy type of an E_∞

ring space again an E_∞ ring space?

University of Chicago

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THE STEENROD ALGEBRA AND ITS DUAL FOR CONNECTIVE K-THEORY (*)

R. James Milgram

A basic spectrum of modern algebraic topology is the spectrum for connective real K-theory, b_0. Its cohomological structure was described by Adams [1] and Stong [10] at the prime 2 and W. Singer [9] at odd primes. In this note we will only be concerned with 2 where

$$H^*(b_0, Z/(2)) \cong \mathcal{A}(2)/\mathcal{A}(2)\left\{Sq^1, Sq^2\right\},$$

and $\mathcal{A}(2)$ is the mod (2) Steenrod algebra.

Mahowald in [6], [7] and Don Anderson (unpublished) initiated the study of the Steenrod algebra for connective K-theory $\mathcal{A}(b_0) = \left\{b_0, b_0\right\}_*$ and $\pi_*(b_0 \wedge b_0)$. This latter is useful (indeed essential) for constructing and studying the b_0 generalized Adams spectral sequence [3]. Acutally though what is most needed are not the homotopy groups but the homotopy type.

In general, if $\mathcal{M}$ is a spectrum, then the techniques of "classical" homological algebra are applicable to the study of the $\mathcal{M}$-generalized Adams spectral sequence only if

$$\mathcal{M} \wedge \mathcal{M} \simeq \bigvee_J \Sigma^J(\mathcal{M}).$$

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For example, the Eilenberg-MacLane spectra $K(Z/p,0)$, and the Brown-Peterson spectra have this property. It is easily checked though that $b_0 \wedge b_0$ cannot split in this way (already true by the 7 skeleton) so comparatively little attention has been given to the resulting Adams spectral sequence.

In this note we obtain the homotopy type of $b_0 \wedge b_0$ at the prime 2. It turns out to be associated not to b_0 alone but to b_0, b_{sp} and certain spaces obtained from b_0, b_{sp} by using $K(Z/2,0)$ resolutions of b_0, b_{sp}. (Here b_{sp} is the 3 connective cover of b_0, and again from [10]

$$H^*(b_{sp},Z/2) \cong \mathcal{A}(2)/\mathcal{A}(2)\{Sq^1,Sq^5\}.)$$

In terms of these spaces whose precise definitions will be given in 2 our main theorem is

<u>Theorem</u> A. $$b_0 \wedge b_0 \simeq \bigvee_J \Sigma^J K(Z/2,0) \vee b_0 \vee \Sigma^4 b_{sp} \vee \Sigma^8 b_0^{(3)}$$

$$\vee \ldots \vee \Sigma^{8k-4} b_{sp}^{2k-1-\alpha(k-1)} \vee \Sigma^{8k} b_0^{2k-\alpha(k)} \vee \ldots .$$

This decomposition is actually explicit. Let ψ^3 be the Adams operation $\psi^3 : B_0 \longrightarrow B_0, \psi^3|B_{0[8k,8k+1,\ldots)}$ and looped $8k$ times is $9^k\psi^3$; hence localizing at 2, so 3 is a unit, we have $\frac{1}{9^k}\psi^3|B_{0[8k,\ldots)}$ define an operation in the b_0 spectrum which we also denote ψ^3. Consider the polynomial

$$\varphi_n = (\psi^3-1)(\psi^3-9)\ldots(\psi^3-9^{n-1}).$$

We prove

<u>Theorem</u> B. <u>Let</u> $b_O^{(n)}$ <u>be the space which is universal for maps</u> $X \to b_O$ <u>which are trivial with respect to all higher</u> $Z/2$ <u>cohomology operations of order</u> $< n$; <u>then</u>

(i) φ_{2n} <u>admits a factorization</u>

$$b_O \xrightarrow{\theta_{2n}} \Sigma^{8n} b_O^{(2n-\alpha(n))} \xrightarrow{\lambda} \Sigma^{8n} b_O \xrightarrow{j} b_O$$

<u>where</u> λ <u>is the universal map and</u> j <u>the usual inclusion given by Bott periodicity</u>.

(ii) φ_{2n+1} <u>admits a factorization</u>

$$b_O \xrightarrow{\theta_{2n+1}} \Sigma^{8n+4} b_{sp}^{(2n-\alpha(n))} \xrightarrow{\lambda} \Sigma^{8n+4} b_{sp} \xrightarrow{\hat{j}} \Sigma^{8n} b_O \xrightarrow{j} b_O.$$

The θ_n's defined in Theorem B allow us to define maps

$$b_O \wedge b_O \longrightarrow \begin{cases} \Sigma^{8n} b_O^{(2n-\alpha(n))} \\ \Sigma^{8n+4} b_{sp}^{(2n-\alpha(n))} \end{cases}$$

using the ring structure of b_O and the b_O module structure of b_{sp}.

<u>Theorem</u> C. $b_O^{(n)}$ <u>and</u> $b_{sp}^{(n)}$ <u>are filtered modules over</u> b_O (<u>i</u>.<u>e</u>., <u>there are universal maps</u>

$$\mu : b_O^{(i)} \wedge b_O^{(n)} \longrightarrow b_O^{(n+i)}$$

$$\mu : b_O^{(i)} \wedge b_{sp}^{(n)} \longrightarrow b_{sp}^{(n+i)}$$

so the diagrams

$$\begin{array}{ccc} b_O^{(i)} \wedge b_{sp}^{(n)} & \longrightarrow & b_{sp}^{(n+i)} \\ \downarrow{\scriptstyle \lambda\wedge\lambda} & & \downarrow{\scriptstyle \lambda} \\ b_O \wedge b_{sp} & \longrightarrow & b_{sp} \\ \downarrow{\scriptstyle 1\wedge\hat{j}} & & \downarrow{\scriptstyle \hat{j}} \\ b_O \wedge b_O & \longrightarrow & b_O \end{array} \qquad \begin{array}{ccc} b_O^{(i)} \wedge b_O^{(n)} & \longrightarrow & b_O^{(n)} \\ \downarrow{\scriptstyle \lambda\wedge\lambda} & & \downarrow{\scriptstyle \lambda} \\ b_O^{(i)} \wedge b_O & \longrightarrow & b_O \end{array}$$

commute and there are minimal models for the $b_O^{(n)}, b_{sp}^{(n)}$ *so that the maps*

$$b_O \wedge b_O \longrightarrow \Sigma^{8n} b_O^{2n-\alpha(n)}$$

defined as the composites

$$b_O \wedge b_O \xrightarrow{1\wedge\theta_{2n}} b_O \wedge \Sigma^{8n} b_O^{2n-\alpha(n)} \xrightarrow{\mu} \Sigma^{8n} b_O^{2n-\alpha(n)}$$

$$b_O \wedge b_O \xrightarrow{1\wedge\theta_{2n+1}} b_O \wedge \Sigma^{8n+4} b_{sp}^{2n-\alpha(n)} \xrightarrow{\mu} \Sigma^{8n+4} b_{sp}^{2n-\alpha(n)}$$

together with (usual) maps into the Eilenberg-MacLane spectrum $K(Z/2,0)$ *defines the splitting in Theorem* A.

Remark. These theorems represent part of the foundation for ongoing work with Mahowald which will appear in due course. For now let me observe that the splitting above implies that while the techniques of classical homological algebra are not appropriate to the b_0-spectrum we can modify homological algebra by introducing valuated or filtered categories and filtered resolutions which implies filtered Ext groups, etc. With these modifications a beautiful theory seems to emerge which contains deep information about homotopy and is easily handled, using techniques developed in ordinary homological algebra for Abelian rings.

1. Some techniques from homological algebra.

We may write $H^*(b_0, Z/2) \cong \mathcal{A}(2)/\mathcal{A}(2)\bar{\mathcal{A}}_1$ where $\mathcal{A}_1$ is the subalgebra of $\mathcal{A}(2)$ generated by Sq^1 and Sq^2. Under such circumstances we can use change of rings techniques to reduce the study of the ordinary Adams spectral sequence for spaces $X \wedge b_0$ to $\mathrm{Ext}_{\mathcal{A}_1}(H^*(X), Z/2)$.

Lemma 1.1 (Change of rings). Let M have the form $N \otimes (\mathcal{A}(2)/\mathcal{A}(2)\bar{B})$ where N is an $\mathcal{A}(2)$ module and B is a sub-Hopf algebra of $\mathcal{A}(2)$. Suppose, moreover, there is an $\mathcal{A}(2)$ action on B so

$$\begin{array}{ccc} \mathcal{A}(2)\otimes B & \longrightarrow & B \\ \uparrow & & \uparrow s \\ B\otimes B & \longrightarrow & B \end{array}$$

commutes; then $\mathrm{Ext}_{\mathcal{A}(2)}(N\otimes\mathcal{A}(2)/\mathcal{A}(2)B, Z/2) \cong \mathrm{Ext}_B(N, Z/2)$ (*well known*).

$\mathcal{A}_1$ satisfies the hypothesis of 1.1 since we can construct a space X with $H^*(X,Z/2) \cong \mathcal{A}_1$ as a module over $\mathcal{A}_1$. Thus $\mathrm{Ext}_{\mathcal{A}(2)}(H^*(b_0), Z/2) \cong \mathrm{Ext}_{\mathcal{A}_1}(Z/2, Z/2)$ which has the form

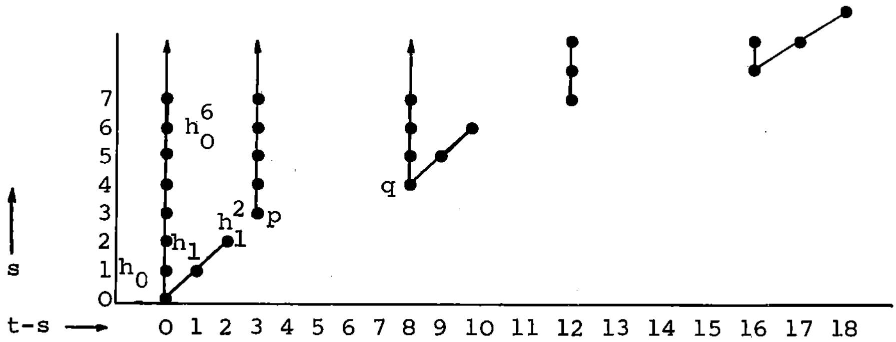


Figura 1.2

Next we recall the result of Wall [11].

Theorem 1.3. *Let* $Q(\in\mathcal{A}_1) = [Sq^1, Sq^2]$; *then* $Q^2 = 0$ *and if* M *is a connected locally finitely generated* $\mathcal{A}_1$*-module, then* M *is* $\mathcal{A}_1$ *free if and only if*

$$H_*(M, Sq^1) \cong H_*(M, Q) = 0.$$

Definition 1.4. *Two modules* M *and* N *over* $\mathcal{A}_1$ *are stably equivalent if there exist connected locally finitely generated projectives* L_1, L_2 *so that*

$$M \oplus L_1 \cong N \oplus L_2 .$$

As a corollary to 1.3 we can easily check that the only connected locally finitely generated projectives are actually free, so we can assume L_1, L_2 are free.

Corollary 1.5 (*Anderson-Brown-Peterson* [4]). M *is stably equivalent to* N *if and only if there is an* $\mathcal{A}_1$*-map* $f : M \to N$ *inducing isomorphisms of* Sq^1 *and* Q *homologies if* M, N *are connected and locally finite*.

The stable equivalence classes of finitely generated $\mathcal{A}_1$-modules form a ring $\mathcal{M}$ under direct sum and tensor product. The O-module acts as additive unit and the module $Z/2$ acts as the multiplicative unit. Adams and Priddy have extensively analyzed $\mathcal{M}$. However, we do not need their results here. What we need is the subring generated by $\overline{\mathcal{A}}_1$.

Here are minimal models for modules in the stable classes of $\underbrace{(\overline{\mathcal{A}}_1) \otimes \ldots \otimes \overline{\mathcal{A}}_1}_{i\text{-times}}$ for some i.

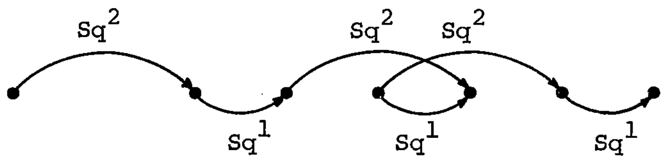
Definition 1.6. *Let* Y_i *be the* $\mathcal{A}_1$ *module with generators*

$$y_0 \; y_4 \; y_8 \cdots y_{4i}$$

and relations $Sq^{2,1,2}y_{4(j-1)} = Sq^1 y_{4j}$, $Sq^1 y_0 = 0$, $Sq^{2,1,2}y_{4i} = 0$.

Here is a picture of Y_1

(1.7)

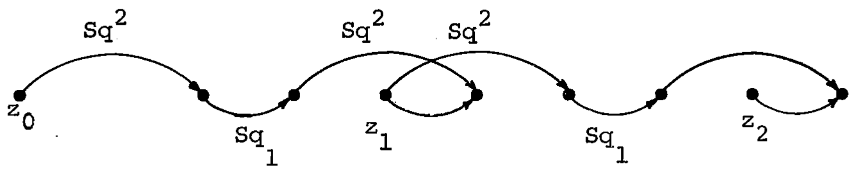


Again we have

Definition 1.8. *Let* Z_i *be the* $\mathcal{A}_1$ *module with generators* $z_0, z_4, \ldots, z_{4i}$ *and relations*

$$Sq^1(z_0) = 0, \; Sq^2(z_{4i}) = 0, \; Sq^{2,1,2}z_{4j} = Sq^1 z_{4(j+1)}$$

(1.9)



Definition 1.10. (a) *Let* S_i *be the* $\mathcal{A}_1$ *module with generators* $s_0, s_4, \ldots, s_{4i}$, h *and with relations*

$$Sq^1(s_{4(r+1)}) = Sq^{2,1,2}(s_{4r}), \; Sq^1(s_0) = 0$$

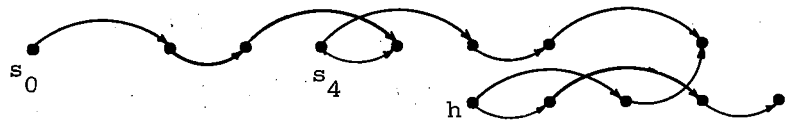
$$Sq^3(h) = Sq^{2,1,2}(s_{4i}).$$

(b) Let T_i be the $\mathcal{A}_1$-module with generators $t_0, \ldots, t_{4i}, k$ with

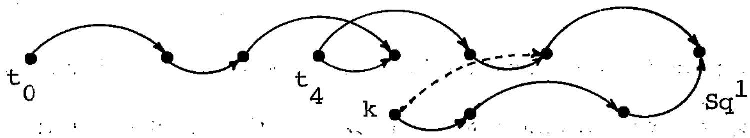
$$Sq^1(t_{4(r+1)}) = Sq^{2,1,2}(t_{4r}), \quad Sq^1(t_0) = 0$$

$$Sq^3 t_{4i} = Sq^2 k.$$

(1.10) S_1



(1.11) T_1



Lemma 1.12. All four of these modules are minimal in their stable classes and any $\mathcal{A}_1$-map $T_1 \to T_i$, $S_i \to S_i$, $Z_i \to Z_i$, $Y_i \to Y_i$ which is non-trivial in degree 0 is an isomorphism.

Proof. From the relations $Sq^{2,1,2}(y_{4(r-1)}) = Sq^1 y_{4r}$ it follows that the y_i, z_i, s_i, t_i all map non-trivially. This is the proof for Y_i, Z_i. For k in T_i we see that k goes either to k or $k + Sq^1(t_i)$ and in either case the map is an isomorphism, similarly for S_i.

Lemma 1.13. (a) The Sq^1 homology of any of these modules is generated by the bottom class,

(b)
$$H_*(Y_i, Q) = Z/2\{Sq^2 y_{4i}\}$$
$$H_*(Z_i, Q) = Z/2\{z_{4i}\}$$
$$H_*(S_i, Q) = Z/2\{h+Sq^2(s_{4i})\}$$
$$H_*(T_i, Q) = Z/2\{Sq^1 k+Sq^2 t_4\}$$

(obvious).

Theorem 1.14. (a) Y_{2i+1} is a minimal model for $\bar{\mathcal{A}}^{4i+3}$.

(b) Z_{2i} is a minimal model for $\bar{\mathcal{A}}^{4i}$.

(c) T_{2i} is a minimal model for $\bar{\mathcal{A}}^{4i+1}$.

(d) S_{2i} is a minimal model for $\bar{\mathcal{A}}^{4i+2}$.

(The proof is a tedious but routine construction. We take, for example, the obvious map of free $\mathcal{A}_1$ modules onto Z_{2i} and verify the kernel is T_{2i}. Similarly, for T_{2i}, S_{2i}, and Y_{2i+1}. Then check the first three stages of an actual resolution of $Z/2$, note that the bar resolution has kernel $\bar{\mathcal{A}}^i$ at stage i and verify that the stable class of the kernel at stage i does not depend on the particular resolution.)

As an easy corollary we have

Corollary 1.15.
$$\{S_\ell\} = \{Y_0 \otimes Y_\ell\}$$
$$\{Y_\ell\} = \{Y_0 \otimes Z_\ell\}$$
$$\{T_{\ell+1}\} = \{Y_0 \otimes S_\ell\}$$
$$\{Z_{\ell+1}\} = \{Y_0 \otimes T_\ell\}$$

<u>and in general this set of stable modules is closed under tensor products</u>.

2. <u>The spaces associated to a resolution of</u> $Z/2$ <u>over</u> $\mathcal{A}_1$

The map $\partial_0 : \mathcal{C}_0 \longrightarrow Z/2$. $\mathcal{C}_0$ can be written as $\mathcal{A}_1(e_1) + \dots + \mathcal{A}_1(e_n)$ and associated to ∂_0 we choose a map

$$d_0 : b_0 \longrightarrow \bigvee_1^n \Sigma^{\dim(e_r)} K(Z/2,0)$$

defined by $d_0^* \iota(e_r) = \partial_0(e_r) I$. The cofiber of d_0 and the associated map

$$M(d_0) \xrightarrow{k} \Sigma b_0$$

provides the first filtration space $b_0^{(1)}$ of b_0. Next, associated to $\partial_1 : \mathcal{A}_1(f_1) \oplus \dots \oplus \mathcal{A}_1(f_m) \longrightarrow \Sigma \mathcal{A}_1(e_i)$ there is a map

$$d_1 : \bigvee_1^n \Sigma^{\dim(e_r)} K(Z/2,0) \longrightarrow \vee \Sigma^{\dim f_j} K(Z/2,0)$$

defined by

$$(2.1) \qquad d_1^*(\iota_j) = \Sigma < \partial_1(f_j),\ e_k > \iota_k .$$

Since $\partial_0 \partial_1 = 0$, $d_1 d_0 \simeq 0$ and there is a lifting $\ell : b_0^{(1)} \longrightarrow \vee \Sigma^{\dim f_j} K(Z/2,0)$. The cofiber of ℓ is $b_0^{(2)}$ and the composite

(2.2) $$b_0^{(2)} \xrightarrow{k} \Sigma b_0^{(1)} \xrightarrow{\Sigma k} \Sigma^2 b_0$$

gives the second filtration of b_0. We now iterate the procedure obtaining the sequence

(2.3) $$\longrightarrow \Sigma^{-i} b_0^{(i)} \longrightarrow \Sigma^{-i+1} b_0^{(i-1)} \longrightarrow \cdots \longrightarrow b_0$$

associated to our resolution of $Z/2$ over $\mathcal{A}_1$.

Observe that the cohomology maps (mod 2) in (2.3) are all trivial and the cofiber at each stage is a wedge of suspensions of $K(Z/2,0)$. Hence (2.3) represents an Adams resolutions of b_0.

Theorem 2.4. *Let the resolution of* $Z/2$ *over* $\mathcal{A}_1$ *be minimal; then*

$$H^*(b_0^{(4i+3)}) \cong \mathcal{A}(2)/\mathcal{A}(2)\mathcal{A}_1 \otimes Y_{2i+1}$$

$$H^*(b_0^{(4i)}) \cong \mathcal{A}(2)/\mathcal{A}(2)\mathcal{A}_1 \otimes Z_{2i}$$

as modules over $\mathcal{A}_1$

Indeed, it is easily seen by induction that, for example, as a module over $\mathcal{A}(2)$, $H^*(b_0^{(4i)})$ is given by

$$\mathcal{A}(2)(e_0) \oplus \mathcal{A}(2)(e_4) \oplus \cdots \oplus \mathcal{A}(2)(e_{8i})$$

subject to relations $Sq^1(e_0) = 0$, $Sq^{2,1,2}(2_{4r}) = Sq^1(e_{4(r+1)})$, and $Sq^2(e_{8i}) = 0$ and similarly for $H^*(b_0^{(4i-1)})$.

Corollary 2.5. For the minimal resolution

$$\operatorname{Ext}^{s}_{\mathcal{A}(2)} H^{*}(b_0^{j}, Z/2) \cong \operatorname{Ext}^{s+j}_{\mathcal{A}_1}(Z/2, Z/2).$$

Thus, for example, in the case of $b_0^{(4)}$ the Ext groups have the form

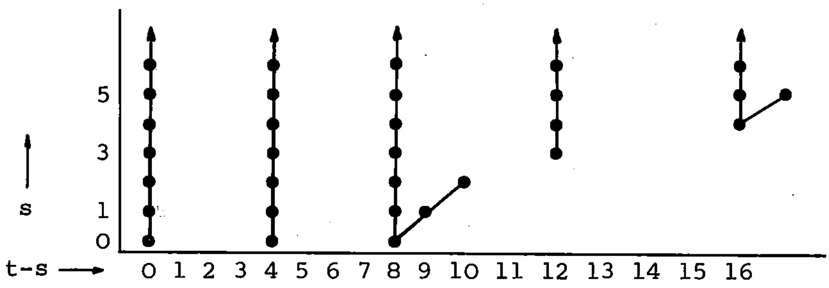


Figure 2.6

The Adams resolution (2.3) has certain universal properties. Thus, for example, by obstruction theory as in [8, Chapter 2] from the map we obtain the lifted maps

$$\mu_{ij} : b_0^{(i)} \wedge b_0^{(j)} \longrightarrow b_0^{(i+j)}$$

Moreover, for any other Adams resolution of b_0 we obtain similar maps and maps

$$\lambda_i \bar{b}_0^{(i)} \longrightarrow b_0^{(i)}$$

so the diagrams

$$\begin{array}{ccc} \bar{b}_0^{(i)} \wedge \bar{b}_0^{(j)} & \longrightarrow & \bar{b}_0^{(i+j)} \\ \downarrow & & \downarrow \\ b_0^{(i)} \wedge b_0^{(j)} & \longrightarrow & b_0^{(i+j)} \end{array}$$

commute up to filtration degree -1 homotopies.

3. The Adams map ψ^3

In the introduction we described how the Adams operation ψ^3 is defined in b_0. Here we give some more explicit properties of certain polynomials in ψ^3.

Definition 3.1. The Postnikov degree k of a map $\varphi: X \to X$ is the biggest Postnikov connective cover of X through which φ factors:

$$\begin{array}{ccc} & & X_k \\ & \overset{\tilde{\varphi}}{\nearrow} & \downarrow (j) \\ X & \xrightarrow[\varphi]{} & X \end{array}$$

Definition 3.2. The Adams degree s of φ is the largest Adams filtration of X through which φ factors:

$$\begin{array}{ccc} & & X^{(s)} \\ & \overset{\tilde{\varphi}}{\nearrow} & \downarrow \\ X & \xrightarrow[\varphi]{} & X \end{array}$$

Example 3.3. ψ^3-1 has Adams and Postnikov degrees 3 and 4, respectively, hence factors through a map θ:

$$\begin{array}{ccc} & & \Sigma^4 b_{sp} \\ & \overset{\theta}{\nearrow} & \downarrow i \\ b_O & \xrightarrow[\psi^3-1]{} & b_O \end{array}$$

Lemma 3.4. $\theta^*(I_{sp}) = Sq^4(I)$.

Proof. $(\psi^3-1)(q) = 8q$. Hence in dimension 4, $(\psi^3-1)^*$ is multiplication by 8. On the other hand, the cohomology map of a torsion free class to the i^{th} stage of any Adams resolution takes the generator to 2^i times the corresponding generator in X^i. Thus i^* is already multiplication by 8 in dimension 4 so (3.4) follows.

Next consider the operation φ_n defined in the introduction.

Lemma 3.5. φ_{n*} *is* 0 *in homotopy in dimensions* $<4n$ *and in dimensions* $\geq 4n$ *is multiplication by* λ_i *with each* λ_i *divisible by* $2^{4n-\alpha(n)}$ *and* $\lambda_{4n} = 2^{4n-\alpha(n)}$ (odd).

In particular, this shows that the Adams filtration of $\varphi_n \leq 4n-\alpha(n)$ and the Postnikov filtration $\leq 4n$.

Proof. The generator in dimension $4j$ in homotopy g_j is mapped by ψ^3 to $9^j g_j$, hence

$$\varphi_n(g_j) = (9^j-1)\dots(9^j-9^{n-1})g_j$$

$$= (9^j-1)(9^{j-1}-1)\dots(9^{j-n+1}-1)9^r(g_j)$$

which equals 0 if $j \leq n-1$. Also on the torsion ψ^3 is the identity so φ_n (torsion) $\equiv 0$. Now, it is well known [2] that $(9^i-1) = 2^{3+\nu_2(i)}$ (odd) where $\nu_2(i)$ is the exponent of 2 in the prime factorization of i. Thus the exponent of 2 in dimension $4i$ is $(\nu_2(i)+3) + (\nu_2(i-1)+3) + \dots + (\nu_2(i-n+1)+3)$ but $\nu_2(i) + \dots + \nu_2(i-n+1) = \nu_2(i!/(i-n)!)$. Now $r! = 2^{r-\alpha(r)}$ (odd) so

$$\nu_2(i!/(i-n)!) = n-\alpha(i)+\alpha(i-n)$$

which takes its minimum when $i = n$ since $\alpha(n)+\alpha(m) \geq \alpha(n+m)$ for all positive n, m.

Now Theorem B amounts to proving that these bounds are best possible. Special circumstances gave the result for ψ^3-1 since the groups in which obstructions could appear were all zero, but to prove the general result we cannot hope to argue one stage at a time. Instead it will depend on making extensive calculations with Adams spectral sequences.

4. Mahowald's theorem and the homotopy type of $b_0 \wedge b_0$

In this section we recall Mahowald's theorem [7] giving the structure of $H^*(b_0, Z/2)$ as a module over $\mathcal{A}_1$ and using it and Theorem B gives the proof of Theorems A and C.

Mahowald's result depends on

Theorem 4.1 (Anderson, Brown, Peterson [4]).

$$H^*(\mathcal{A}(2)/\mathcal{A}(2)\bar{\mathcal{A}}_1, Sq^1) \cong P(\xi_1^4)$$

$$H^*(\mathcal{A}(2)/\mathcal{A}(2)\bar{\mathcal{A}}_1, Q) \cong E(\xi_2^2, \xi_3^2, \ldots, \xi_n^2 \ldots).$$

Proof. Let $\mathcal{A}(2)^*$ be given as $P(\xi_1, \ldots, \xi_n \ldots)$ with $\psi(\xi_i) = \Sigma \xi_j \otimes \xi_{n-j}^{2^{n-j}}$. Then the Q homology can be calculated by dualizing $\langle \bar{Q}, \xi^I \rangle = \Sigma \langle \bar{Q}, \xi_1^I \rangle \otimes \xi_2^I$ where $\psi \xi^I = \Sigma \xi_1^I \otimes \xi_2^I$. This is a derivation and on generators $\bar{Q}(\xi_i) = \xi_{i-2}$, $i > 2$, $\bar{Q}(\xi_2) = 1$. But as a coalgebra over $\mathcal{A}(2)^*$, $H_*(b_0, Z/2) = P(\xi_1^4, \xi_2^2, \xi_3, \ldots)$. The second statement now follows. Similarly for the first $\overline{Sq}^1(\xi_i) = \xi_{i-1}^2$, $i > 1$, $\overline{Sq}^1(\xi_1) = 1$ and this again gives a derivation so (3.1) follows.

Next we have

Theorem 4.2 (Mahowald). There exist $\mathcal{A}_1$ modules M_{2^i}, $i \geq 2$, and surjections $\varphi_i : \mathcal{A}(2)/\mathcal{A}(2)\bar{\mathcal{A}}_1 \longrightarrow \Sigma^{2^i} M_{2^i}$ satisfying

(1) $M_{2^i} \cong Y_{2^{i-2}-1}$

(2) $H_*(\mathcal{A}(2)/\mathcal{A}(2)\bar{\mathcal{A}}_1, Sq^1) \xrightarrow{\varphi_*} H_*(M_{2^i}, Sq^1)$.

$$H_*(\mathscr{A}(2)/\mathscr{A}(2)\bar{\mathscr{A}}_1, \mathbb{Q}) \xrightarrow{\varphi_*} H_*(M_{2^i}, \mathbb{Q})$$

<u>are surjections with</u> $\varphi^*(g_{Sq^1}) = \xi_1^{2^i}, \varphi^*(g_{\mathbb{Q}}) = \xi_i^2$.

<u>Proof</u>. Again we work with the dual algebra. Consider the sequence

$$\xi_{i+1}, \xi_i^2, \xi_i\xi_{i-1}^2, \xi_{i-1}^4, \xi_i\xi_{i-2}^4, \xi_{i-1}^2\xi_{i-2}^4, \ldots, \xi_{i-r}\xi_{i-r-2}^{4j}\xi_{i-r-1}^{2s},$$
$$\xi_{i-r-2}^{4j}\xi_{i-r-1}^{2s+2}, \xi_{i-r}\xi_{i-r-2}^{4(j+1)}\xi_{i-r-1}^{2s-2}, \xi_{i-r-2}^{4(j+1)}\xi_{i-r-1}^{2s}, \ldots, \xi_{i-r-2}^{2^{r+3}} \ldots$$

terminating in

$$\xi_3\xi_2^2\xi_1^{2^i-8}, \xi_2^4\xi_1^{2^i-8}, \xi_3\xi_1^{2^i-4}, \xi_2^2\xi_1^{2^i-4}, 0, \xi_1^{2^i}.$$

The sequence is closed under $\overline{Sq}^1$. We show it also closed under $\overline{Sq}^2$. Note $\overline{Sq}^2(ab) = (\overline{Sq}^2a)b + a(\overline{Sq}^1b) + (\overline{Sq}^1a)(\overline{Sq}^1b)$; hence $\overline{Sq}^2(\xi_i) = 0$, $\overline{Sq}^2(\xi_i^{2s}) = s(\xi_{i-1}^4\xi_i^{2s-2})$, and from these two formulas the result follows.

Now (4.2) may be extended. Consider, for example, the composite $\varphi_{ij} : \mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1 \xrightarrow{\psi^*} \mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1 \otimes \mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1 \longrightarrow M_{2^i} \otimes M_{2^j}$. Clearly, $\varphi_{ij}^*(g\otimes g_{Sq^1}) = \{\xi_1^{2^i}\cdot\xi_1^{2^j}\}$, $\varphi_{ij}^j(g\otimes g_{\mathbb{Q}}) = \{\xi_i^2\xi_j^2\}$. More generally, on iterating we obtain $\mathscr{A}_1$-maps $\varphi_{i_1,i_2,\ldots,i_r} \mathscr{A}/\mathscr{A}\bar{\mathscr{A}}_1 \longrightarrow M_{2^{i_1}} \otimes \cdots \otimes M_{2^{i_1}}$ with

$$\varphi^*_{i_1\ldots i_r}(g\otimes\cdots\otimes g_{Sq^1}) = \left\{\xi^{2^{i_1}}_{i_1}\cdots\xi^{2^{i_r}}_{1}\right\}$$

$$\varphi^*_{i_1\ldots i_r}(g\otimes\cdots\otimes g_Q) = \left\{\xi^2_{i_1}\cdots\xi^2_{i_r}\right\}.$$

Of course, we may substitute for these tensor products their minimal stable $_1$-module representatives, and we assume that done. Consequently we have

Theorem 4.3 (Mahowald). *There is a free $\mathcal{A}_1$-module F and an isomorphism of $\mathcal{A}_1$-modules*

$$F\oplus\sum_{1<i_1<\cdots<i_r}\Sigma^{2^{(i_1+\cdots+i_r)}}\left\{Y_{2^{i_1-2}-1}\otimes\cdots\otimes Y_{2^{i_r-2}-1}\right\}$$

$$\cong\mathcal{A}(2)/\mathcal{A}(2)\bar{\mathcal{A}}_1.$$

Using (4.3) together with Theorem B we prove Theorems A and C as follows. Note that in the minimal resolution of $Z/2$ the corresponding space $b_0^{(n)}$ have $\mathcal{A}_1$-module structure $\{\bar{\mathcal{A}}^n\}\otimes\mathcal{A}/\mathcal{A}\bar{\mathcal{A}}_1$ where $\bar{\mathcal{A}}^n$ is represented by its minimal model. Moreover, an easy calculation shows that

$$\left\{\begin{aligned}&\theta_1 : b_0\longrightarrow\Sigma^4 b_{sp}\\ &\theta_{2^{i-2}} : b_0\longrightarrow\Sigma^{2^i} b_0^{(2^{i-2}-1)}\end{aligned}\right\}\ \text{satisfies}\ \theta^*_{2^{i-2}}(\bar{g}_{Sq^1}) = (\bar{\xi}_1^{2^i}),$$

since this latter class is the restriction of the integral generator and the calculation in (3.5) shows $\theta^*(\iota_{2^i}) = (\text{odd})$ (integral generator) But from this an easy calculation shows $\theta_{2^{i-2}_*}(g_{\mathbb{Q}}) = (\xi_i^2)$. Next we construct maps

$$\theta_{2n} : b_O \longrightarrow \Sigma^{2n} b_O^{4n-\alpha(n)}$$

and similarly the θ_{2n+1}. For example, to construct θ_6 use the composite

$$(4.4) \qquad b_O \xrightarrow{\theta_4} \Sigma^{10}(b_O^7) \xrightarrow{\hat{\theta}_2} \Sigma^{24} b_O^{(10)}$$

where $\hat{\theta}_2$ is the lifting in the Adams filtration of θ_2 through seven stages. Next we construct out of these maps the desired map $b_O \wedge b_O \longrightarrow \bigvee \Sigma^{8n} b_O^{4n-\alpha(n)} \vee \Sigma^{8n+4} b_{sp}^{4n-\alpha(n)}$ using the composites

$$b_O \wedge b_O \xrightarrow{\theta_{2n} \wedge 1} \Sigma^{8n} b_O^{4n-\alpha(n)} \wedge b_O \longrightarrow \Sigma^{8n} b_O^{4n-\alpha(n)}$$

Then the calculations above show μ^* for Sq^1 and $\mathbb{Q}$ homology are both isomorphisms. Also by minimality μ^* is an injection in cohomology with $\mathcal{A}(2)$-free cokernel. We extend now to a homotopy equivalence by simply mapping into $\Sigma^n(K(Z/2,0))$'s 1 for each free generator in the cokernel.

5. The proof of Theorem B

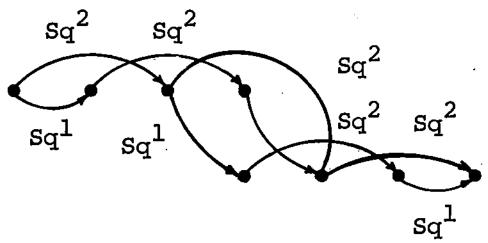
We begin by calculating $\mathrm{Ext}^{s^*}_{\mathcal{A}(2)}(Z/2, H^*(Db_O \wedge b_O))$ for

$s > 0$. Here Db_0 is the Spanier-Whitehead dual of b_0.

Lemma 5.1. *If as a module over* $\mathcal{A}_1$, $H^*(X) = M_1 \oplus M_2$, *then as a module over* $\mathcal{A}_1$, $H^*(D(X)) = D(M_1) \oplus D(M_2)$. *Moreover, the dual of* $\mathcal{A}_1$ *is again* $\mathcal{A}_1$.

Proof. As the splitting is clear we need only show the second statement. We write $\mathcal{A}_1$

(5.2)



This picture is symmetric and that is the result.

Lemma 5.3. (a) *The dual of* Y_i *has generators* $\theta_0, w_1, \ldots, w_i$ *with relations*

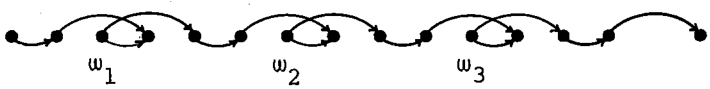
$$Sq^{2,1}\theta_1 = Sq^1\theta_2, \ldots, Sq^{2,1,2}w_{j-1} = Sq^1 w_j, \quad j < i.$$

(b) *The dual of* Z_i *has generators* $z_0, \ldots, z_{i-1}$ *with relations*

$$Sq^{2,1}z_1 = 0, \; Sq^{2,1,2}(z_{j-1}) = Sq^1 z_j, \quad j < i-1.$$

Proof. (a) The module described has picture

(5.4)



which is clearly the reverse of Y_i. (b) Is similar.

Lemma 5.5. *Let L_i be the module with generators $e_0, e_1, \ldots, e_i$ and relations*

$$Sq^{2,1,2}(e_j) = Sq^1(e_{j+1}), \quad j < i;$$

then there are exact sequences

$$\text{(a)} \quad 0 \longrightarrow \Sigma M \longrightarrow D(T_i) \longrightarrow L_i \longrightarrow 0,$$

$$\text{(b)} \quad 0 \longrightarrow N \longrightarrow D(S_i) \longrightarrow \Sigma L_i \longrightarrow 0,$$

where

$$M = \mathcal{A}_1/\mathcal{A}_1(Sq^1, Sq^{2,1,2}) = Y_0$$

$$N = \mathcal{A}_1/\mathcal{A}_1 Sq^3.$$

Proof. Turn diagrams (1.10), (1.11) on end; then N and M are the modules on the second lines in each case.

Remark 5.6. The extensions are given by $Sq^1(e_0) = \Sigma I$ in (a) and $Sq^1(\Sigma e_0) = Sq^2 I$ in (b).

Similarly we have

Lemma 5.7. (a) *There is an exact sequence*

$$0 \longrightarrow P \longrightarrow D(Y_i) \longrightarrow \Sigma^2 L_{i-1} \longrightarrow 0$$

where

$$P = \mathcal{A}_1/\mathcal{A}_1 Sq^2.$$

(b) <u>There is an exact sequence</u>

$$0 \longrightarrow \Sigma(Z/2) \longrightarrow D(Z_i) \longrightarrow L_{i-1} \longrightarrow 0$$

<u>and the extensions are given by</u> $Sq^1 e_0 = Sq^{2,1}\Sigma I$ <u>in</u> (a), $Sq^1(e_0) = \Sigma I$ <u>in</u> (b). (From (5.3).)

<u>Lemma</u> 5.8. $\mathrm{Ext}^{**}_{\mathcal{A}_1}(L_i, Z/2)$ <u>has generators</u> $e_0, \ldots, e_i$ <u>and relations</u>

$$qe_r = h_0^3 e_{r+1}$$

$$h_1 e_r = 0$$

<u>all</u> r. <u>In particular</u>, $\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(L_i, Z/2) = 0$ <u>for</u> $t-s > 4i$.

<u>Proof</u>. There is an exact sequence

$$\mathcal{A}_1/\mathcal{A}_1 Sq^1 \longrightarrow L_i \longrightarrow \Sigma^4 L_{i-1},$$

consequently a long exact sequence of Ext groups. Note using change of rings $\mathrm{Ext}_{\mathcal{A}_1}(\mathcal{A}_1/\mathcal{A}_1 Sq^1, Z/2) = \mathrm{Ext}_{E(Sq^1)}(Z/2, Z/2) = P(h_0)e_0$. Induction produces all of (5.8) except $qe_r = h_0^3 e_{r+1}$. But through a range $L_i \cong \{\bar{A}^{-4i}\}$ and the desired relation holds for $\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(\bar{A}^{-4i}, Z/2) \cong \mathrm{Ext}^{s+4i,t}_{\mathcal{A}_1}(Z/2, Z/2)$, $s > 0$.

Now from (5.1) we have

Theorem 5.9. As a module over $\mathcal{A}_1$

$$H^*(D(b_0),Z/2) \cong S^0 \bigvee_{\alpha(j)\equiv 0(4)} \Sigma^{(-8j+2\alpha(j)-1)} D(Z_{(j-\alpha(j)/2)})$$

$$\bigvee_{\alpha(j)\equiv 1(4)} \Sigma^{-8j+2\alpha(j)-1} D(Y_{(j-(\alpha(j)+1/2))})$$

$$\bigvee_{\alpha(j)\equiv 2(4)} \Sigma^{-8j+2\alpha(j)-2} D(S_{(j-\alpha(j)/2)}{}^{-1})$$

$$\bigvee_{\alpha(j)\equiv 3(4)} \Sigma^{-8j+2\alpha(j)-3} D(T_{(j-(\alpha(j)+1/2))})$$

$$\bigvee F.$$

Here $j = 1,2,3,\ldots,$ and F is a free $\mathcal{A}_1$-module. Also, $D(\ )$ has its lowest generator in dimension 0.

Thus, ignoring the free part the first few terms are

(5.10) $\quad (S^0) \vee \Sigma^{-7}D(Y_0) \vee \Sigma^{-15}D(Y_1) \vee \Sigma^{-22}D(S_1) \vee \Sigma^{-31}D(Y_3) \vee \cdots .$

Proof. Let $j = 2^{i_1} + \ldots + 2^{i_r}$, then stably

$$(5.11)\quad Y_{(2^{i_1}-1)} \otimes \cdots \otimes Y_{(2^{i_r}-1)} \cong \begin{cases} Z_{j-\frac{1}{2}r} & r \equiv 0(4) \\ Y_{j-\frac{1}{2}(r+1)} & r \equiv 1(4) \\ S_{j-\frac{1}{2}(r+2)} & r \equiv 2(4) \\ T_{j-\frac{1}{2}(r+1)} & r \equiv 3(4) \end{cases}$$

Now, the dimension of Z_k is $4k+1$, that of Y_k is $4k+3$, that of S_k is $4k+6$, that of T_k is $4k+5$. Thus, for example, the dual of

$$\Sigma^{4j} Z_{j-\frac{1}{2}\alpha(j)} \cong \Sigma^{-4j-4(j-\frac{1}{2}\alpha(j))-1} D(Z_{(j-\frac{1}{2}\alpha(j))}) =$$

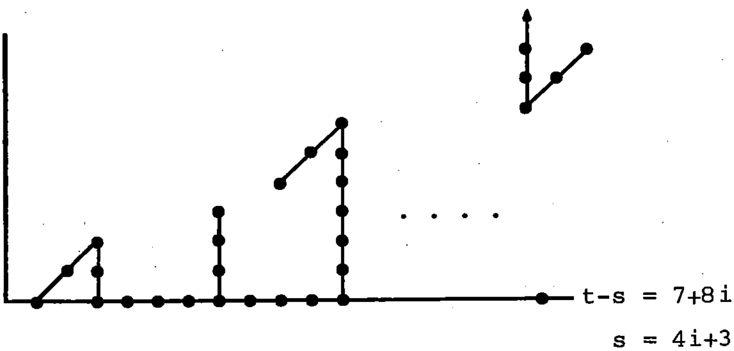
$$= \Sigma^{-8j+2\alpha(j)-1} D(Z_{j-\frac{1}{2}\alpha(j)})$$

and these are the first wedge summands in (4.11). The others are handled similarly.

We next need

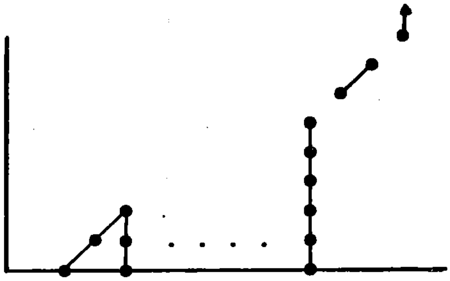
Lemma 5.12. (a) $\mathrm{Ext}_{\mathcal{A}_1}(D(Y_{2i+1}, Z/2))$ has the form

(5.13)



(b) $\mathrm{Ext}_{\mathcal{A}_1}(D(Y_{2i}), Z/2)$ <u>has the form</u>

(5.14)



<u>Proof</u>. Immediate from (5.7), (5.8).

Similar calculations can be made in the same way for $D(Z)$, $D(S)$, $D(T)$. We leave the details to the reader.

What we need to observe is that after suspending the correct number of times to put this into $DH^*(b_0)$ the first spike generator for the piece $\{D\Sigma^{8r}(X_r)\}$ occurs in $t-s = -8r$ and in s filtration $4r-\alpha(r)$. Similarly, the first spike for $D(\Sigma^{8r+4}(W_r))$ occurs in $t-s = -8r-4$ and s filtration $4r-\alpha(r)$, and since $f(r) = (4r-\alpha(r))$ is a monotone increasing function of r it is easily checked that these generating spikes and indeed all the spikes represent infinite cycles in the Adams spectral sequence.

To complete the proof we need

<u>Theorem</u> 5.15. <u>Let</u> L, M <u>be connected modules of locally-finite type over</u> $\mathcal{A}(2)$; <u>then</u>

$$\mathrm{Ext}_{\mathcal{A}(2)}(L,M) \cong \mathrm{Ext}_{\mathcal{A}(2)}(D(M)\otimes L, Z/2).$$

Proof. Let

$$\longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow L \longrightarrow 0$$

be an (2) free resolution of L which we may also suppose to be of locally finite type. Then Ext (L,M) is the homology of the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(C_0,M) \xrightarrow{\delta_1} \mathrm{Hom}_{\mathcal{A}}(C_1,M) \longrightarrow \cdots .$$

On the other hand, we have

Lemma 5.16. $\mathrm{Hom}_{\mathcal{A}}(L,M) \cong \mathrm{Hom}_{\mathcal{A}}(D(M)\otimes L, Z/2)$ where the isomorphism is explicitly given by the formula

$$h(f)(g\otimes \ell) = \langle g, f(\ell)\rangle .$$

(Here we regard D(M) as Hom(M,Z/2) with $\mathcal{A}$-module action given by $\alpha(g)(m) = g(\chi(\alpha)m)$ where χ is the canonical anti-automorphism.)

Proof. $h(f) \in \mathrm{Hom}_{Z/2}(D(M)\otimes L, Z/2)$ is certainly well defined. We must check that it is an $\mathcal{A}(2)$-map. Notice

$$h(f)\alpha(g\otimes \ell) = h(f)\Sigma\alpha_i(g)\otimes \alpha_i'(\ell)$$

$$= \Sigma\langle \alpha_i(g), f\alpha_i'(\ell)\rangle .$$

By assumption f is an $\mathcal{A}$-map so this last sum is

$$\Sigma\langle \alpha_i(g), \alpha_i' f(\ell)\rangle = \Sigma\langle g, \chi(\alpha_i)\alpha_i' f(\ell)\rangle$$

$$= \langle g, (\Sigma\chi(\alpha_i)\alpha_i') f(\ell)\rangle$$

but $\Sigma\chi(\alpha_i)\alpha_i' = 0$ for $\alpha \neq 1$ so this is $\epsilon(\alpha)\langle g, f(\ell)\rangle$ where ϵ is the augmentation. But this is by definition the action of $\mathcal{A}$ on $Z/2$.

We now define a map

(5.17) $$k : \mathrm{Hom}_{\mathcal{A}}(M \otimes L, Z/2) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M, \mathrm{Hom}_{Z/2}(L, Z/2))$$

by the rule

$$\langle (kr)(m), \ell\rangle = \langle r, m \otimes \ell\rangle.$$

We verify that k is indeed an $\mathcal{A}$-map

$$\langle [k(r)\alpha + \alpha k(r)](m), \ell\rangle =$$

$$= \langle r, (\alpha m) \otimes \ell\rangle + \langle r, (m \otimes (\chi(\alpha)\ell))\rangle =$$

$$= \langle\, , \alpha m \otimes \ell + m \otimes \chi(\alpha)\ell\rangle$$

and in general

$$\alpha(m) \otimes \ell + m \otimes \chi(\alpha)\ell$$

is decomposable over $\bar{\mathcal{A}} = \ker(\epsilon)$ [5]. Hence since $r(\theta(x)) = \epsilon(\theta)r(x) = 0$ for $\theta \in \bar{\mathcal{A}}$ it follows that $k(r)\alpha + \alpha k(r) \equiv 0$ all

r so k is indeed an $\mathcal{A}$-map.

The proof of (5.16) is now completed by (routinely) verifying that $h \cdot k = 1$ restricted to $D(M) \otimes L$ and checking that $kTh = 1$ on embedding

$$M \subset \mathrm{Hom}_{Z/2}(\mathrm{Hom}_{Z/2}(M, Z/2), Z/2)$$

The proof of (5.15) is now completed on noting that given an $\mathcal{A}$-map $f : L \longrightarrow L'$ there is the natural map

(5.18) $$\mathrm{Hom}_{\mathcal{A}}(L', M) \xrightarrow{D(f)} \mathrm{Hom}_{\mathcal{A}}(L, M)$$

and the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(L', M) & \xrightarrow{D(f)} & \mathrm{Hom}_{\mathcal{A}}(L, M) \\ \downarrow h \cong & & \downarrow h \\ \mathrm{Hom}_{\mathcal{A}}(D(M) \otimes L', Z/2 & \xrightarrow{D(1 \otimes f)} & \mathrm{Hom}_{\mathcal{A}}(D(M) \otimes L, Z/2) \end{array}$$

commutes. On applying this to the chain complex defining $\mathrm{Ext}_{\mathcal{A}}(L, M)$ we see at once that it is isomorphic to the chain complex defining $\mathrm{Ext}_{\mathcal{A}}(D(M) \otimes L, Z/2)$ and the result follows.

The results on $\mathrm{Ext}_{\mathcal{A}(2)}(H^*(b_0), H^*(b_0))$ obtained above (5.9-5.14) give in particular

5.20. $\mathrm{Ext}^{**} = 0$ if $t-s = -1$, and for $t-s = 0$, $\mathrm{Ext}^{s,s}$ is a free module over $P(h_0)$ having generators $e_0, e_1, e_2, \ldots$

where the s-degree of e_i is $4i-\alpha(i)$.

We now complete the proof of Theorem B. It has already been observed that the spikes are infinite cycles. Now, the φ_n are all linearly independent (since $\varphi_{i*} = 0$ in dimensions less than $4i$ but not in $4i$). φ_1 has s-degree 3 and $\varphi_2 = a + b\varphi_1 + \tau$ where by (5.20) τ must have s-degree ≥ 7. Checking on homotopy in dimension zero, $\tau_*(I) + aI \equiv 0$. But $\tau_*(I)$ must be divisible by 2^7, hence $a = 2^7 a'$. Again in dimension 4

$$0 = 2^7 a' + 8b + \tau_*(q).$$

Here since q has filtration 3, $\tau_*(q) = 2^4 b'(q)$ and we see that b is divisible by 2^4. Hence φ_2 actually has filtration 7 and we may repeat the argument with φ_3, etc. This shows the φ_i lift to the correct Adams filtrations. We must still show their Postnikov filtrations are correct.

For example, we check on φ_2. Note that we can choose $e_0, e_1, \ldots$ so that $e_i = p(f_i)$ for $i > 1$ in Ext. Now multiplication by p corresponds to composition with the periodicity map in homotopy. Hence, having seen that e_0, e_1 are represented by 1, φ_1 we note that τ in the argument above can be chosen so $\tau = p(f)$. With this modification $\tau_*(I) = \tau_*(q) = 0$ so a and b in the argument above also $= 0$ and $\varphi_2 = p(f)$. Next we show φ_3 represented by this composite

$$b_O \xrightarrow{\varphi_1} b_{sp} \xrightarrow{\hat{\varphi}_2} \Sigma^8 b_{sp}^{(3)}$$

and the remaining generators $e_4, e_5, \ldots$ can be assumed to lie in the image of p^2, etc.

Theorem B follows.

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ON NOVIKOV'S EXT^1 MODULO AN INVARIANT PRIME IDEAL

Haynes R. Miller and W. Stephen Wilson(*)

This note is a statement of some results on $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*/I_n)$ which we talked about informally at the summer 1974 homotopy-theory conference at Northwestern University. Proofs will appear elsewhere. For details on the Brown-Peterson spectrum BP and on BP^*BP and BP_*BP, we refer the reader to [2, 11, 1].

We shall use the generators v_i of Hazewinkel [3], so that

$$BP_* \simeq Z_{(p)}[v_1, v_2, \ldots]$$

with $|v_n| = 2p^n - 2$, and $BP_* \simeq BP^{-*}$. The ideals

$$I_n = (p, v_1, \ldots, v_{n-1}) \qquad 0 \le n \le \infty$$

are the prime ideals of BP_* invariant under the coaction of BP_*BP (or the action of BP^*BP); see [5, 9, 4]. We point out that

$$\mathrm{Ext}^{**}_{BP^*BP}(BP^*, BP^*/I_n) \simeq \mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP_*/I_n),$$

(*) Both authors were partially supported by the NSF at the time of this research.

and henceforth denote this algebra by

$$\mathrm{Ext}^{**}(BP_*, BP_*/I_n).$$

Multiplication by v_n on BP_*/I_n is a BP_*BP-comodule map. In fact, we have

<u>Theorem</u> (Landweber [14]; see also Johnson-Wilson [4]). For $0 < n < \infty$,

$$\mathrm{Ext}^{0,*}(BP_*, BP_*/I_n) \simeq \mathbb{F}_p[v_n].$$

Thus $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ splits up as an $\mathbb{F}_p[v_n]$-module into a direct sum of v_n-torsion and v_n-torsion-free submodules. For p odd, we describe the v_n-torsion summand completely, and exhibit all but one generator for the v_n-torsion-free summand.

The short exact sequence of comodules (where $v_0 = p$)

$$0 \longrightarrow BP_*/I_n \xrightarrow{v_n} BP_*/I_n \longrightarrow BP_*/I_{n+1} \longrightarrow 0$$

gives rise to the "Bockstein" exact couple

$$\begin{array}{ccc} \mathrm{Ext}^{**}(BP_*, BP_*/I_n) & \xrightarrow{v_n} & \mathrm{Ext}^{**}(BP_*, BP_*/I_n) \\ {\scriptstyle \delta_n} \nwarrow & & \swarrow {\scriptstyle \rho_n} \\ & \mathrm{Ext}^{**}(BP_*, BP_*/I_{n+1}) & \end{array}$$

in which δ_n has bidegree $(1,2-2p^n)$.

Henceforth let p be an odd prime. Recall [1] that $BP_*BP \simeq BP_*[t_1,t_2,\ldots]$, $|t_n| = 2p^n-2$. In the cobar construction for BP_*BP ([7]) with coefficients in BP_*/I_n, $n>0$, $[t_1^{p^i}]$ is cycle representing a nonzero class

$$h_i \in \mathrm{Ext}^{1,p^i q}(BP_*,BP_*/I_n),$$

$q = 2p-2$. Clearly h_i is taken to h_i by the reduction ρ_n.

Note that

$$\mathrm{Ext}^{**}(BP_*,BP_*/I_\infty) \simeq \mathrm{Ext}^{**}_{P_*}(\mathbb{F}_p,\mathbb{F}_p)$$

where P_* is the Hopf algebra of Steenrod reduced powers. Thus $\mathrm{Ext}^{1,*}(BP_*,BP_*/I_\infty)$ is additively generated by $\{h_i : i \geq 0\}$ [6]. (At the other extreme recall that Novikov [10] has computed $\mathrm{Ext}^{1,*}(BP_*,BP_*/I_0)$.)

<u>Theorem</u> A. Let p be odd and $0<n<\infty$. All relations in the $\mathbb{F}_p[v_n]$-submodule of $\mathrm{Ext}^{1,*}(BP_*,BP_*/I_n)$ generated by $\{h_i : i \geq 0\}$ are consequences of

$$v_n^{p^s} h_{s+n} = v_n^{p^{s+1}} h_s \qquad s \geq 0.$$

<u>Corollary</u> A'. The h_i for $0 \leq i < n$ generate distinct free $\mathbb{F}_p[v_n]$-module summands.

The next theorem describes the v_n-torsion submodule of $\mathrm{Ext}^{1,*}(BP_*,BP_*/I_n)$, $0<n<\infty$. For $r>0$, write $r = ap^s$ with

$(a,p) = 1$, and if $s \neq 0$ write $s = kn + i + 1$ with $0 \leq i < n$.

Let

$$q(r) = q_n(r) = \begin{cases} p^s & \text{if } a = 1 \\ p^s + (p-1)\sum_{\ell=0}^{k-1} p^{\ell n+i} & \text{if } a \neq 1 \end{cases}$$

In particular, for $n = 1$ with $s > 1$ and $a \neq 1$, $q(ap^s) = p^s + p^{s-1} - 1$.

<u>Theorem</u> B. Let p be odd and $0 < n < \infty$. The v_n-torsion submodule of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ is a sum of cyclic $\mathbb{F}_p[v_n]$-modules on generators

$$c_n(r) \in \mathrm{Ext}^{1, 2r(p^{n+1}-1)-2q(r)(p^n-1)}(BP_*, BP_*/I_n)$$

satisfying, for a such that $(a,p) = 1$ and $a \neq 1$:

(i) $v_n^{q(r)} c_n(r) = 0$

$v_n^{q(r)-1} c_n(r) = \delta_n(v_{n+1}^r) \neq 0$

(ii) $h_{s+n} = c_n(p^s) + v_n^{p^s(p-1)} h_s \quad s \geq 0$

(iii) $\rho_n(c_n(p^s)) = h_{s+n}$

$\rho_n(c_n(ap^0)) = a v_{n+1}^{a-1} h_n$

$$\rho_n(c_n(ap^s)) = \begin{cases} 2a v_2^{ap^s - p^{s-1}} h_0 & \text{if } n = 1 \text{ and } s > 1. \\ a v_{n+1}^{ap^s - p^{s-1}} h_i & \text{otherwise.} \end{cases}$$

Most of our understanding of the v_n-torsion-free part of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ derives from the following theorem of Morava.

<u>Theorem</u> (Morava [8]). Let p be odd. The rank of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ over $\mathbb{F}_p[v_n]$ is 1 for $n = 1$, and $n+1$ for $1 < n < \infty$.

Corollary A' gives us all but one generator of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ mod v_n-torsion if $n > 1$. For the last generator we can only offer:

<u>Conjecture</u>. For p odd and $1 < n < \infty$, there is an element $w_n \in \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ generating a free $\mathbb{F}_p[v_n]$-module summand and reducing to

$$\rho_n(w_n) = v_{n+1}^{1+p+\cdots+p^{n-2}} h_{n-1}.$$

Our principal evidence for this conjecture is its truth for $n = 2$ and 3.

These results have applications in stable homotopy. It is immediate from Theorem B that $\delta_0\delta_1(v_2^t) \neq 0$ in $\mathrm{Ext}^{2,*}(BP_*, BP_*)$ for $t > 0$. This implies the theorem of L. Smith [12] that $\beta_t \neq 0$ in π_*^S for $t > 0$.

Recall [10] that the image of

$$\rho_0 : \mathrm{Ext}^{1,*}(BP_*, BP_*) \longrightarrow \mathrm{Ext}^{1,*}(BP_*, BP_*/(p))$$

is generated by $\left\{v_1^k h_0 : k \geq 0\right\}$. Since $\mathrm{Ext}^{2,*}(BP_*, BP_*)$ is p-torsion, the exact sequence

$$\mathrm{Ext}^{1,*}(BP_*,BP_*) \xrightarrow{\rho_0} \mathrm{Ext}^{1,*}(BP_*,BP_*/(p)) \xrightarrow{\delta_0} \mathrm{Ext}^{2,*}(BP_*,BP_*) \xrightarrow{p} \mathrm{Ext}^{2,*}(BP_*,BP_*)$$

allows us to compute the kernel of multiplication by p in $\mathrm{Ext}^{2,*}(BP_*,BP_*)$. This gives a complete list of cyclic $Z_{(p)}$-module summands, but no information on their orders. Using this list it is easy to see that $\delta_0\delta_1\delta_2(v_3) \neq 0$ in $\mathrm{Ext}^{3,*}(BP_*,BP_*)$. This implies the result of E. Thomas and R.S. Zahler [13] that $\gamma_1 \neq 0$ in π_*^S.

In a following note with D.C. Johnson and R.S. Zahler we describe this technique in more detail and use it to show the nontriviality of a sporadic but infinite collection of γ_t's.

Acknowledgement. Raph Zahler first noticed that $\delta_1(v_2^{ap^s})$ is divisible by $v_1^{p^s-1}$. This result helped stimulate our interest in Ext^1 and we would like to thank Raph for bringing it to our attention.

PRINCETON UNIVERSITY

PRINCETON UNIVERSITY AND THE INSTITUTE FOR ADVANCED STUDY

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AN ALGEBRAIC VERSION OF THE INCOMPRESSABILITY THEOREM OF S. WEINGRAM

John C. Moore

In his paper "On the incompressability of maps" ([14]), Weingram defined a map $f: X \longrightarrow Y$ to be incompressible if it is impossible to factor f up to homotopy through a finite complex. He then proved that if X is $\Omega(S^{2n+1})$, $n > 0$, Y is $K(\pi, 2n)$ where π is a cyclic group, and f is essential, then f is incompressible. An easy application of this result obtained for him the theorem of W. Browder to the effect that if G is an H-space whose singular homology thinks it is a finite complex, then the Hurewicz morphism $\pi_{2n}(G) \longrightarrow H_{2n}(G)$ is zero for $n > 0$. The object of this paper is to show how to prove similar results with appropriate coalgebras replacing spaces.

1. Notation and Conventions

Throughout this paper the ground ring R will be $\mathbb{Z}_{(p)}$, the integers localized at the prime p. Module will mean R-module, coalgebra will mean supplemented differential graded algebra over R.

If C is a coalgebra, $\Omega(C)$ will denote its loop algebra, i.e. the algebra obtained by applying the "cobar" construction to C. If A is an algebra, then B(A) will denote the classifying coalgebra of A, i.e. the "bar" construction applied to A. The notation is in accord with usual notation, e.g. [1], [2].

For $n > 0$, let S^n denote the exterior algebra with primitive elements having 1-free generator ι_n in dimension n.

Observe that $\Omega(S^{n+1})$ is a tensor algebra with 1-generator in degree n, taking this to be primitive, $\Omega(S^{n+1})$ becomes a Hopf algebra and there is natural imbedding of coalgebras $j_n : S^n \longrightarrow \Omega(S^{n+1})$.

2. Construction of an incompressible morphism

Our next objective is to construct an algebraic substitute for $K(\pi, 2n)$ where π is cyclic. To this end suppose that $n > 0$. Now S^{2n-1} admits a unique multiplication making it into a Hopf algebra. Since S^{2n-1} is an algebra $B(S^{2n-1})$ is defined, and it is the tensor coalgebra with primitive elements having 1-free generator $\sigma(\iota_{2n-1})$ the suspension of ι_{2n-1}. Since S^{2n-1} is a commutative algebra $B(S^{2n-1})$ is Hopf algebra. As an algebra it is the algebra with divided powers generated by $\sigma(\iota_{2n-1})$.

For $m \in \mathbb{Z}$, $m \geq 0$, there is a unique morphism of Hopf algebras $f_m : S^{2n-1} \longrightarrow S^{2n-1}$ such that $f_m(\iota_{2n-1}) = p^m \iota_{2n-1}$. The next step is to make f_m into the inclusion morphism of a fibration. Hence let $W(S^{2n-1})$ be the bundle space of the universal bundle with fibre S^{2n-1} and base $B(S^{2n-1})$. Thus $W(S^{2n-1})$ is a twisted tensor product, $S^{2n-1} \otimes_{\tau} B(S^{2n-1})$, where the twisting morphism τ is characterized by $\tau(\sigma(\iota_{2n-1})) = \iota_{2n-1}$. Now indeed $W(S^{2n-1})$ is a Hopf algebra having commutative multiplication and commutative comultiplication. Notice that there is a canonical imbedding of Hopf algebras $\iota_{2n-1} : S^{2n-1} \longrightarrow W(S^{2n-1})$. Let $\tilde{S}^{2n-1} = S^{2n-1} \otimes W(S^{2n-1})$, and let $\tilde{f}_m : S^{2n-1} \longrightarrow \tilde{S}^{2n-1}$ be the composite morphism

$$S^{2n-1} \xrightarrow{\Delta} S^{2n-1} \otimes S^{2n-1} \xrightarrow{f_m \otimes \bar{\iota}_{2n-1}} S^{2n-1} \otimes W(S^{2n-1}) = \tilde{S}^{2n-1}.$$

There is a commutative diagram of Hopf algebras

$$\begin{array}{ccc} & & \tilde{S}^{2n-1} \\ & \overset{\tilde{f}_m}{\nearrow} & \\ S^{2n-1} & & \Big\downarrow h \\ & \underset{f_m}{\searrow} & \\ & & S^{2n-1} \end{array}$$

where h is projection on the first factor. Now $\tilde{S}^{2n-1}$ considered as an S^{2n-1} module via $\tilde{f}_m$ is free. Let $A(2n-1,m)$ be the cokernel of $\tilde{f}_m$ in the category of abelian groups over the category of commutative coalgebras. Thus $A(2n-1,m) = R \otimes_{S^{2n-1}} \tilde{S}^{2n-1}$ is bicommutative or abelian Hopf algebra. The algebraicsubstitute for $K(R/p^m R,2n)$ is $B(A(2n-1,m))$.

An algebraic substitute for the fibration sequence

$$K(R,2n) \xrightarrow{p^m} K(R,2n) \longrightarrow K(R/p^m R,2n)$$

is also needed. This is essentially the sequence

$$B(S^{2n-1}) \xrightarrow{B(\tilde{f}_m)} B(\tilde{S}^{2n-1}) \longrightarrow B(A(2n-1,m)).$$

of abelian Hopf algebras.

(*) In order to avoid technical difficulties it will be assumed that in the rest of this paper all algebras and coalgebras have underlying R-modules which are flat and that they are chain equivalent over R to coproducts of elementary complexes.(*)

2.1 Definition. The morphisms of coalgebras $f_0, f_1 : C' \to C''$ are *homotopic* if there exists a coalgebra D and morphisms of coalgebras $\iota_0, \iota_1 : C' \to D$, $\pi : D \to C'$, and $F : D \to B\Omega C''$ such that

1) π is a chain equivalence,
2) $\pi\iota_0 = 1_{C'} = \pi\iota_1$, and
3) $\beta(C'')f_0 = F\iota_0$ and $\beta(C'')f_1 = F\iota_1$, where

$\beta(C'') : C'' \to B\Omega(C'')$ is the canonical chain equivalence.

The morphism $f : C' \to C''$ is *incompressible* if f cannot be factored up to homotopy through a coalgebra C such that for some integer N, $H_g(C) = 0$ for $g > N$.

2.2 Theorem. If $n, m \in \mathbb{Z}$, $n > 0$, $m > 0$, then any morphism of algebras $g : \Omega(S^{2n+1}) \longrightarrow B(A(2n-1,m))$ is a morphism of Hopf algebras, and if such a morphism has the property that $H_{2n}(g) \neq 0$ it is incompressible as a morphism of coalgebras.

Proof. Observe that $B(\tilde{S}^{2n-1})$ considered as a $B(S^{2n-1})$ module via $B(\tilde{f}_m)$ is free. Thus if C is the cokernel of $B(\tilde{f}_m)$ as morphism of abelian Hopf algebras, the sequence $B(S^{2n-1}) \longrightarrow B(\tilde{S}^{2n-1}) \longrightarrow C$ has a Serre spectral sequence such that one can repeat the calculation of Weingram ([4]) for the fibration $K(\mathbb{Z},2n) \longrightarrow K(\mathbb{Z},2n) \longrightarrow K(\mathbb{Z}/p^m\mathbb{Z},2n)$ The morphism $C \longrightarrow B(A(2n-1,m))$ is a homotopy equivalence of coalgebras and is surjective. Further since $\Omega(S^{2n+1})$ is a tensor algebra with one primitive generator, one verifies readily that any morphism g as above lifts to a morphism $\tilde{g} : \Omega(S^{2n+1}) \longrightarrow B(\tilde{S}^{2n-1})$ of abelian Hopf algebras. The proof of the theorem is now easily completed exactly by Weingram's method.

It remains to discuss applications of this result. No proofs will be given during this process.

3. Discussion of factoring the incompressible morphism

A homotopy commuted coalgebra C is a coalgebra C together with a morphism of algebras $\Omega(C) \longrightarrow \Omega(C) \otimes \Omega(C)$ which gives $\Omega(C)$ the structure of a Hopf algebra. Morphisms of homotopy commuted coalgebras are defined in the expected manner.

It will be shown elsewhere that if A is a connected Hopf algebra, then $B(A)$ is in a natural way a simply connected homotopy commuted coalgebra. This results in an adjoint relationship between the category of simply connected homotopy commuted coalgebras and the category of connected Hopf algebras given by the functors $\Omega(\ )$ and $B(\ )$. The techniques used are extensions of those used earlier ([1]).

If X is a differential graded R-module, with $X_0 = 0$, and X_q free, let $T'(X)$ denote the connected Hopf algebra whose underlying coalgebra is the tensor coalgebra of X, and whose multiplication is the shuffle multiplication. If A is a connected Hopf algebra there is a natural bijection between the morphisms of differential modules $Q(A) \longrightarrow X$ and the morphisms of Hopf algebras $A \longrightarrow T'(X)$. Thus for C a homotopy commuted coalgebra there is a natural bijection between the morphisms of differential modules of degree -1, $C \longrightarrow X$ and the morphisms of homotopy commuted coalgebras $C \quad BT'(X)$.

Let $S'(X)$ denote the standard commutative subcoalgebra of X for X as in the preceding paragraph. Now $S'(X)$ has a Hopf algebra structure such that it is a sub Hopf algebra of $T'(X)$. Indeed as an algebra $S'(X)$ is just the algebra with divided powers generated by X. If A is a connected Hopf alge bra with commutative diagonal, then any morphism $A \to T'(X)$ factors uniquely through $S'(X)$. Thus if C is a homotopy commuted coalgebra such that $\Omega(C)$ has a commutative diagonal, there is a natural bijection between the morphisms of differential modules of degree -1, $C \to X$ and the morphisms of homotopy commuted coalgebras $C \to BS'(X)$. Any commutative coalgebra C has a natural homotopy commuted coalgebra structure. Using this structure $\Omega(C)$ is primitively generated. Notice that $\Omega(S^{2n+1})$ has a commutative coalgebra structure and thus there is a bijection between the morphism $\Omega(S^{2n+1}) \to X$ of degree -1, and the morphisms of homotopy commuted coalgebras $\Omega(S^{2n+1}) \to BS'(X)$. Now if C is a homotopy commuted coalgebra, let $\Omega^A(C)$ denote the Hopf algebra obtained from $\Omega(C)$ dividing by the commutator ideal. Now any morphism of Hopf algebras $\Omega(C) \to T'(X)$ factors through $\Omega^A(C)$. Notice that $Q\Omega(C) = Q\Omega^A(C) = s^{-1}(J(C))$ where $J(C)$ is the augmentation coideal of C, and $s^{-1}(X)$ is the desuspension of X for any differential module X. If X is such that $X_q = 0$ for $q > 2n-1$, $q \equiv -1 \bmod 2n$, then any morphism of homotopy commuted coalgebras $\Omega S^{2n+1} \to BS'(S)$ is a morphism of Hopf algebras.

Suppose now that G is a simply connected Hopf algebra whose coalgebra is homotopy commuted. One verifies easily that if $G \to X$ is a morphism of degree -1, which annililates the square of the augmentation ideal, then $G \to BT'(X)$ is morphism of Hopf algebras. Further in this case if $\Omega B(G)$ is the Hopf algebra obtained from G as indicated earlier, then $\Omega B(G)$ has a homotopy commuted diagonal. It then follows that any differential morphism of degree -2 $B(G) \longrightarrow X$ induces a morphism of Hopf algebras with homotopy commuted diagonal $\Omega B(G) \longrightarrow BT'(X)$.

Suppose that X is such that $X_g = 0$ for $g \neq 2n-1, 2n$, X_{2n-1} is free with basis x, X_{2n} is free with basis y and $dy = p^m x$. Now $S'(X)$ is the Hopf algebra $A(2n-1,m)$ of the preceding paragraph and the morphism of Hopf algebras $S'(X) \longrightarrow T'(X)$ has a canonical retraction of abelian Hopf algebras $BT'(X) \longrightarrow BS'(X)$ of the natural morphism $BS'(X) \longrightarrow BT'(X)$. However for $m > 0$ this retraction is not a morphism of coalgebras which are homotopy commuted.

Suppose now that G is a Hopf algebra with homotopy commuted diagonal and that G is simply connected. Suppose further that $H_g(G) = 0$ for $g > N$ for some integer N. The preceding considerations combine to imply that if $\Omega S^{2n+1} \xrightarrow{f} G$ is either a morphism of homotopy commuted coalgebras or a morphism of algebras and $H_g(G) = 0$ for $0 < g \leq n$ then $H_{2n}(f) = 0$. Thus if N_0 is the first strictly positive integer such that $H_{N_0}(G) \neq 0$, then N_0 is odd and $H_g(G) = 0$ for q even and $q < 2N_0$.

PRINCETON UNIVERSITY

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DIEUDONNE MODULES FOR ABELIAN HOPF ALGEBRAS

Preliminary Report in Honor of

SAMUEL EILENBERG

Douglas C. Ravenel(*)

By Abelian Hopf algebra we mean graded connected biassociative strictly bi-commutative Hopf algebra of finite type over a perfect field k of characteristic p. Let $\underline{A}$ denote the category of such objects. $\underline{A}$ is known to be abelian ([12]) and our purpose here is to show that it is isomorphic to a certain category of modules. An analogous theorem for the nongraded case was proved long ago by Dieudonné, and the modules that he used have been studied extensively (see [1], Chapter V, and [4]). I am grateful to Bill Singer for first bringing this work to my attention and suggesting the problem of carrying it over to the graded case.

The ring D in question is a noncommutative power series over $W(k)$ (the Witt ring of k) in two variables F and V subject to the relations

$$FV = VF = p$$

$$Fw = w^{\sigma}F, \quad Vw^{\sigma} = wV$$

(*) Research partially supported by N.S.F.

for $w \in W(k)$, where w^{σ} denotes the action of the Frobenius automorphism of k lifted to $W(k)$.

In our case we will obtain modules over a commutative graded ring $E = W(k)[[F,V]]/(FV-p)$ where $\dim F = 1$, $\dim V = -1$. F will be seen to correspond to the Frobenius endomorphism of a Hopf algebra A which sends $x \in A$ to x^p, while V corresponds to the dual of F, commonly known as the Verschiebung.

The relation between abelian Hopf algebras and E-modules will be described in Theorem 3" below, which is our main result.

Our first result is a decomposition theorem.

<u>Definition</u>. Let n be an integer prime to p. An Abelian Hopf algebra is of <u>type</u> n if each of its primitives and generators has dimension np^i for some i. Let $\underline{T_n A} \subset A$ denote the full subcategory of type n Abelian Hopf algebras.

<u>Theorem</u> 1. There is a canonical categorical splitting $A \cong \prod_{(n,p)=1} \underline{T_n A}$, i.e.

a) Every Abelian Hopf algebra is canonically a direct product of type n Abelian Hopf algebras.

b) There are no nontrivial maps between a type n Hopf algebra and a type m Hopf algebra for $m \neq n$.

c) Moreover, $\underline{T_1 A} \cong \underline{T_n A}$ $\forall n$

Such a decomposition is well-known for the Hopf algebra $H_*(BU;k)$ (see [3] for example) The general decomposition is established by showing that the endomorphism ring of $H_*(BU;k)$ acts canonically on any abelian Hopf algebra. Part (b) follows from the fact that a Hopf algebra map sends primitives to primitives. Part (c) is trivial.

We now construct a set of projective generators for $\underline{T_1A}$. Let $B_n \in \underline{A}$ be $k[b_1,b_2,\ldots,b_n]$ with $\dim b_i = i$ and coproduct $\psi b_i = \sum_{s+t=i} b_s \otimes b_t$ where $b_0 = 1$. Let W_n be the type 1 factor of B_{p^n}. It is a polynomial algebra $k[w_0,w_1,\ldots,w_n]$ with $\dim w_i = p^i$. The coproduct is obtained lifting to $W(k)$ and defining the Witt polynomials $f_m(w) = \sum_{i=0}^{m} p^i w_i^{p^{m-i}}$, $0 \le m \le n$, to be primitive.

<u>Theorem</u> 2. W_n is a projective object in $\underline{A}$, and its dual W_n^* is therefore injective.

<u>Proof</u>. Let S_r be the simple object $k[x_r]/x_r^p$, $\dim x_r = r$. Any Abelian Hopf algebra can be built up out of these simple objects by multiple extensions, so it suffices to show $\mathrm{Ext}^1_{\underline{A}}(W_n,S_r) = 0 \;\forall\, r$, which is a simple calculation.

Now let $\underline{W} \subset \underline{T_1A}$ denote the full subcategory whose objects are the W_n. Let $\underline{FW}$ denote the category of contravariant functors from $\underline{W}$ to the category of finite $W(k)$ modules. This category is abelian. We define a functor

$$\underline{D} : \underline{T_1A} \longrightarrow \underline{FW}$$

by

$$\underline{D}(A)(W_n) = \mathrm{Hom}_{\underline{A}}(W_n, A).$$

Now we can state our main result:

Theorem 3. The functor $\underline{D}$ defined above is an equivalence of abelian categories.

The proof is analogous to that of Theorem V, §1,4.3 of [1].

Theorem 3 can be described in a more useful way by analyzing the structure of $\underline{W}$. Let $V_n : W_{n-1} \hookrightarrow W_n$ be the inclusion and let $F_n : W_{n+1} \longrightarrow W_n$ be defined by $F_n(w_i) = w_{i-1}^p$. Note that $V_n F_{n-1} = F_n V_{n+1} = p$. Then we have

Lemma 4. The endomorphism ring of W_n is $W(k)/p^{n+1}$ and these endomorphisms along with the F_n and V_n generate all of the morphisms of $\underline{W}$.

Hence Theorem 3 can be paraphrased as

Theorem 3'. A type 1 Abelian Hopf algebra is characterized by a sequence of $W(k)$ modules $W_n(A) = \mathrm{Hom}(W_n, A)$ and maps $F_n : W_n(A) \longrightarrow W_{n+1}(A)$ and $V_n : W_n(A) \longrightarrow W_{n-1}(A)$ where $V_n F_{n-1} = F_n V_{n+1} = p$.

If we identify $f \in W_n(A)$ with the element $f(w_n) \in A$, we have $(F_n f)(w_{n+1}) = f(w_n)^p \in A$, i.e. F_n corresponds to the Frobenius endomorphism of A, while V_n corresponds similarly to the dual endomorphism, i.e. the Verschiebung.

To make this more concise let $\underline{E}_0^+$ denote the where A_0 is projective and A_1 is polynomial. (If A is not finitely generated, one can still construct and A_0 and A_1 but they need not be of finite type).

This is a consequence of

Theorem 6. $\mathrm{Ext}^2_{\underline{A}}(B,A) = 0$ for all A iff B is polynomial.

We will conclude by identifying some well-known Hopf algebra functors with standard functors from homological algebra. It is convenient at this point to embed $\underline{E}_0^+$ in $\underline{E}$, the full category of graded E-modules and maps of all degrees. Hence for $M, N \in \underline{E}$, $\mathrm{Hom}_{\underline{E}}(M,N)$ is also an E-module. Moreover, if N is nonnegative and M does not have any generators in positive dimensions then $\mathrm{Hom}_{\underline{E}}(M,N)$ will also be nonnegatively graded. Define modules $P = E/VE$, $R = E/FE$.

Theorem 7. Let $A \in \underline{T_1 A}$. Then $\mathrm{Hom}_{\underline{E}}(P,C(A))$ is isomorphic to the abelian restricted Lie algebra of primitives of A (where F corresponds to the restriction), and $\mathrm{Ext}^1_{\underline{E}}(T,C(A))$ is isomorphic to the abelian restrict Lie coalgebra (with V corresponding to the corestriction) of decomposable elements of A.

The functors $\mathrm{Ext}^1_{\underline{E}}(P,C(A))$ and $\mathrm{Hom}_{\underline{E}}(R,C(A))$ are the functors $\hat{P}$ and $\hat{Q}$ respectively defined in [6] and also in [5]

§3. Hence an extension in $\underline{T_1 A}$ induces six term exact sequences relating these functors as was shown in [6]. (Note that $Ext^2_{\underline{E}}(P,-) = Ext^2_{\underline{E}}(R,-) = 0$). It is evident that the connecting homomorphisms of these sequences must be E-module maps, i.e. they must preserve the restriction and corestriction respectively. Hence the argument of 4.10 of [6] (which leads to contradictions of Theorems 2 and 4) is incorrect.

COLUMBIA UNIVERSITY

N.B. These results were also obtained by C. Schoeller, "Etude de la Categorie des Algebres de Hopf Commutatives Connexes sur un Corps", Manuscripta Math. 3(1970), 133-155.

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THE ϵ SEQUENCES IN THE STABLE HOMOTOPY OF SPHERES

Larry Smith

Recollections

Let p be an odd prime and $V(0) = S^0 \cup_p e^1$ the stable Moore space for p. There is a fundamental map

$$\alpha : S^{2p-2}V(0) \longrightarrow V(0)$$

introduced by Adams [1], Toda [8], and Yamammoto [9] which was used by Toda and Adams to define non-zero elements of order p

$$\alpha_t \in \Pi^s_{2t(p-1)-1}$$

by the composition

$$\begin{array}{ccc} S^{2t(p-1)}V(0) & \xrightarrow{\alpha^t} & V(0) \\ \text{include bottom cell} \big\uparrow & & \big\downarrow \text{collapse onto } p \text{ cell} \\ S^{2t(p-1)} & \xrightarrow[\alpha_t]{} & S^1 \end{array}$$

If we let $V(1) = V(0) \cup_\alpha cS^{2p-2}V(0)$ be the mapping cone of α, then for $p > 3$ there is a basic map [41]

$$\beta : S^{2p^2-2}V(1) \longrightarrow V(1)$$

and for $p > 3$ the elements

$$\beta_t \in \Pi_{2t(p^2-1)-2p}$$

defined by the composition (N.B. the cell structure of $V(1)$ is $S^0 \cup_p e^1 \cup_\alpha e^{2p-1} \cup_p e^{2p}$)

$$\begin{array}{ccc} S^{2t(p^2-1)}V(1) & \xrightarrow{\beta^t} & V(1) \\ \big\uparrow\cup & & \big\downarrow \\ S^{2t(p^2-1)} & \xrightarrow{\beta_t} & S^{2p} \end{array}$$

have been shown to be non-zero and of order p for all $t > 0$ [41].

The spaces $V(0)$, $V(1)$, etc., and the maps α, β, etc., have a natural origin in complex cobordism theory. Specifically, recall that $\Omega_*^U(X)$ is the bordism module [6] of singular weakly almost complex manifolds on X. The coefficient ring Ω_*^U of bordism classes of closed weakly almost complex manifolds was determined by Milnor and Novikov who showed

$$\Omega_*^U \simeq \mathbb{Z}[x_2, x_4, \ldots]; |x_{2k}| = 2k.$$

For each prime p there are polynomial generators

$x_{2p^i-2} = [V^{2p^i-2}] \in \Omega^U_{2p^i-2}$, $i = 1, \ldots$ called <u>Milnor</u> <u>manifolds</u> that play a very special role throughout the theory. (N.B. Milnor manifolds are not unique, but are sufficiently characterized by the characteristic number conditions

$$c_E[V^{2p^i-2}] \equiv 0 \bmod p \quad \text{all} \quad E$$

$$c_{\Delta^i_{p-1}}[V^{2p^i-2}] \equiv \pm p \bmod p^2$$

In addition to the examples of such manifolds in [6] we note that the hypersurfaces

$$Q_n(p) = \left\{[z] \in \mathbb{C}P(n+1) \,\middle|\, z_0^p + \ldots + z_{n+1}^p = 0\right\}$$

are Milnor manifolds whenever $n = p^i-1$.)

The spaces $V(0), V(1), \ldots$ arise naturally when one considers the question:

(*) What <u>cyclic</u> Ω^U_* modules can be realized as $\tilde{\Omega}^U_*(X)$ for some finite cw spectrum X?

To make a long story short the cyclic modules

$$V(n) = \Omega^U_*/(p, [V^{2p-2}], \ldots, [V^{2p^n-2}])$$

are fundamental in this connection. Clearly

$$V(0) = \tilde{\Omega}_*^U(V(0))$$

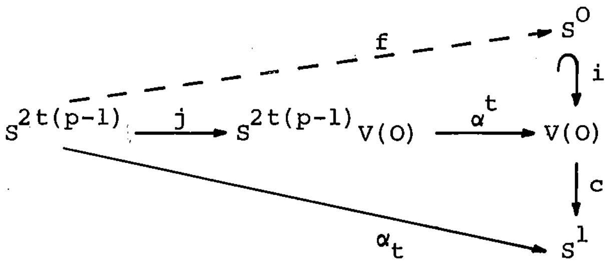
and one way to characterize α [4] is

$$\alpha_*(\Sigma^{2p-2}1) = [V^{2p-2}]$$

so that

$$\tilde{\Omega}_*^U(V(1)) \simeq V(1).$$

Let me quickly sketch the proof of [41] that $\alpha_t \neq 0 \in \Pi^s_{2t(p-1)-1}$. Introduce the diagram



Assuming $\alpha_t = 0$ implies the existence of the dotted map f. Applying $\Omega_*^U(\ )$ to the diagram and chasing around leads to the conclusion

$$0 \neq [V^{2p-2}]^t \in \mathrm{Im}\left\{\Pi_*^s \longrightarrow \Omega_*^U\right\}.$$

This however is impossible as $\Pi_{2t(p-2)}$ is a finite group and Ω_*^U is torsion free. Thus $\alpha_t \neq 0$.

Likewise the map

$$\beta : S^{2p^2-2}V(1) \longrightarrow V(1)$$

may be characterized by

$$\beta_* \Sigma^{2p^2-2} 1 = [V^{2p^2-2}] \in \tilde{\Omega}^U_*(V(1)).$$

The proof that $\beta_t \neq 0 \in \Pi^s_{2t(p^2-1)-2p}$ depends on a knowledge of the Hurecwicz map

$$\Pi^s_*(V(\tfrac{1}{2})) \longrightarrow \Omega^U_*(V(\tfrac{1}{2}))$$

where $V(\tfrac{1}{2}) = S^0 \cup_p e^1 \cup_\alpha e^{2p-1}$ and proceeds as for the proof of the non triviality of α_t.

Notice that in this context the maps

$$\alpha^{(t)} : S^{2t(p-1)} \xrightarrow{\alpha^t j} V(0)$$

$$\beta^{(t)} : S^{2t(p^2-1)} \xrightarrow{\beta^t j} V(1)$$

are fundamental; they are characterized by

$$[S^{2t(p-1)}; \alpha^{(t)}] = [V^{2p-2}]^t \in \tilde{\Omega}^U_*(V(0))$$

$$[S^{2t(p^2-1)}; \beta^{(t)}] = [V^{2p^2-2}]^t \in \tilde{\Omega}^U_*(V(1))$$

in the usual notations for singular bordism.

In many ways the element β_p is anamolous. Namely, if one consults Toda's tables one finds an element ϵ_{p-1}, and

$\beta_p = \epsilon_{p-1}$, at least up to a non-zero multiple because the p component of $\Pi^S_{2p(p^2-1)-2p}$ is cyclic of order p. If one is a homotopy theorist this instantly raises the question of what role is played by the other ϵ's, namely $\epsilon_1, \ldots, \epsilon_{p-2}$ (For future reference note $\epsilon_i \in \Pi^S_{2(p-1)(p^2+1)-2}$, $i = 1,\ldots,p-1$.)

The answer lies in examining

$$\mathrm{Im}\left\{\Pi^S_*(V^{(s)}(1)) \longrightarrow \tilde{\Omega}^U_*(V^{(s)}(1))\right\}$$

where

$$V^{(s)}(1) = \mathrm{Cone}\left\{\alpha^s : S^{2s(p-1)}V(0) \longrightarrow V(0)\right\}$$

The study of

$$\mathrm{Im}\left\{\Pi^S_*(V^{(s)}(1)) \longrightarrow \tilde{\Omega}^U_*(V^{(s)}(1))\right\}$$

is a natural outgrowth of the iceberg of results about $V(0)$, $V(1)$, etc., I have left hidden. As a first approximation it is clear from the Ballantine lemma [II; Lemma 3.1] that for $s < p+1$ the classes $[V^{2p^2-2}]^{tp} \in \Omega^U_*(V^{(s)}(1))$, $t = 1,\ldots$ are the only ones that stand a chance of being spherical. Moreover when $s = 1$, these classes are spherical and are used to define the elements $0 \neq \beta_{tp} \in \Pi^S_{2(p-1)[tp^2+(t-1)p+p-1]-2}$

Statement of Results

Theorem 1. For $p > 3$ there is for each $r = 1,\ldots,p-1$ a map

$$\Delta_r : S^{2p(p^2-1)}V^{(p-r)}(1) \longrightarrow V^{(p-r)}(1)$$

such that the diagram

$$\begin{array}{ccc} S^{2p(p^2-1)}V^{(p-r)}(1) & \xrightarrow{\Delta_r} & V^{(p-r)}(1) \\ \Big\uparrow j & & \Big\downarrow c \\ S^{2p(p^2-1)} & \xrightarrow{\epsilon_r} & S^{2(p-r)(p-1)+2} \end{array}$$

is commutative and

$$[S^{2p(p^2-1)}, \Delta_r j] = [V^{2p^2-2}] + [M] \in \tilde{\Omega}^U_*(V^{(p-r)}(1))$$

where $[M] \in ([\mathbf{C}P(p-1)]^{p-r-1}) \subset \Omega^U_*$. (Note that the complex $V^{(s)}(1)$ has a cell decomposition $S^0 \cup_p e^1 \cup_{\alpha_s} e^{2s(p-1)+1} \cup_p e^{2s(p-1)+2}$.)

Theorem 2. For $p > 3$ there is no map

$$\Delta^{(0)} : S^{2p(p^2-1)} \longrightarrow V^{(p)}(1)$$

such that $[S^{2p(p^2-1)};\Delta^{(0)}] = [V^{2p^2-2}] + [D]$ in $\tilde{\Omega}^U_*(V^{(p)}(1))$ for any decomposable $[D]$.

Proceeding as with the α's and β's we introduce for each $p > 3$ elements

$$\epsilon_r(t) \in \Pi^s_{2(p-1)[tp^2+(t-1)p+r]-2} \qquad \begin{cases} r = 1,\dots,p-1 \\ t > 0 \end{cases}$$

by the commutative diagram

$$\begin{array}{ccc} S^{2tp(p^2-1)}V^{(p-r)}(1) & \xrightarrow{\Delta_r^t} & V^{(p-r)}(1) \\ \Big\uparrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle c} \\ S^{2tp(p^2-1)} & \xrightarrow[\epsilon_r(t)]{} & S^{2(p-r)(p-1)+2} \end{array}$$

<u>Theorem</u> 3. Let $p > 3$ be a prime. Then the elements

$$\epsilon_r(t) \in \Pi^s_{2(p-1)[tp^2+(t-1)p+r]} \qquad \begin{cases} r = 1,\dots,p-1 \\ t > 0 \end{cases}$$

are non-zero of order p. Moreover

$$\epsilon_r(1) = \epsilon_r$$

$$\epsilon_{p-1}(t) = \beta_{tp}$$

Outline of Proofs

We do the proofs in reverse order.

The Proof of (3). Fix r, t, p and let

$$f = \Delta_r^t j : S^{2tp(p^2-1)} \longrightarrow V^{(p-r)}(1),$$

and introduce the diagram where the column is cofibration

$$\begin{array}{ccc} & & V^{(p-r)}(\tfrac{1}{2}) \\ & \overset{g}{\nearrow} & \downarrow i \\ S^{2tp(p^2-1)} & \xrightarrow{\;f\;} & V^{(p-r)}(1) \\ & \underset{\epsilon_r(t)}{\searrow} & \downarrow c \\ & & S^{2(p-r)(p-1)+2} \end{array}$$

Supposing $\epsilon_r(t) = 0$ gives a lifting g as indicated by the dashed arrow. Letting $\gamma \in \Omega_0^U(V^{(p-r)}(\tfrac{1}{2}))$ be a generator we see

$$g_*[S^{2t(p)(p^2-1)}] = ([V^{2p^2-2}]+[M])^t\gamma.$$

Let

$$X = V^{(p-r)}(\tfrac{1}{2}) \cup_g e^{2t(p)(p^2-1)+1}$$

and let the image of γ in $\Omega_*^U(X)$ be ζ. Then we see by fiddling with exact sequences

$$\mathrm{Ann}(\zeta) \ni p, [\mathbb{C}P(p-1)], ([V^{2p^2-2}]^p + [M])^t$$

and so [3]

$$\text{girth } \mathrm{Ann}(\zeta) \geq 3$$

and hence [3; 5] hom $\dim_{\Omega^U_*} \Omega^U_*(X) \geq 3$.

On the other hand as an outgrowth of our study initiated in [3] of homological properties of complex bordism modules, P.E. Conner and I undertook a detailed study of homological properties of $\Omega^U_*(Y)$ where Y was a cell complex sith only a few cells [4]. Now

$$X = S^0 \cup_p e^1 \cup e^{2(p-r)(p-1)+1} \cup e^{2tp(p^2-1)+1}$$

is just such a complex and we found for such a complex X that hom $\dim_{\Omega^U_*} \Omega^U_*(X) = 0,1,2$ contrary to the conclusion reached above. Therefore the assumption $\epsilon_r(t) = 0$ must be false. ■

<u>The Proof of</u> (2) Supposing such a map exists one concludes as in (1) that the composite

$$\epsilon_0 : S^{2p(p^2-1)} \xrightarrow{\Delta^{(0)} j} V^{(p)}(1) \xrightarrow{c} S^{2p(p-1)+2}$$

represents a non-zero element of $\Pi^s_{2(p-1)[p^2]-2}$ But Toda has shown [7] this group has trivial p component. ■

The Proof of (1). I do not really have a good proof of (1), so I will skip the details referring them to [5]

One begins by proving:

Theorem A. There is a map ϵ such that

$$\begin{array}{ccc} S^{2p(p^2-1)} & \xrightarrow{\ \epsilon\ } & V^{(p-1)}(1) \\ & \searrow^{\epsilon_1} & \downarrow c \\ & & S^{2(p-1)^2+2} \end{array}$$

commutes.

This is proved by obstruction theory using Toda's tables and drudgery; and to quote Liulevicius, "It turns out there is no obstruction to drudgery"

Next we observe that there is a diagram

$$\begin{array}{ccccc} S^{2s(p-1)}V(0) & \xrightarrow{\ \alpha^s\ } & V(0) & \longrightarrow & V^{(s)}(1) \\ \downarrow \alpha & & \downarrow & & \downarrow q \\ S^{2(s-1)(p-1)}V(0) & \xrightarrow{\ \alpha^{s-1}\ } & V(0) & \longrightarrow & V^{(s-1)}(1) \end{array}$$

so we get a map

$$q_r : V^{(p-1)}(1) \longrightarrow V^{(p-r)}(1)$$

and we next prove:

Theorem B. The diagram

$$\begin{array}{ccc} S^{2p(p^2-1)} & \xrightarrow{\epsilon} & V^{(p-1)}(1) \\ \Big\downarrow{\scriptstyle \epsilon_r} & & \Big\downarrow{\scriptstyle q_r} \\ S^{2(p-r)(p-1)+2} & \xleftarrow{c} & V^{(p-r)}(1) \end{array}$$

commutes.

This is proved by first showing $cq_r\epsilon \neq 0$ by manipulations with Toda brackets and then consultation of Toda's tables.

Notice that when $r = p-1$

$$[S^{2p(p^2-1)}; q_r\epsilon] = [V^{2p^2-2}]^p \in \Omega^U_*(V(1))$$

because $\epsilon_{p-1} = \beta_p$. Using induction we get

Theorem C. $[S^{2p(p^2-1)}, q_r\epsilon] = [V^{2p^2-2}]+[M]$ in $\tilde{\Omega}^U_*(V^{(p-r)}(1))$.

Finally by obstruction theory it is shown that $q_r\epsilon$ extends to $V^{(p-r)}(1)$. ■

INDIANA UNIVERSITY

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THE GEOMETRIC DIMENSION OF COMPLEX VECTOR BUNDLES

V.P. Snaith

1. Introduction

In this paper we obtain necessary conditions for the existence of an n-dimensional trivial real vector sub-bundle of a complex vector bundle $\pi : E \longrightarrow X$ in terms of some K-theory classes, e(E). We will work with Real, equivariant K-theory, $KR^*_{Z_2}(X)$ [2]. We will use $n \subset E$ to signify that $\mathbb{R}^n \times X$ is a vector sub-bundle of E. The method and its motivation are described below.

In [10] necessary conditions for $n \subset E$, when E is Spin(k) bundle, are obtained by studying the homomorphism $K^O(BSpin(k)) \longrightarrow K^O(BSpin(k-m))$ to obtain necessary conditions in terms of K-theory for the lifting of the classifying map $E : X \longrightarrow BSpin(k)$. In these calculations it is the behaviour of the exotic Spin representations which is responsible for the success of the method. This suggests that there should exist a construction using Clifford algebras and Clifford bundles. If $n \subset E$ then C(E), the Clifford bundle of E, has a natural $C_n = C(\mathbb{R}^n)$, structure. However, if $p : V \longrightarrow X$ is a C_n vector bundle and H is the real one dimensional non-trivial representation of Z_2 Clifford multiplication gives an isomorphism $V \longrightarrow V \otimes H$ of Z_2 vector bundles over the complement of X in $X \times nH$, so that $[V](1-H) \in \mathrm{im}\left\{K_{Z_2}(X \times nH) \longrightarrow K_{Z_2}(X) = K(X) \otimes R(Z_2)\right\}$. Applying this line of motivation to $C(E)$ we construct a class $e(E) \in K^*_{Z_2}(E) \cong K^*_{Z_2}(X)$ which is simple to compute and enjoys the property that $e(E) \in \mathrm{im}\left\{KR^*_{Z_2}(E \times nH) \longrightarrow K^*_{Z_2}(E)\right\}$ if $n \subset E$, where E is given the Real involution which sends $v \in E_x$ to $(-v) \in E_x$. The image of $K^*_{Z_2}(E \times nH) \longrightarrow K^*_{Z_2}(E)$ is well known and this gives immediate restrictions on n. Also in the case when X is torsion free there exists a straightforward method for determining $\mathrm{im}\left\{KR^*_{Z_2}(E \times nH) \longrightarrow K^*_{Z_2}(E \times nh)\right\}$. Let H_k be the Hopf bundle over $\mathbb{F}P^k (\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H})$. We apply this method to the bundles $mH_k \longrightarrow \mathbb{C}P^k$ and $mH_k \longrightarrow \mathbb{H}P^k$. For the most important case, $E = mH_k \longrightarrow \mathbb{R}P^k$, the class $e(E)$ is zero. Nevertheless it is possible to outline a programme for obtaining restrictions on $n \subset (mH_k \longrightarrow \mathbb{R}P^k)$. Finally it should be pointed out that the method is stable in the sense that it obtains restrictions on n such that $n+8t \subset E \oplus \mathbb{R}^{8t}$ and hence restrictions on the geometric dimension of E [9].

It has been pointed out to me that the method employed here has been used, in a more sophisticated manner, in [5] to study tangent k-planes with finite singularities. The accuracy of §§3.13 and 3.15 owes much to the efforts of Don Davis.

2. The classes $e(E)$

Let $\pi : E \longrightarrow X$ be a complex vector bundle. The spaces E and X will be considered as Z_2-spaces with trivial action and as Real spaces with an involution which is trivial on X and is multiplication by minus one on the fibres of E. Suppose that the complex dimension of E is $2m$. From the real Clifford bundle, $C(E)$, [1] with respect to some real metric on E and let $v \circ w$ denote the Clifford product of $v, w \in C(E_x)$, where $E_x = \pi^{-1}(x)$ and $x \in X$. The Clifford bundle $C(E)$ is the quotient of the tensor algebra bundle $T(E) = \bigoplus_{k \geq 0} E^{\otimes k}$ by the ideal generated by $(v \otimes v + \langle v, v \rangle 1)$. Denote by $C^e(E)$ and $C^o(E)$ the quotients of $\bigoplus_{k \text{ even}} E^{\otimes k}$ and $\bigoplus_{k \text{ odd}} E^{\otimes k}$ respectively. Now consider the real exterior algebra bundle $\Lambda(E) = \bigoplus_{i \geq 0} \lambda^i_{\mathbb{R}}(E)$ and let $v \wedge w \in \lambda^{i+j}(E)$ denote the exterior product of $v \in \lambda^i(E)$ and $w \in \lambda^j(E)$. For $v \in E_x$ we may form the exterior product map $d_v = (v \wedge -) \otimes 1_{\mathbb{C}} : \Lambda(E) \otimes_{\mathbb{R}} C \longrightarrow \Lambda(E) \otimes_{\mathbb{R}} \mathbb{C}$ and $D_v = (d_v - d_v^*)$ where d_v^* is the adjoint of dv with respect to the natural complex inner product on $\Lambda(E) \otimes_{\mathbb{R}} \mathbb{C}$. Since $D_v^2 = -1$ the map $D : v \longrightarrow \mathrm{End}(\Lambda(E) \otimes_{\mathbb{R}} \mathbb{C})$ extends to $D : C(E) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathrm{End}(\Lambda(E) \otimes_{\mathbb{R}} \mathbb{C})$ and it is well known [6] that the map $\tilde{D} : C(E) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \Lambda(E) \otimes_{\mathbb{R}} \mathbb{C}$ defined by $\tilde{D}(z) = D(z)(1)$ is an isomorphism of vector bundles which takes left Clifford multiplication by $v \in E_x$ to D_v and satisfies $\tilde{D}(C^e(E) \otimes_{\mathbb{R}} \mathbb{C}) = \Lambda^e(E) \otimes_{\mathbb{R}} \mathbb{C}$, $\tilde{D}(C^o(E) \otimes_{\mathbb{R}} \mathbb{C}) = \Lambda^o(E) \otimes_{\mathbb{R}} \mathbb{C}$.

Also the complex vector bundle, E, has a natural orientation and hence a non-zero section $f : X \to C^{o}(E) \otimes_{\mathbb{R}} \mathbb{C}$ and for $x \in X f(x) \circ f(x) = 1$ since, in local coordinates, $f(x)$ is the Clifford product of the $4m$ distinct basis elements of E_x. Denote by C_+ and C_- the eigenspaces of the involution on $C(E) \otimes_{\mathbb{R}} \mathbb{C}$ given by Clifford multiplication on the left by f. Put $C_+^e = (C^e(E) \otimes_{\mathbb{R}} \mathbb{C}) \cap C_+$, $C_+^o = C^o \cap C_+$, $C_-^e = C^e \cap C_-$ and $C_-^o = C^o \cap C_-$. Let $v \in E_x$ and define

$$\delta_v : C_+^e(E_x) \longrightarrow C_-^o(E_x)$$

by

$$\delta_v(z \otimes \lambda) = (v \circ z) \otimes i\lambda (z \in C(E_x);\ i, \lambda \in \mathbb{C} \text{ and } i^2 = -1).$$

The same formula defines a Z_2-equivariant map $\delta_v : C_+^o \otimes_{\mathbb{C}} H \to C_-^e \otimes_{\mathbb{C}} H$ where H is the one dimensional involution representation and Z_2 acts trivially on $C(E)$. If $C(E) \otimes_{\mathbb{R}} \mathbb{C}$ and $C(E) \otimes_{\mathbb{R}} H$ are given the Real involution which is complex conjugation, $(\bar{\ })$, in each fibre then

$$\delta_{(-v)}(z \otimes \bar{\lambda}) = (-v \circ z) \otimes i\bar{\lambda}$$

$$= (v \circ z) \otimes (\overline{i\lambda})$$

so that δ gives homomorphisms of Real vector bundles

$$\delta : \pi^* C_+^e \longrightarrow \pi^* C_-^o ,$$

$$\delta : \pi^* C_+^o \otimes H \longrightarrow \pi^* C_-^e \otimes H$$

if E is given the Real involution $\bar{v} = -v$, $(v \in E_x)$. Suppose now that $s_1, \ldots, s_n : X \longrightarrow E$ are linearly independent sections of E. Let $(v, y_1, \ldots, y_n)$ be a point in the Z_2 space $E_x \times nH$ with trivial action on the first factor. Define homomorphisms

$$\varepsilon_{(v,y)} : C_+^e \longrightarrow C_+^o \otimes H$$

and

$$\varepsilon_{(v,y)} : C_-^o \longrightarrow C_-^e \otimes H$$

by

$$\varepsilon_{(v,y)}(z \otimes \lambda) = \left(z \circ \left[\sum_{i=1}^{n} y_i s_i(x) \right] \right) \otimes \lambda .$$

Since

$$\varepsilon_{(v,-y)}(z \otimes \lambda) = -\varepsilon_{(v,y)}(z \otimes \lambda)$$

and

$$\varepsilon_{(-v,y)}(z \otimes \bar{\lambda}) = \varepsilon_{(v,y)}(z \otimes \bar{\lambda})$$

ε gives homomorphisms of Z_2-equivariant, Real vector bundles over $E \times nH$,

$$\epsilon : \pi^* C^e_+ \to \pi^* C^o_+ \otimes H \quad \text{and} \quad \epsilon : \pi^* C^o_- \to \pi^* C^e_- \otimes H.$$

The δ and ϵ homomorphisms commute, being left and right multiplication in a bundle of algebras and we may combine them into the following complex of Real Z_2-vector bundles over $E \times nH$,

$$(2.1) \qquad 0 \to C^2_+ \xrightarrow{(\delta,\epsilon)} C^o_- \oplus C^o_+ \otimes H \xrightarrow{\binom{-\epsilon}{\delta}} C^e_- \otimes H \to 0.$$

The complex (2.1) is exact whenever either one of δ or ϵ is, which happens when $(v, y_1, \ldots, y_n) \neq \underline{0}$.

The complex (2.1) represents an element in $KR_{Z_2}(E \times nH)$.

Definition 2.2.

(i) Let E be a complex vector bundle of complex dimension $2m$. Define $e(E) \in K_{Z_2}(E)$ to be the element represented by the complex (2.1) with $n = 0$.

(ii) Let E' be a complex vector bundle of dimension $(2m+1)$ and put $E = E' \oplus (X \times \mathbb{C})$. Define $e(E') \in K_{Z_2}(E \oplus 2H) \cong K_{Z_2}(E')$ to be the element represented by the complex (2.1) with $n = 2$ and s_1, s_2 given by the canonical sections of $X \times \mathbb{C}$. From the preceeding discussion we have

Proposition 2.2.

(i) If $n \subset E$ and $\dim E = 2m$ then

$$e(E) \in \operatorname{im} KR_{Z_2}(E \times nH) \longrightarrow K_{Z_2}(E) \ .$$

(ii) If $n \subset E'$ and $\dim_{\mathbb{C}} E' = 2m+1$ then

$$e(E') \in \operatorname{im}\left\{KR_{Z_2}(E' \times \mathbb{C} \times (n+2)H) \longrightarrow K_{Z_2}(E' \times \mathbb{C} \times 2H)\right\}.$$

We now proceed to the determination of $e(E)$ and $e(E')$ in the following manner. Firstly we observe that the complex (2.1) may be defined for any complex vector bundle E using $\epsilon = 0$ and using eigenspaces obtained from $(i)^{t^2} f : X \longrightarrow C^0(E) \otimes_{\mathbb{R}} \mathbb{C}$, $(t = \dim E)$. This complex represents an element $e(E) \in K_{Z_2}(E)$ which will not in general belong to $\operatorname{im}\left\{KR_{Z_2}(E) \longrightarrow K_{Z_2}(E)\right\}$. Also this characteristic class $e(E)$ is easily seen to satisfy $e(E_1 \oplus E_2) = e(E_1) \cdot e(E_2)$. Since the orientation of complex vector bundles is preserved under maps of complex vector bundles $e(E)$ is a natural exponential characteristic class on complex vector bundles and can be evaluated for a line bundle over $\mathbb{C}P^t$ and then calculated by means of the splitting principle.

<u>Proposition</u> 2.3. In

$$K_{Z_2}(H_t) \cong K(\mathbb{C}P^t) \otimes R(Z_2) = \frac{Z[H_t]}{(H_t - 1)^{t+1}} \otimes \frac{Z[H]}{(H^2 - 1)}$$

$$e(H_t) = 1 \otimes H - H_t^{-1} \otimes 1.$$

Proof. Since $H_t = (S^{2t+2} \times \mathbb{C})/\sim$, where $(z,v)\sim(z\bar{\lambda},\lambda v)$ $(\lambda \in S^1; v \in \mathbb{C}; z \in S^{2t+2})$, it suffices to consider the exterior complex of the real vector space $\mathbb{R}^2 = \mathbb{C}^1$

$$0 \longrightarrow \mathbb{C} \xrightarrow{\alpha} \mathbb{C}^2 \xrightarrow{\beta} \lambda^2\mathbb{C}^2 = \mathbb{C} \longrightarrow 0$$

on which S^1 acts via matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ on $\mathbb{C}^2$ and trivially on the other vector spaces. Left Clifford multiplication, $\Lambda^e \longrightarrow \Lambda^o$, is given by $\alpha+\beta^*$. At the point $v = x_1+ix_2 = (x_1,x_2) \in \mathbb{R}^2$, α and β are given by

$$\alpha(1) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \beta(y_1,y_2) = x_1y_2-x_2y_1$$

from which left multiplication by v is seen to be

$$\begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix} : \Lambda^e \longrightarrow \Lambda^o.$$

Let $e_1 = (1,0)$ and $e_2 - (0,1)$ then we must find the eigenspaces of $\tau = i(e_1 \circ e_2 \circ -) : \Lambda^e \longrightarrow \Lambda^e$. It is easily verified that $\bar{e}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \in \Lambda^e_+$ and $\bar{e}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \in \Lambda^e_-$ are the basis elements from which it follows that the homomorphism $\alpha+\beta^* : \Lambda^e \longrightarrow \Lambda^o$ of S^1 vector spaces splits as the sum of

(2.3.1) $$(\Lambda^e_- \longrightarrow \Lambda^o_-) \equiv 0 \longrightarrow \mathbb{C} \xrightarrow{(v\cdot\_)} \mathbb{C} \longrightarrow 0$$

and

$$(2.3.2) \qquad (\Lambda^e_- \longrightarrow \Lambda^o_+) \equiv 0 \longrightarrow \mathbb{C} \xrightarrow{(v\cdot\_)} \mathbb{C} \longrightarrow 0$$

where $\lambda \in S^1$ acts as $\bar{\lambda}$ on the right hand term of (2.3.1) and as λ on the right hand term of (2.3.2). Upon quotienting out the S^1 action (2.3.1) becomes the Thom class of H_t^{-1}, $\underline{\Lambda}(H_t^{-1})$, which is $(-H_t^{-1})\underline{\Lambda}(H_t) \in K(H_t)$ and (2.3.2) becomes $\underline{\Lambda}(H_t)$, the Thom class of H_t. The result now follows from the fact that the complex (2.1) represents the element

$$(\Lambda^e_+ \longrightarrow \Lambda^o_-) \otimes 1 + (\Lambda^e_- \longrightarrow \Lambda^o_+) \otimes H.$$

<u>Corollary</u> 2.4. If E is a complex vector bundle $(\dim_{\mathbb{C}} E = t)$ then $e(E) = (-1)^t(\lambda^t_{\mathbb{C}}(E))^{-1}[\Lambda^e(E) \otimes 1 - \Lambda^o(E) \otimes H]$ in $K_{Z_2}(E) \equiv K(X) \otimes R(Z_2)$, where $\Lambda^e(E) \sum_i \lambda^{2i}_{\mathbb{C}} E$ and $\Lambda^o(E) = \sum_i \lambda^{2i+1}_{\mathbb{C}} E$.

<u>Proof</u>. Thom classes and the classes $e(E)$ are both exponential. Thus the result follows for $E = \sum_1^t L_i$, a sum of lines, from the expansion

$$\begin{aligned} e(E) &= \Pi(1 \otimes H - L_i^{-1} \otimes 1) \\ &= (\Pi L_i^{-1})\Pi(L_i \otimes H - 1 \otimes 1) \\ &= (-1)^t(\lambda^t_{\mathbb{C}}(E))^{-1}[\Lambda^e(E) \otimes 1 - \Lambda^o(E) \otimes H]. \end{aligned}$$

The general result now follows from the splitting principle.

Lemma 2.5. The element obtained by putting $E' = 0$ in Definition 2.2 (ii) is (up to sign) the Thom class in $K_{Z_2}(X \times \mathbb{C} \times 2H) \cong K(X) \otimes R(Z_2)$.

Proof. The element $e(\underline{0})$ is represented by a complex of Z_2-vector bundles over $\mathbb{C} \times 2H$ of the form

$$(2.5.1) \qquad 0 \longrightarrow \Lambda_+^e \oplus \Lambda_-^e \xrightarrow{\ \psi\ } \Lambda_-^o \oplus \Lambda_+^o \longrightarrow 0$$

where, over $(u_1, u_2) \in \mathbb{C} \times 2H$,

$$\psi(u_1, u_2) = \begin{pmatrix} (-\circ u_1) & i(u_2 \circ -) \\ (iu_2 \circ -) & (u_1 \circ -) \end{pmatrix}$$

Hence $\psi(u_1, u_2)^* \psi(u_1, u_2) = \|u_1 u_2\|^2$ and the complex (2.5.1) is, up to sign, that obtained from the action of C_4 on $\Lambda_{\mathbb{C}}^*(\mathbb{C}^2)$, which is the construction of the Thom class [6].

Combining the results of §§2.2-2.5 we obtain:

Theorem 2.6. Let E be a complex vector bundle over X with $\dim_{\mathbb{C}} E = t$. Let $\Lambda(E) \in K(E)$ be the Thom class.

Suppose that $n \subset E$ then

(i) if t is even

$$(\lambda^t_{\mathbb{C}}(E))^{-1}[\Lambda^e(E) \otimes 1 - \Lambda^o(E) \otimes H]\underline{\Lambda}(E)$$

is in

$$\text{im}\left\{KR_{Z_2}(E \times nH) \longrightarrow K_{Z_2}(E)\right\}$$

and

(ii) if t is odd

$$(\lambda^t_{\mathbb{C}}(E))^{-1}[\Lambda^e(E) \otimes 1 - \Lambda^o(E) \otimes H]\underline{\Lambda}(E) \cdot \underline{\Lambda}(\mathbb{C}) \cdot \underline{\Lambda}(H \otimes_{\mathbb{R}} \mathbb{C})$$

is in

$$\text{im}\left\{KR_{Z_2}(E \times \mathbb{C} \times (n+2)H) \longrightarrow K_{Z_2}(E \times \mathbb{C} \times 2H)\right\}.$$

(In (i) and (ii) the spaces E and $E \times \mathbb{C}$ have the Real involution which is multiplication by minus one in each fibre.)

3. The torsion free case

In this section we consider the image of the complexification, $c : KR^*_{Z_2} \longrightarrow K^*_{Z_2}$, in the case when $K^*_{Z_2}$ is a torsion free group and we apply this to determine the image groups in Theorem 2.6 for bundles $mH_t \longrightarrow \mathbb{C}P^t$ and $mH_t \longrightarrow \mathbb{H}P^t$.

Throughout this section liberal use is made of the Thom isomorphisms for KR_{Z_2}-theory for which the reader is referred to [3 and 4].

It is convenient to consider $KR^*_{Z_2}$ and $K^*_{Z_2}$ as being graded by Z_8. Recall $KR^*(pt) = Z[\eta_1,\eta_4]/I$ where $\deg \eta_i = -i$ and I is the ideal generated by $2\eta_1$, η_1^3, $\eta_1\eta_4$ and η_4^2-4. Also $K^*(pt) = Z[\mu_2]/(\mu_2^4-1)$ where $\deg \mu_2 = -2$ and $c(\eta_4) = 2\mu_2^2$. Put $\mathbb{R}^{p,q} = \mathbb{R}^q \oplus \mathbb{R}^p$, considered as a Real space with involution $\tau(v_1 \oplus v_2) = v_1 \oplus (-v_2)$. Let $r : K^*_{Z_2}(X) \longrightarrow KR^*_{Z_2}(X)$ denote Realification. Let X be a Real space with involution, τ.

<u>Definition</u> 3.1. Let $K^*_{Z_2}(X)$ be a torsion free group. A decomposition $K^*_{Z_2}(X) = (M_+ \oplus M_- \oplus T \oplus T^*) \otimes K^*(pt)$ will be called a <u>*-decomposition</u> if * acts like $(+1)$ on M_+, (-1) on M_- and interchanges T and T^*. Here x^* denotes $\tau^*(\bar{x})$.

<u>Proposition</u> 3.2 [11].

(i) Let $K^*_{Z_2}(X) = (M_+ \oplus M_- \oplus T \oplus T^*) \otimes K^*(pt)$ be a *-decomposition. Assume given elements $h_1,\dots,h_n \in KR^*_{Z_2}(X)$ such that $c(h_1),\dots,c(h_n)$ is a basis for the $K^*(pt)$-module, $K^*(pt) \otimes (M_+ \oplus M_-)$. Then, as a $KR^*(pt)$-module, $KR^*_{Z_2}(X) = F \oplus r(K^*_{Z_2}(pt) \otimes T)$, where F is the free $KR^*(pt)$-module on $h_1,\dots,h_n$.

(ii) If, in part (i), $K^{-1}_{Z_2}(X) = 0$, then there exists homogeneous elements $h_1,\dots,h_n \in KR^{even}_{Z_2}(X)$ as in the assumption of part (i).

Remark 3.3. In $\pi : E \to X$ is a complex vector bundle and $K^*(X)$ is torsion free then $K^*_{Z_2}(E \times nH)$ and $K^*_{Z_2}(E \times \mathbb{C} \times (n+2)H)$ are both free of torsion. Hence the images of

$$KR_{Z_2}(E \times nH) \xrightarrow{c} K_{Z_2}(E \times nH) \to K_{Z_2}(E)$$

and

$$KR_{Z_2}(E \times \mathbb{R}^{2,0} \times (n+2)H) \xrightarrow{c} K_{Z_2}(E \times \mathbb{C} \times (n+2)H) \to K_{Z_2}(E \times \mathbb{C} \times 2H)$$

can be calculated, using Proposition 3.2. In order to apply Proposition 3.2 it is necessary to know which elements are in the image of the complexification homomorphism, c. For example, in Proposition 3.2 (ii), if $x \in K^0 \cap M_+$ we need to know which one of x and $x \cdot \mu_2^2$ is in $\mathrm{im}(c)$.

The remainder of this section will deal with the application of the method to $mH_t \to \mathbb{C}P^t$ and $mH_t \to \mathbb{H}P^t$.

Lemma 3.4. Let $\pi : L \to X$ be a complex Z_2-line bundle. Suppose the space L has Real involution which is either trivial or multiplication by minus one on fibres, then

$$\underline{\Lambda}(L)^* = (-L^{-1})\,\underline{\Lambda}(L) \in K_{Z_2}(L).$$

Proof. The Thom class is represented by the complex

$$0 \to \pi^*(X \times \mathbb{C}) \xrightarrow{d} \pi^*(L) \to 0$$

where d, over $v \in L_x$, is multiplication by v. Hence the *-operation gives the Thom complex of $\bar{L} = L^{-1}$ and $L \cdot \underline{\Lambda}(\bar{L}) = (-\underline{\Lambda}(L))$.

<u>Corollary</u> 3.5. For $\pi : E = mH_t \longrightarrow \mathbb{C}P^t$ the *-operation is described as follows:

(i) $K^1_{Z_2}(E \times 2rH) = 0$,

$$K^0_{Z_2}(E \times 2rH) \cong Z[H_t]/(H_t-1)^{t+1} \otimes Z[H]/(H^2-1), \text{ and}$$

$$(H_t^j \otimes w)^* = (-1)^{m+r} H_t^{-j-m} \otimes wH^r.$$

(ii) $K^1_{Z_2}(E \times (2r+1)H) = 0$,

$K^0_{Z_2}(E \times (2r+1)H) \longrightarrow K^0_{Z_2}(E \times 2rH)$ is a monomorphism which commutes with * and has image

$$K^0_{Z_2}(E \times 2rH) \cdot (H-1).$$

Also

$$K^0_{Z_2}(E \times (2r+2)H) \longrightarrow K^0_{Z_2}(E \times (2r+1)H)$$

is onto.

<u>Proof</u>. The statements about the groups and homomorphisms are to be found in [2, pp. 105 and pp. 102]. The calculation of * follows from the facts:

(a) $H_t \in K^0(\mathbb{C}P^t)$ satisfies $\bar{H}_t = H_t^{-1}$,

(b) the Thom class of $E \times (rH \otimes_{\mathbb{R}} \mathbb{C})$ is

$\underline{\Lambda}(H_t)^m \underline{\Lambda}(H \otimes_{\mathbb{R}} \mathbb{C})^r$,

(c) $H_t^j \otimes w$ corresponds to $(H_t^j \otimes w) \cdot \underline{\Lambda}(E \times (rH \otimes_{\mathbb{R}} \mathbb{C}))$

and $*$ is a ring homomorphism.

Corollary 3.6. For $\pi : E = mH_t \longrightarrow \mathbb{H}P^t$ the $*$-operation is described as follows:

(i) $K^1_{Z_2}(E \times 2rH) = 0$,

$K^0_{Z_2}(E \times 2rH) \cong Z[H_t]/(H_t - 2)^{t+1} \otimes Z[H]/(H^2 - 1)$ and

$(x \otimes w)^* = (-1)^r x \otimes wH^r$.

(ii) $K^1_{Z_2}(E \times (2r+1)H) = 0$,

$K^0_{Z_2}(E \times (2r+1)H) \longrightarrow K^0_{Z_2}(E \times 2rH)$

is a monomorphism which commutes with the $*$-operation and has image $K^0_{Z_2}(E \times 2rH) \cdot (H-1)$. Also

$$K^0_{Z_2}(E \times (2r+2)H) \longrightarrow K^0_{Z_2}(E \times (2r+1)H)$$

is onto.

<u>Proof</u>. The Real maps

$$\begin{array}{ccc} m(H_{2t+1}+\bar{H}_{2t+1}) & \longrightarrow & mH_t \\ \downarrow & & \downarrow \\ \mathbb{C}P^{2t+1} & \longrightarrow & \mathbb{R}P^t \end{array}$$

induce monomorphisms.

From Corollary 3.5, we obtain the $*$-decompositions of $K^0_{Z_2}(mH_t \times nH)$ which are set out in the table below for $mH_t \quad \mathbb{C}P^t$. Firstly we need some notation. Put

$$T_1(a,b) = \left\{H_t^{j-u} \otimes w \,|\, w \in Z[H]/(H^2-1),\ a \leq j \leq b\right\},$$

$$T_2(a,b) = \left\{H_t^{j-u} \otimes 1;\ a \leq j \leq b\right\} \quad \text{and}$$

$$T_3(a,b) = \left\{H_t^{j-u} \otimes (1-H);\ a \leq j \leq b\right\}.$$

In the following table the $*$-decomposition of $K_{Z_2}(mH_t \times (2q+1)H)$ is described in terms of its image in $K_{Z_2}(mH_t \times 2qH)$.

TABLE 3.7

*-DECOMPOSITION OF $K^{O}_{Z_2}(mH_t \times nH)$ FOR $mH_t \longrightarrow \mathbb{C}P^t$

m	t	n	M_+	M_-	T
2u	2s	$8\ell, 8\ell+4$	$H_t^{-u} \otimes w$	O	$T_1(1,s)$
2u	2s	$8\ell+2, 8\ell+6$	O	O	$T_2(-s,s)$
2u	2s+1	$8\ell, 8\ell+4$	$H_t^{-u} \otimes w$	$(1-H_t)^{2s+1} \otimes w$	$T_1(1,s)$
2u	2s+1	$8\ell+2, 8\ell+6$	O	O	$T_2(-s,s+1)$
2u+1	2s	$8\ell, 8\ell+4$	O	$(1-H_t)^{2s} \otimes w$	$T_1(0,s-1)$
2u+1	2s	$8\ell+2, 8\ell+6$	O	O	$T_2(-s,s)$
2u+1	2s+1	$8\ell, 8\ell+4$	O	O	$T_1(0,s)$
2u+1	2s+1	$8\ell+2, 8\ell+6$	O	O	$T_2(-s-1,s)$
2u	2s	odd	$H_t^{-u} \otimes (1-H)$	O	$T_3(1,s)$
2u	2s+1	odd	$H_t^{-u} \otimes (1-H)$	$(1-H_t)^{2s+1} \otimes (1-H)$	$T_3(1,s)$
2u+1	2s	odd	O	$(1-H_t)^{2s} \otimes (1-H)$	$T_3(0,s-1)$
2u+1	2s+1	odd	O	O	$T_3(0,s)$

The following table gives generators for the *-decomposition of $K^0_{Z_2}(mH_t \times nH)$ for $mH_t \longrightarrow \mathbb{H}P^t$. We use the same convention as in Table 3.7 about the representation of elements when n is odd.

TABLE 3.8

*-DECOMPOSITION OF $K^0_{Z_2}(mH_t \times nH)$ FOR $mH_t \longrightarrow \mathbb{H}P^t$

n	M_+	M_-	T
$8\ell+2, 8\ell+6$	0	0	$H_t^j \otimes 1 (0 \le j \le t)$
$8\ell, 8\ell+4$	$H^j \otimes w (0 \le j \le t)$	0	0
odd	$H_t^j \otimes (1-H)(0 \le j \le t)$	0	0

We now determine in which dimensions (modulo eight) the elements of M_+ and M_- of Table 3.7, 3.8 are real (that is, in the image of the complexification homomorphism).

<u>Lemma</u> 3.9. The following elements are real.

(a) $\pi : mH_t \longrightarrow \mathbb{C}P^t$:

(i) $\mu_2^{\beta}(1-H_t)^t \otimes 1 \in K^{2\beta}_{Z_2}(mH_t \times nH)$

if $n \equiv 4 \pmod 8$ and $\beta+m \equiv t+2 \pmod 4$

or $n \equiv 0 \pmod 8$ and $\beta+m \equiv t \pmod 4$.

(ii) $\mu_2^{\beta}(1-H_t)^t \otimes (1-H) \in K_{Z_2}^{2\beta}(mH_t \times nH)$

if $n \equiv 3,5 \pmod 8$ and $\beta+m \equiv t+2 \pmod 4$

or $n \equiv 1,7 \pmod 8$ and $\beta+m \equiv t \pmod 4$.

(b) $\pi : mH_t \quad \mathbb{H}P^t$:

(i) $\mu_2^{2\beta}(2-H_t)^t \otimes 1 \in K_{Z_2}^{4\beta}(mH_t \times nH)$

if $n \equiv 4 \pmod 8$ and $\beta+m \equiv t+1 \pmod 2$

or $n \equiv 0 \pmod 8$ and $\beta+m \equiv t \pmod 2$.

(ii) $\mu_2^{2\beta}(2-H_t)^t \otimes (1-H) \in K_{Z_2}^{4\beta}(mH \times nH)$

if $n \equiv 3,5 \pmod 8$ and $\beta+m \equiv t+1 \pmod 2$

or $n \equiv 1,7 \pmod 8$ and $\beta+m \equiv t \pmod 2$.

<u>Proof</u>. As a Real spaces $(mH_t|\mathbb{C}P^t)-(mH_t|\mathbb{C}P^{t-1}) = \mathbb{R}^{2m,2t}$

and

$$(mH_t|\mathbb{H}P^t)-(mH_t|\mathbb{H}P^{t-1}) = \mathbb{R}^{4m,4t}$$

and

$$KR(\mathbb{R}^{a,b}) \cong KR^{a-b}(pt) \longrightarrow K(\mathbb{R}^{a+b})$$

is an isomorphism if and only if $a-b \equiv 0 \pmod 8$. Now the Z_2-representations $8H$ and $\mathbb{R}^4 \otimes 4H$ have Spin (8)-structures [3] and hence a Thom isomorphism for KR_{Z_2}. Hence the results (a)(i) and (b)(i) follow from the fact that the images of

$$K_{Z_2}^{2\beta}(\mathbb{R}^{2m,2t} \times nH) \longrightarrow K_{Z_2}^{2\beta}((mH_t|\mathbb{C}P^t) \times nH)$$

and

$$K_{Z_2}^{4\beta}(\mathbb{R}^{4m,4t} \times nH) \longrightarrow K_{Z_2}^{4\beta}((mH_t|\mathbb{R}P^t) \times nH)$$

are generated, in the dimensions cited, by the elements in the statement of the theorem. The results for $n \equiv 3$ or $7 \pmod 8$ follow by restriction from the cases $n \equiv 4$ or $8 \pmod 8$. The results for $n \equiv 1$ or $5 \pmod 8$ follows by lifting the classes generators in $KR_{Z_2}^{4}((8\ell+4))H)$ and $KR_{Z_2}^{0}(8\ell H)$ to $KR_{Z_2}^{4}((8\ell+5)H)$ and $KR_{Z_2}^{0}((8\ell+1)H)$. That this lifting can be achieved is easily seen by considering the $KO_{Z_2}^{*}$ exact sequences for the pairs given by unit disc and sphere in the Z_2-representation $nH(n = 4,5,8$ and $9)$ [2, pp. 106] and using the well-known facts [8] that $KO^4(\mathbb{R}P^4) = Z \oplus Z_2$, (generated by η_4 and $\eta_4(1-H_4)$) and $K^0(\mathbb{R}P^8) = Z \oplus Z_{16}$ (generated by 1 and $(1-H_8)$).

<u>Lemma</u> 3.10. For $\pi : 2uH_t \longrightarrow \mathbb{C}P^t$ the following elements are Real.

(i) $H_t^{-u} \otimes 1 \in K_{Z_2}^{4u+\ell}(2uH_t \times \ell H)$ if $\ell \equiv 0 \pmod 4$.

(ii) $H_t^{-u} \otimes (1-H) \in K_{Z_2}^{4u+\ell+1}(2uH_t \times \ell H)$ if $\ell \equiv 3,7 \pmod 8$.

(iii) $H_t^{-u} \otimes (1-H) \in K_{Z_2}^{u+\ell-1}(2uH_t \times \ell H)$ if $\ell \equiv 1,5 \pmod 8$.

Proof. Part (ii) follows from (i) by restricting the Real elements and Part (iii) follows from (i) by lifting the Real elements, as in the proof of 3.9. Also the Thom isomorphisms for the Z_2-representations $8H$ and $\mathbb{R}^4 \otimes 4H$ imply we need only prove (i) when $\ell = 0$. Let $E_1 = uH_t \otimes_{\mathbb{R}} \mathbb{C}$ with Real involution given by complex conjugation. Let $E_2 = 2uH_t$ with trivial involution and let $E_3 = 2uH_t$ with the antipodal involution in each fibre. There is a commutative diagram

$$\begin{array}{ccc}
KR^0(E_1) & \xrightarrow{\ c\ } & K^0(E_1) \\
\cong \downarrow & & \cong \downarrow \alpha_1 \\
KR^0(E_1 \oplus E_1) & \xrightarrow{\ c\ } & K^0(E_1 \oplus E_1) \\
\Vert & & \uparrow \\
KR^0(E_2 \oplus E_3) & & \\
\cong \uparrow & & \cong \uparrow \alpha_2 \\
KR^{4u}(E_3) & \xrightarrow{\ c\ } & K^{4u}(E_3)
\end{array}$$

in which the vertical maps are Thom isomorphisms [3]. However α_1 preserves Thom classes, $\alpha_2(\mu_2^{2u} H_t^{-u} \underline{\Lambda}(E_3)) = \underline{\Lambda}(2E_1)$ and $\underline{\Lambda}(E_1)$ is Real in the top line.

Lemma 3.11. For $\pi : mH_t \longrightarrow \mathbb{H}P^t$ the following elements are

Real.

(i) If $n \equiv 0(2)$, $\mu_2^{2m+n} H_t^{2j} \otimes 1 \in K_{Z_2}^{4m+2n}(mH_t \times 2nH)$ $(0<2j\leq t)$

and

$$\mu_2^{2m+n+4}(2H_t^{2j}-H_t^{2j+1}) \otimes 1 \in K_{Z_2}^{4m+4+2n}(nH_t \times 2nH).$$

(ii) If $n \equiv 0(2)$ and $\varepsilon = \pm 1$,

$$\mu_2^{2m+n} H_t^{2j} \otimes (1-H) \in K_{Z_2}^{4m+2n}(mH_t \times (2n+\varepsilon)H)$$

and

$$\mu_2^{2m+n+4}(2H_t^{2j}-H_t^{2j+1}) \otimes (1-H) \in K_{Z_2}^{4m+4+2n}(mH_t \times (2n+\varepsilon)H).$$

<u>Proof</u>. As usual (ii) follows from (i) by restriction, lifting of the Real elements. Since H_t is a symplectic bundle, H_t^{2j} is the complexification of a real bundle. The Thom classes represented by $\mu_2^{2m+n}(1 \otimes 1) \in K_{Z_2}^{4m+2n}(mH_t \times 2nH)$, are Real because of the Real equivariant Thom isomorphism theorem for Spin (8k)-representations [3]. In fact, since it is only necessary to determine where each Thom class is Real in $K_{Z_2}^0$ or in $K_{Z_2}^4$, the question of determining which of the two possible dimensions occurs can be resolve by restricting via $\mathbb{C}P^{2t+1} \longrightarrow \mathbb{H}P^t$ and using Lemma 3.10. Finally the Thom class, which is Real, in $K^4(H_t)$ restricts to $\mu_2^2(2-H_t) \in K^4(\mathbb{H}P^t)$. Forming products of

the Real elements described above yields all the Real elements in the statement of (i).

We now compute the elements in Theorem 2.6 for sums of the canonical bundles over $\mathbb{C}P^t$ and $\mathbb{H}P^t$.

Lemma 3.12.

(a) For $E = mH_t \longrightarrow \mathbb{H}P^t$ with $m > t$,

$$(\lambda_{\mathbb{C}}^{2m}(E))^{-1}[\Lambda^e(E)\otimes 1-\Lambda^o\otimes H] =$$

$$= \sum_{j=0}^{t} \binom{m}{j} (H_t-2)^j 2^{2(m-j)-1} \otimes (1-H).$$

(b) For $E = mH_t \longrightarrow \mathbb{C}P^t$ with $m > t$,

$$(\lambda_{\mathbb{C}}^{m}(E))^{-1}[\Lambda^e(E)\otimes 1-\Lambda^o(E)\otimes H] =$$

$$= \sum_{j=0}^{t} \binom{m}{j} H_t^{-m}(H_t-1)^j 2^{m-j-1} \otimes (1-H).$$

Proof.

(a) This follows from the equations,

$$\Lambda^e(E)-\Lambda^o(E) = (\Lambda^e(H_t)-\Lambda^o(H_t))^m = (2-H_t)^m = 0 \in K^o(\mathbb{H}P^t)$$

and

$$\Lambda^e(E)+\Lambda^o(E) = (2+H_t)^m.$$

(b) Is equally simple.

From 2.6, 3.2, and 3.7-3.12 it is routine to obtain the following results, for which we will only work through one case. We use the notation of [7] and [10].

Theorem 3.13 (Davis-Mahowald [7]). Let $E = pH_k \longrightarrow \mathbb{H}P^k$. If, for some $s \geq 0$ and $0 \leq \epsilon \leq 3$,

$$8s+4(p-m)-\epsilon \subset pH_k \oplus (\mathbb{H}P^k \times \mathbb{R}^{8s})$$

$$\binom{n}{k} \equiv 0 \begin{cases} \text{modulo } 2^{2(k-m)}, \text{ if } (k-m) \text{ is even} \\ \text{modulo } 2^{2(k-m)+2-\epsilon}, \text{ if } \epsilon \neq 0 \text{ and } (k-m) \text{ is odd} \\ \text{modulo } 2^{2(k-m)+1}, \text{ if } \epsilon = 0, (k-m) \text{ odd.} \end{cases}$$

Example 3.14. For $E = 4kH_t \longrightarrow CP^t$, suppose $8\ell \subset E$. Notice that, in any $*$-decomposition in Table 3.7, elements in M_- can only signify the existence of Real elements in $KR^2_{Z_2}$ or $KR^6_{Z_2}$ and these can give rise to two-torsion in $KR^0_{Z_2}$. Hence M_- can be ignored when calculating the image of

$$c : KR^0_{Z_2}(E \times 8\ell H) \longrightarrow K^0_{Z_2}(E \times 8\ell H).$$

By Lemma 3.10, M_+ is Real in $K^0_{Z_2}(E \times 8\ell H)$ in this case and

$$(\text{im } c) \subset \left\{H_t^{-2k} \otimes w, x+x^* \,\middle|\, w \in R(Z_2), x \in K^0_{Z_2}\right\} \subset K^0_{Z_2}(E \times 8\ell H).$$

Also

$$K^{0}_{Z_2}(E \times 8\ell H) \longrightarrow K^{0}_{Z_2}(E) \cong K^{0}(\mathbb{C}P^{t}) \otimes R(Z_2)$$

has image

$$(1-H)^{4\ell} K^{0}_{Z_2}(E) = 2^{4\ell-1} K^{0}(\mathbb{C}P^{t}) \otimes (1-H).$$

From §§2.6 and 3.12, if $8\ell \subset E$ then each coefficient of

$$\sum_{j=0}^{t} \binom{4k}{j} 2^{4k-j-1} (H_t-1)^{j} \in K^{0}(\mathbb{C}P^{t})$$

is congruent to zero modulo $2^{4\ell-1}$. For example,

$$\binom{4k}{t} \equiv 0 \pmod{2^{t-4(k-\ell)}}.$$

The computation of §3.14 is typical of those required to prove the following theorem.

<u>Theorem</u> 3.15. Let $E = pH_t \longrightarrow \mathbb{C}P^{t}$.

If, for some $s \geq 0$ and $\epsilon = 0$ or 1,

$$8s+2(p-m)-\epsilon \subset pH_t \oplus (\mathbb{C}P^{t} \times \mathbb{R}^{8s})$$

then

$$\binom{p}{t} \equiv 0 \text{ modulo } 2^{t-m}.$$

4. Non-immersion results for real projective spaces

In this section I will outline briefly a programme for obtaining restrictions on the geometric dimension of $mH_t \longrightarrow \mathbb{R}P^t$. This programme is speculative in that, although the strategy is clear the technical details are not yet optimal. However, there is reason to believe that the following programme will yield very efficient non-immersion results. I will describe the method associated with $\mathbb{H}P^t$, although $\mathbb{C}P^t$ could also be used.

Let $E_1 \equiv pH_t \longrightarrow \mathbb{H}P^t$ and let $E_2 = 4pH_t \longrightarrow \mathbb{R}P^{4t+3}$. Let $f : \mathbb{R}P^{4t+3} \longrightarrow \mathbb{H}P^t$ be the canonical fibring. Consider the following diagram.

(4.1)
$$\begin{array}{ccc} & & KR^O_{Z_2}(E_1 \times nH) \xrightarrow{\ \beta\ } K^O_{Z_2}(E_1) \\ & & \downarrow f^* \\ KR^O_{Z_2}(E_2 \times (n+r)H) & \xrightarrow[\alpha]{} & KR^O_{Z_2}(E_2 \times nH). \end{array}$$

Let $s_1, \ldots, s_n$ be sections of E_1 and let $\bar{s}_1, \ldots, \bar{s}_{n+r}$ be sections of E_2, where <u>no relation</u> between the s_i and the $\bar{s}_i$ is assumed. The complex (2.1) defines an element, S, of $KR^O_{Z_2}(E_1 \times nH)$ and an element, $\bar{S}$, of $KR^O_{Z_2}(E_2 \times (n+r)H)$. The element γ (for example, restrictions on the k-divisibility of $\psi^k\gamma$) may be obtained. The equation $\beta(S) = e(E_1)$ and restrictions on γ lead to bounds for $(n+r)$ whose efficiency depends on the efficiency of the estimates of 3, which seems to be very good (c.f. [7]).

The programme sketched above will be developed elsewhere [12].

EMMANUEL COLLEGE
CAMBRIDGE

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SOME HOMOTOPY GROUPS MOD 3

Martin Tangora

Introduction

This is a preliminary report of some calculations of the 3-primary component of the stable homotopy groups of spheres. Oka has published these groups as far as the 76-stem, and I extend the range to 103.

The prime 3 has some special interest because many of the powerful new methods do not apply: see the talks by Smith, Toda and Zahler.

One reason for stopping at 103 is that several difficult questions arise in the 104-stem: is $\beta_2^4 = 0$? is $\beta_1^3\beta_5 = 0$? Another reason is that that range happens to be self-contained by my methods, which is rather unusual: one so often needs information from higher stems to settle questions in lower ones.

In the 106-stem another interesting question arises, which might be thought of as the mod 3 analogue of the θ_j question mod 2, namely, is $\langle h_2, h_2, h_2\rangle$ a permanent cycle in the Adams spectral sequence? I hope this will be tractable but I do not have any information yet.

During and after the meeting I had communications from Toda, Oka, and Osamu Nakamura reporting that Nakamura has also calculated these homotopy groups, stopping at the same dimension. I am pleased to report that our proofs are sometimes different but our results are always the same.

My original motivation for this work was an endeavor to prove Toda's "important relation" $\alpha_1\beta_1^3 = 0$ by an Adams spectral sequence argument. I have not had any success, and a helpful conversation with J. Frank Adams has made me wonder whether it was a realistic program.

This research was done at Oxford's Mathematical Institute. I am deeply indebted to the Science Research Council, to the University of Oxford, and to Professor Ioan James for their support and hospitality.

Methods

The results are obtained by means of the classical Adams spectral sequence, i.e., for ordinary (co-)homology.

J. Peter May, in his 1964 Princeton thesis, showed how to calculate the additive structure of the E_2 term, the cohomology of the mod 3 Steenrod algebra, and he gave the calculation through the 88-stem. Those results contain contradictions to Toda's important relation, but May corrected them and greatly extended them around 1966. I am greatly indebted to May for mailing me an extra copy of his unpublished revision, which presents the initial term of the May spectral sequence mod 3 as far as the 177-stem. This was the basis for my calculations of the additive structure of E_2.

Not having access to May's revision, Nakamura went over most of the same ground independently, and obtained the additive structure of E_2 (mod 3) through the 158-stem. I have not found any discrepancies.

For the multiplicative structure of E_2, including Massey products (and matrix Massey, or messy, products), I have been using the mod 3 lambda algebra, as revised and improved by Bousfield and Kan ([2], especially pp. 101-102). (The original version [3] had a mistake in sign, and was not well arranged for calculation.) The usual computational tricks for the mod 2 lambda algebra [11] carry over to odd primes, mutatis mutandis. The resulting algebra is too big to use for the additive structure of E_2 but I have found it invaluable in retrieving products.

After the meeting Sholom Rosen wrote me about a new approach which may be even better for this than the lambda algebra, but I have not yet put it to work.

The differentials which are implied by the solution of the Hopf Invariant One Problem, by Adams' calculations of the image of J, and by Toda's relation will be taken for granted, and then all other differentials are obtained independently. This means in particular that I have reproduced all of Oka's calculations for stems through 76, and verified his results by arguments which are in a profound sense essentially the same as his, but are quite different formally.

Whenever possible I prefer to prove differentials by the most elementary means: simple manipulation of the multiplicative structure of E_2. This requires extensive knowledge of that structure. I often resort to Moss's theorems but rarely if ever to Steenrod operations in Ext. The philosophy is that the fewer fancy tools you need in your argument, the more confidence you have in your result.

Generators for homotopy groups

Here is a list of the 3-primary stable stems from 77 to 103. I write Π_n for the 3-component of the stable n-stem. The sign $\doteq$ will be used to denote equality up to sign, in order that mod 2 specialists will not be put off by the presence of non-zero scalar coefficients.

Notation is the ancient curse of the subject; what seems like a sensible notation at one stage of development becomes unreasonable at another stage. I have followed the notation of Toda and Oka for homotopy elements as far as possible, and for E_2 my notation is essentially that of May's unpublished revision.

The following stems contain nothing of order 3:

77, 80, 88, 89, 96, 97, 98.

The following contain only elements in the image of J:

79, 83, 87, 103.

Other stems:

$\pi_{78} = Z_3$ generated by $\beta_2^3 \doteq \beta_1\lambda \doteq \alpha_1\mu$.

$\pi_{81} = Z_3+Z_3$. One survivor in filtration $s = 3$, from

$$f_2 = h_2 \begin{matrix} h_1 & b_1 \\ -h_2 & b_0 \end{matrix} \quad \text{(where } b_i = \langle h_i, h_i, h_i \rangle\text{)},$$

giving a generator which I call γ; the other in $s = 4$, $\langle \alpha_1, \alpha_1, \beta_5 \rangle$, which I call μ_2 by (weak) analogy with Oka's $\mu = \langle \alpha_1, \alpha_1, \lambda \rangle$.

$\pi_{82} = Z_3$. The survivor is $a_0^2\langle h_1, h_2, h_2 \rangle$ and it seems hard to write as a Toda bracket. Let me call it ν (for "new").

$\pi_{84} = Z_3+Z_3$ with generators $\alpha_1\gamma$ and $\alpha_1\mu_2 \doteq \beta_1\beta_5$.

$\pi_{85} = Z_3+Z_3$. One generator in filtration 6, which can be represented by $\langle \beta_1, 3\iota, \beta_5 \rangle$ and possibly (I doubt it) by $\alpha_1\nu$. Call it ϵ_2' because of its resemblance to $\epsilon' \doteq \langle \beta_1, 3\iota, \beta_2 \rangle$. The other generator is $\beta_1\mu$.

$\pi_{86} = Z_3$ generated by ε_{121} - analogue of ε_1 in π_{38} (mod 3) and corresponding to the element discussed by Zahler in his talk (for primes ≥ 5). It can be expressed as $\langle \alpha_1, 3\iota, \nu \rangle$.

$\pi_{90} = Z_3$ generated by ε_{221} (following Zahler's notation), alias β_6, alias $\langle \alpha_1, 3\iota, \varepsilon_{121} \rangle$.

$\pi_{91} = Z_3+Z_3+Z_3$ with generators $\beta_1\gamma$, $\beta_1\mu_2$, and α_{23}.

$\pi_{92} = Z_3+Z_3$. One generator in filtration $s = 4$, indecomposable, probably $\langle \beta_1, 3\iota, \gamma \rangle$. The other in $s = 7$, identified with $\langle \beta_1, \alpha_2, \beta_5 \rangle$. But I haven't proved yet that $\beta_1\pi_{82} = 0$, which leaves these Toda brackets in limbo.

$\pi_{93} = Z_9$. This is only the third element of order 9 in Coker J; the others were Toda's φ in π_{45} and Oka's μ in π_{75}. Call the generator φ_2; it lies in $\langle \alpha_1, \alpha_1, \varepsilon_{121} \rangle$ and in $\langle \alpha_1, \alpha_2, \nu \rangle$. Moreover $3\varphi_2 \doteq \alpha_1\beta_6 \doteq \alpha_2\varepsilon_{121} = \alpha_3'\nu$. The analogy with φ is striking.

$\pi_{94} = Z_3+Z_3$. One generator is $\alpha_1\beta_1\gamma \doteq \beta_2\lambda \doteq \langle \varphi, \alpha_1, \varphi \rangle$ (I must warn you about these relations: see the proof below that $\beta_2^5 = 0$). The other is $\beta_1^2\beta_5 \doteq \alpha_1\beta_1\mu_2$.

$\pi_{95} = Z_3 + Z_9$ generated by $\beta_1 \epsilon_2'$ and by α_{24}'.

$\pi_{99} = Z_3 + Z_3$. One survivor in $s = 5$ appears to contain $\langle \beta_1, \alpha_2, \gamma \rangle$; the other is α_{25}.

$\pi_{100} = Z_3$ generated by $\beta_2 \beta_5$.

$\pi_{101} = Z_3 + Z_3$ generated by $\beta_2 \mu$ and $\beta_1^2 \mu_2$. With the right choice of γ you have $\beta_1^2 \gamma = \beta_2 \mu$.

$\pi_{102} = Z_3 + Z_3$ isomorphic with π_{92} under multiplication by β_1; the element at $s = 6$ is also divisible by α_1.

Two key relations

Some important relations are implicit in the preceding table.

For one, the assertion $\pi_{77} = 0$ implies the relation $\alpha_1 \beta_5 = 0$, which was Oka's stopping place. This is equivalent to the Adams differential $d_3(q) = h_0(Ac)$, where (Ac) is the survivor in $s = 5$, $t-s = 74$ corresponding to β_5, and $q = \langle a_0, h_0, h_1, h_2, c \rangle$, $c = \langle h_0, h_1, h_1 \rangle$ ($\{c\} = \beta_2$ in π_{26}). (The long Massey product given for q is an example of a fruit of the lambda algebra.) I found this differential the most elusive in the entire range (not counting those which were taken for granted, as explained above under Methods).

An outline of proof is as follows:

$$\alpha_1\beta_1\beta_5 \doteq \{h_0b_0(Ac)\} \doteq \{c(Mh_0)\}$$

where all you need to know about (Mh_0) is that $d_6(Mh_0) \doteq b_0^6$; hence we can show that $\alpha_1\beta_1\beta_5 \doteq \langle\beta_1^3,\beta_1^3,\beta_2\rangle$. Multiplying once more by β_1 we get

$$\alpha_1\beta_1^2\beta_5 \doteq \langle\beta_1^4,\beta_1^3,\beta_2\rangle \doteq \epsilon'\langle\alpha_1,\beta_1^3,\beta_2\rangle \in \epsilon'\pi_{60} = 0$$

But in the spectral sequence the vanishing of $\alpha_1\beta_1^2\beta_5$ is clearly equivalent to that of $\alpha_1\beta_5$; in other words, $d_3(b_0^2q) \neq 0$ if and only if $d_3(q) \neq 0$.

I remark that Nakamura's proof of this relation is different in detail, but similar in spirit: he also proves first that $\alpha_1\beta_1^2\beta_5 = 0$ but by means of a different Toda bracket for $\alpha_1\beta_1\beta_5$, namely $\langle\varphi,\alpha_4,\beta_1\rangle$.

One can also show that $\langle\beta_1^3,\beta_1^3,\beta_2\rangle = 0$ by means of Toda's relation $\beta_1^3 \doteq \langle\alpha_1,3\iota,\beta_2\rangle$.

The assertion $\pi_{88} = 0$ contains the relation $\beta_1\beta_2^3 = 0$, which is equivalent to $d_4(h_1q) = c^3b_0$. One deduces $d_4(h_1q) \neq 0$ from $d_4(g_2q) \neq 0$ in a straightforward way (here $g_2 = \langle h_0,h_0,h_1\rangle$). The latter differential is obtained by showing that $\beta_1^2\mu = 0$ in π_{95} by means of some Toda bracket manipulations, as follows. First,

$$\langle\beta_1,\epsilon_1,\beta_2\rangle \doteq \mu \bmod 3\mu$$

since

$$3\langle\beta_1,\epsilon_1,\beta_2\rangle \doteq \beta_1\langle\epsilon_1,\beta_2,3\iota\rangle \doteq \epsilon_1\langle\beta_2,3\iota,\beta_1\rangle \doteq \epsilon_1\epsilon' \doteq 3\mu \neq 0.$$

Then

$$\begin{aligned}\beta_1^2\mu &\doteq \beta_1^2\langle\beta_1,\epsilon_1,\beta_2\rangle \pmod 0\\ &\doteq \langle\beta_1^3,\epsilon_1,\beta_2\rangle\\ &\doteq \langle\langle\beta_2,3\iota,\alpha_1\rangle,\epsilon_1,\beta_2\rangle\\ &\doteq \langle\beta_2,\epsilon_2,\beta_2\rangle \pm \langle\beta_2,3\iota,0\rangle = 0 \bmod 0.\end{aligned}$$

This shows that the d_4 in question is non-zero.

On the height of β_2

Recall that Toda showed $\beta_i^{p+1} = 0$ for all $i \geq 2$ and for all primes $p \geq 5$ ([10], Theorem 5.8). For $p = 3$ he did not get a sharp result by those methods. We can easily see that for $p = 3$, $\beta_2^6 = 0$, since it is clear in π_{78} that β_2^3 is divisible by α_1, and $\alpha_1^2 = 0$.

We can sharpen this to $\beta_2^5 = 0$ as follows. We don't know much about π_{130} (where β_2^5 occurs), but we can show that in π_{107} the only element which is not in the kernel of $\alpha_1\beta_1^2 : \pi_{107} \longrightarrow \pi_{130}$ is $\beta_2\mu_2$. (I ask the reader's faith for this, as I do not wish to over-expose my partial results beyond 103.)

Note that $\beta_2\gamma$ is only defined modulo $\beta_2\mu_2$, since γ has only been defined modulo μ_2. Thus for a certain choice γ_0 of γ we will have $(\beta_2\gamma_0)(\alpha_1\beta_1^2) = 0$ in π_{130}.

The main idea now is to factor β_2^5 into the above expression. Write

$$\beta_2^5 = \beta_2^2\beta_2^3 \doteq \langle\alpha_1,\alpha_1,\varphi\rangle\beta_2^3 = \alpha_1\langle\alpha_1,\varphi,\beta_2\rangle\beta_2^2.$$

Since $\alpha_1\beta_2^2 \doteq \beta_1\varphi$,

$$\beta_2^5 \doteq \beta_1\varphi\langle\alpha_1,\varphi,\beta_2\rangle = \beta_1\langle\varphi,\alpha_1,\varphi\rangle\beta_2.$$

Now $\langle\varphi,\alpha_1,\varphi\rangle$, which is in π_{94} (mod 0), can be shown to be an element of $\left\{cb_1^2\right\} = \left\{h_0b_0f_2\right\}$, which also contains $\alpha_1\beta_1\gamma$ (for all choices of γ) and so

$$\langle\varphi,\alpha_1,\varphi\rangle \doteq \alpha_1\beta_1\gamma_0 \bmod \beta_1^2\beta_5$$

and so finally

$$\begin{aligned}\beta_2^5 &\doteq \beta_1(\alpha_1\beta_1\gamma_0)\beta_2 \quad (\text{mod } \beta_1^3\beta_2\beta_5 = 0)\\ &= 0.\end{aligned}$$

There still remains the question whether $\beta_2^4 = 0$ in π_{104}. This seems difficult and I remain neutral. Even the corresponding question in E_2, whether $c^4 = 0$, is slippery, but Nakamura has proved that $c^4 \neq 0$.

A conjecture

Recall that Smith's proof that β_i is essential for all i ([8], §5) holds only for primes $p \geq 5$. This leaves open the question of the β_i for $p = 3$.

Oka observed that $\pi_{58} = 0$ and thus $\beta_4 = 0$, but he found β_5 in π_{74}, and Nakamura and I have found β_6 in π_{90}. What about the rest?

In the May spectral sequence it leaps to the eye that for $i = 2,3,4$, β_{i+3} arises from β_i through multiplication by an element which May calls a_2 (and I call A), which is not an element of E_2 but which may be regarded as an operator there.

From the point of view of the Adams spectral sequence, β_4 fails to be essential because the candidate for β_4, namely $a_0^2 e_1 = a_0^2 \langle h_1, h_1, h_2 \rangle$, is not a cycle: $d_3(a_0^2 e_1) \doteq h_0 b_0^2 b_1$. Now the candidate for β_7 in the 106-stem bears a close resemblance to $A(a_0^2 e_1)$, and it appears plausible that d_3 of this element should be non-zero, hitting $A(h_0 b_0^2 b_1)$.

This leads me to conjecture that $\beta_7 = 0$. In the absence of any evidence to the contrary, I will go ahead and conjecture that for $p = 3$ we have $\beta_{3i+1} = 0$ for all $i \neq 0$.

Reading E_∞ horizontally

Instead of inspecting the E_∞ term of the Adams spectral sequence stem by stem, we can inspect it row by row. Joel Cohen has ventured to conjecture that only a finite number of elements will survive in any given row, i.e., for any given Adams filtration.

The most interesting question seems to be that mentioned in the introduction, whether $b_i = \langle h_i, h_i, h_i \rangle$ (analogue of θ_j ?) is a surviving cycle. I don't know yet about b_2 in the 106-stem; recall that the differential on b_1 in the 34-stem was the old problem finally resolved by Toda in 1967 [9]. If b_2 survives, so also will $h_0 h_3$ in the 110-stem (just as, for $p = 2$, θ_j gives rise to η_j).

On the next page we give a table of those elements of E_∞ which correspond to indecomposable homotopy elements, through π_{103}. The asterisks stand for homotopy elements as yet unnamed. The well-known α family are omitted from this table.

Note the gap from 45 to 68. Remember it whenever you are about to make an optimistic conjecture.

s																	
s = 7															*?		
6												ϵ_2'		ϵ_{221}		φ_2	
5								β_5	μ				ϵ_{121}				*
4							λ			μ_2	ν				*		
3			ϵ'		ϵ_2	φ				γ							
2	β_1	β_2		ϵ_1													
t-s =	10	26	37	38	42	45	68	74	75	81	82	85	86	90	92	93	99

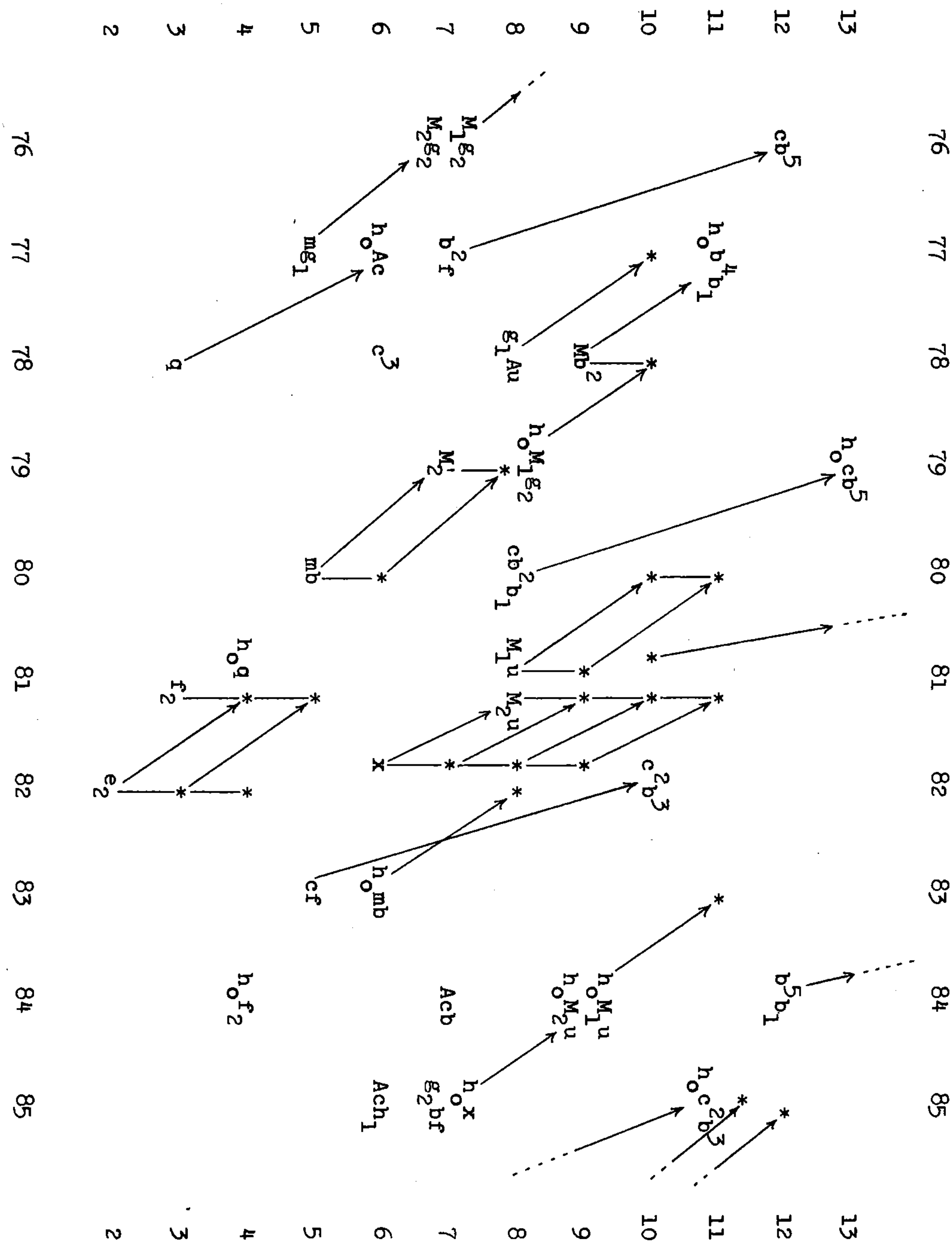
TABLE OF HOMOTOPY-INDECOMPOSABLE ELEMENTS OF E_∞, EXCLUDING Im J; $t-s \leq 103$

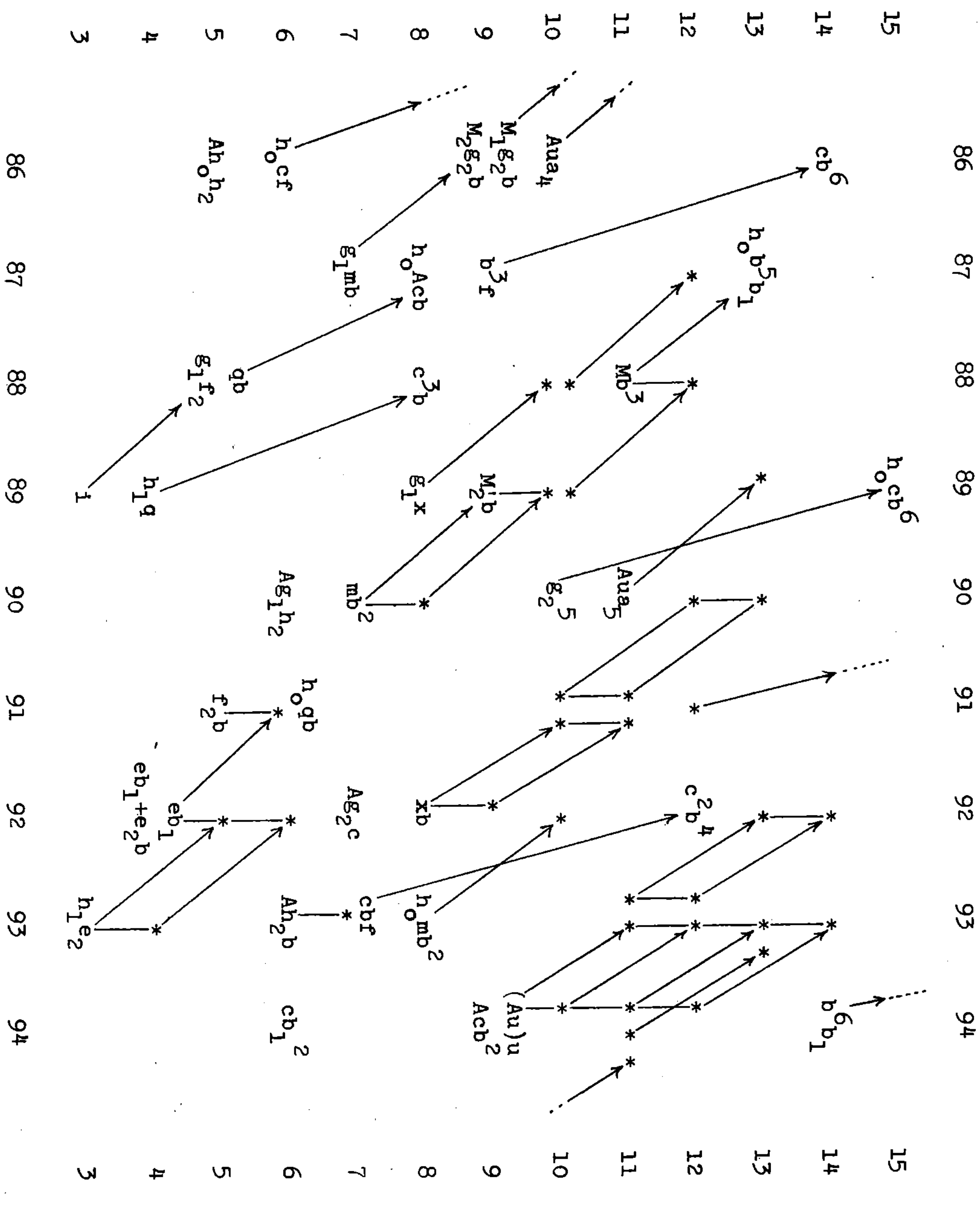
Spectral sequence tables

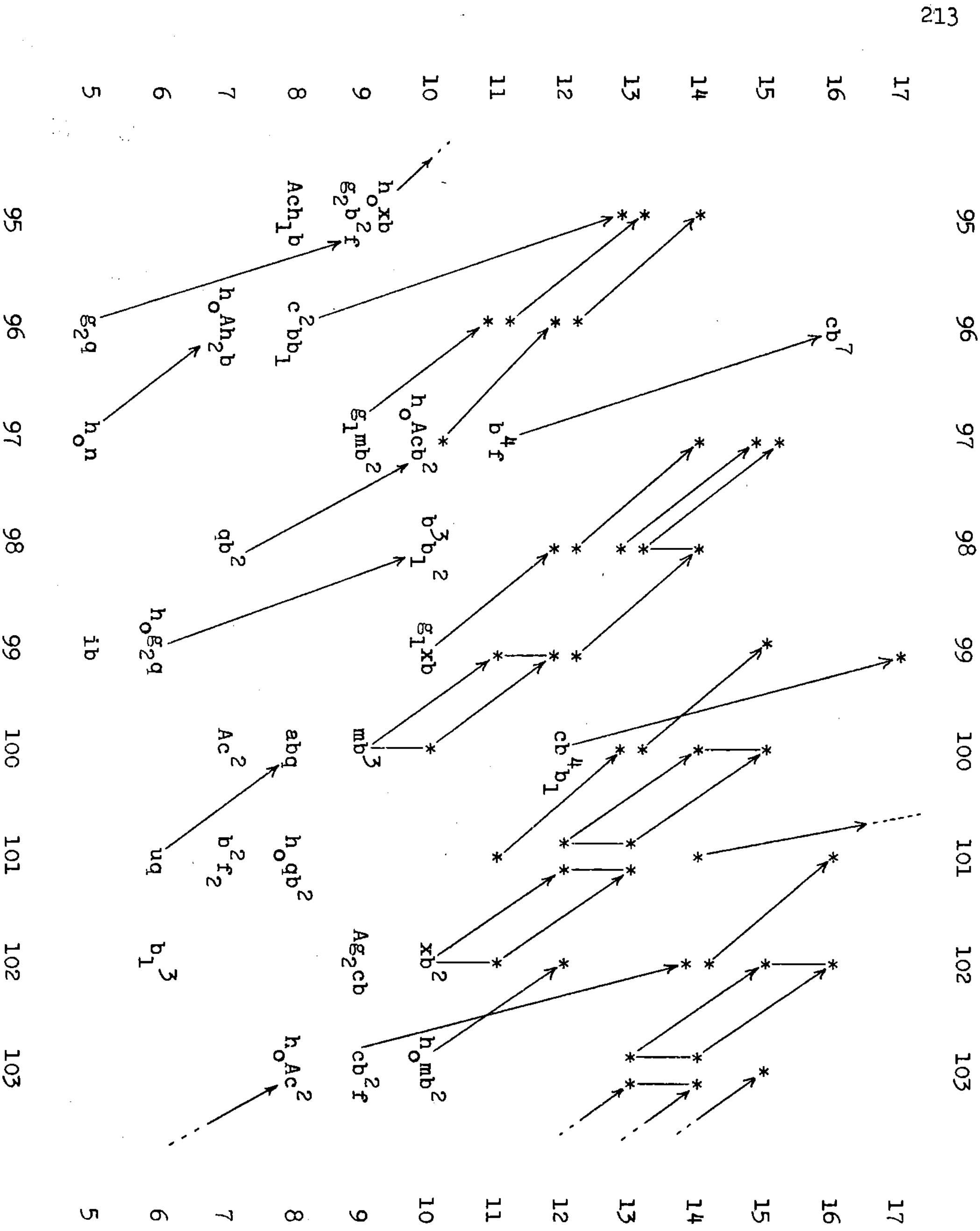
By popular demand, we append tables of the Adams spectral sequence for $p = 3$ in the range $76 \leq t-s \leq 103$. As usual, these are essentially tables of E_2 with all the differentials drawn in as arrows. Vertical lines connect elements of E_2 which are related by multiplication by a_0. If a differential goes off the edge, its degree is given by the number of dots at the end of the arrow.

Only low filtration degrees are shown, to save space. Above these degrees, the behavior of E_2 in this range of $t-s$ is regular and predictable. Moreover, nothing survives to E_∞ in filtration $s > 9$ $(t-s \leq 103)$ except β_1^5 in π_{50} and the α families in Im J.

A glossary is also included to help in interpreting the names of the elements. The algebraic structure of E_2 still seems very messy. In the tables I have left some elements unnamed, both to reduce crowding and to evade the troublesome business of choosing and explaining a notation. Unnamed elements are represented by asterisks.



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g1mb2
h0Acb2
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abq
Ac2
cb4b1
uq
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h0qb2
b1 3
Ag2cb
xb2
cb2f
h0mb2
h0Ac2

Glossary

This is a brief guide to the elements in the E_2 tables. The arrangement is alphabetical rather than logical. Each indecomposable of E_2 (except the fundamental elements h_i) has been given as a Massey product. Entries in parentheses are operators in E_2, not elements thereof; the Massey product forms given for some of them are not necessarily always valid, but are correct for the cases occurring in the tables.

The sign "=" should always be read "$\doteq$", i.e., equal up to sign.

a_0 arises from τ_0 and corresponds to multiplication by 3

(a) is May's a_1, arising from τ_1^3 (with much abuse of language)

(A) is May's a_2, arising from τ_2^3 (ditto)

a_4, a_5: In general a_i for $i \geq 3$ corresponds to α_i in π_{4i-1}

$b = b_0$; $b_i = \langle h_i, h_i, h_i \rangle$

$c = \langle h_0, h_1, h_1 \rangle$

$e = e_1 = \langle h_1, h_1, h_2 \rangle$; $e_2 = \langle h_1, h_2, h_2 \rangle$

$$f = f_1 = \left\langle h_1 \begin{matrix} h_1 & b_1 \\ -h_2 & b \end{matrix} \right\rangle ; \quad f_2 = \left\langle h_2 \begin{matrix} h_1 & b_1 \\ -h_2 & b \end{matrix} \right\rangle$$

$g_1 = \langle a_0, h_0, h_0 \rangle$; $g_2 = \langle h_0, h_0, h_1 \rangle$

h_i arises from the Steenrod reduced power P^{3^i}

$i = \langle h_0, h_0, e_2 \rangle$

$(m) = \langle\ , a_0 b_1, h_2 \rangle$

$(M) = \langle\ , a_0^2, a_0 f \rangle$

(M_1) = multiplication by (A)b in the May spectral sequence

(M_2) = multiplication by (a)(B) in the May spectral sequence, where (B) is May's b_2^0 and may be thought of as $\langle \xi_2, \xi_2, \xi_2 \rangle$

$M_2' = \langle b, a_0 b_1, a_0 b_1 \rangle$ (and $a_0 M_2' = (M_2) g_2 h_0$)

$(n) = \langle\ , a_0, a_0 h_2 e \rangle$

$q = \langle a_0, h_0, h_1, h_2, c \rangle$

$u = \langle a_0, h_0, h_1, g_1 \rangle$

$x = \langle a_0, h_2 b_1, a_0, a_0^2 h_1 \rangle$

UNIVERSITY OF OXFORD and

UNIVERSITY OF ILLINOIS at Chicago Circle

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CONSTRUCTING SOME SPECTRA

Hirosi Toda

We start from the following

Problem 1. Given integers $n, a > 0$, does there exist spectrum $E = E_p$ for each prime p and a constant c (independent of p) satisfying the followings?

$$\#\text{ of generators of } \sum_{*<ap^n} \pi_*(E)\otimes Z_p < c$$

and

$$\#\text{ of generators of } \sum_{*<ap^n} H_*(E)\otimes Z_p < c.$$

As is easily seen the spectra $E = S^0$, $H_p = K(Z_p)$, BP do not give any answer of the above problem. However, for the case $n = 1$, an easy answer is given by taking $E = V(1)$ or the following $H_p(1)$. We define

$$H_p(1) = \text{fibre of } P^1 : H_p \longrightarrow \Sigma^q H_p = R_1^0\cdot H_p$$

and

$$BP(1) = \text{fibre of } r_1 : BP \longrightarrow \Sigma^q BP = R_1^0\cdot BP,$$

where $q = 2(p-1)$, R_i^j is a symbol having bidegree $(-1, 2p^j(p^i-1))$, r_A is the Landweber-Novikov operation and P^A

is a cohomology operation dual to that of Milnor's (having a similar action to homology to r_A). Then $\pi_*(H_p(1))$ has only two generators 1 and $h = \{R_1^0\}$ of degree 0 and q-1, and

$$H_*(BP(1);Z_p) \cong Z_p[\xi_1^p,\xi_2,\xi_3,\ldots] \otimes \Lambda(\psi_1^0),$$

$$H_*(H_p(1);Z_p) \cong H_*(BP(1);Z_p) \otimes \Lambda(\tau_0,\tau_1',\tau_2',\ldots)$$

where $H_*(H_p;Z_p) = Z_p[\xi_1,\xi_2,\ldots] \otimes \Lambda(\tau_0,\tau_1,\ldots)$ by Milnor, $\tau_i' = \tau_i - \tau_0\xi_i$, $\psi_1^0 = [R_1^0 \cdot \xi_1^{p-1}]$ of degree pq-1 and $H_*(BP;Z_p) = Z_p[\xi_1,\xi_2,\ldots]$ for the mod p reduction ξ_i of $m_i \in H_*(BP) = Z_{(p)}[m_1,m_2,\ldots]$, deg ξ_i = deg $m_i = 2(p^i-1)$, deg $\tau_i = 2p^i-1$. The element ψ_1^0 is detected by the secondary operation Ψ_1^0 associated with the relation $P^{p-1}P^1 = 0$. Taking p sufficiently large $(pq-2 > ap)$ we see that $H_p(1)$ gives an answer to the problem for n = 1. Also we may regard that the spectrum V(1) is the (pq-2) skeleton of $H_p(1)$ for $p \geq 3$.

Now our problem is to construct spectra of such a sort $H_p(n)$, $BP(n)$ for $n = 2,3,\ldots$. We consider

<u>Problem</u> 2. Given positive integer n, can we construct a chain complex C(n) satisfying the following conditions? For $X = H_p$ or BP

$$C(n) = \Lambda(R_i^j; i+j \leq n) \otimes X^*(X),$$

for monomials x, y in $\Lambda(R_j^i)$, $\partial(x \otimes 1) = \Sigma y \otimes f_{x,y}$

$$f_{x,y} \equiv \begin{cases} -1 & \text{if } x = zR_i^j \text{ and } y = zR_k^j R_{i-k}^{j+k} \\ P_i^j \text{ or } r_i^j & \text{if } y = xR_i^j;\ P_i^j = P^A,\ r_i^j = r_A \text{ for } A = p^j \Delta_i, \\ 0 & \text{otherwise} \end{cases}$$

modulo p and higher terms in the subalgebra generated by r_i^j's.

The chain complex is represented by a sequence

$$C(n) : X_0 = X \xrightarrow{\partial_0} X_1 \xrightarrow{\partial_1} X_2 \longrightarrow \cdots \longrightarrow X_{\binom{n+1}{2}} = X$$

where X_r is the product (wedge) of $x \cdot X$ for monomials x in $\Lambda(R_i^j)$ of the length r.

We denote by $H_p(n)$ resp. $BP(n)$ a fibre (tower, relaization or desuspension of iterated cones) of the above sequence $C(n)$ if it exists.

<u>Lemma</u>.

(i) Assume the existence of $C(n)$. Then there exists $BP(n)$ for $p \geq \frac{1}{4}(n^2+n+2)$ and $H_p(n)$ for $p \geq \frac{1}{4}(n^2+3n+4)$ or $p = 3$, $n = 2$. They are unique if the inequalities hold.

(ii) Assume the existence of $BP(n)$. Then there exists a spectral sequence:

$$E_2 = H_*(C(n); Y_*(BP)) \Longrightarrow Y_*(BP(n)),$$

which collapses if $p > \frac{1}{4}(n^2+n+2)$ and if $Y = S, H, H_p, V(m)$ or $= BP$. Similar spectral sequence exists for $H_p(n)$.

In general, a fibre of $C(n)$ exists if $[\Sigma^{k-2} X_r, X_{r+k}] = 0$ for $k \geq 3$. Then the lemma is proved by the fact $BP^*(BP) = 0$ for $* \not\equiv 0 \pmod q$ and also counting the number of Bocksteins in the monomials of $H_p(H_p) = H_*(H_p; Z_p)$ of appropriate degrees.

Corollary. $\pi_*(H_p(n)) = H_*(C(n); Z_p)$, so the number of the generators of $\pi_*(H_p(n))$ is not greater than $2^{\binom{n+1}{2}}$.

In $H_*(BP; Z_p)$ r_i^j acts same as P_i^j. Consider the subalgebra P^* of the mod p Steenrod algebra $H_p^*(H_p)$ spanned by P^A, then the associated graded algebra $E^0(P^*)$ is the envelopping algebra over a Lie algebra mod p spanned by P_i^j with the relation $[P_k^j, P_{i-k}^{j+k}] = P_i^j$. So, modulo p and higher terms, $C(n)$ changes to May's resolution of (non-restricted) Lie algebra $\{P_i^j\}$, and we have

Lemma. There exists spectral sequences:

$$'E_2 = Z_p[\xi_1^{p^n}, \ldots, \xi_n^p, \xi_{n+1}, \ldots] \otimes \Lambda(\psi_i^j; i+j \leq n) \Longrightarrow H_*(C(n); H_*(BP; Z_p))$$

and

$$''E_2 = {}'E_2 \otimes \Lambda(\tau_0, \tau_1', \tau_2', \ldots) \Longrightarrow H_*(C(n); H_*(H_p; Z_p)),$$

where $\deg \psi_i^j = 2p^{j+1}(p^i-1)-1$.

<u>Corollary</u>. If $C(n)$ exists then Problem 1 is affirmative $H_p(n)$.

<u>Note</u> 1. $C(n)$ may be regarded as a sort of (unusual) resolution of $H_p(n)$ or $BP(n)$.

<u>Note</u> 2. Let p be sufficiently large w. r. t. n. If $H_p(n)$ exists and the above associated spectral sequences collapse, then we define a spectrum $VB(n)$ as the $(p^n q-2)$-skeleton of $H_p(n)$:

$$H_*(VB(n);Z_p) \cong \Lambda(\tau_0,\tau_1',\tau_2',\ldots,\tau_n') \otimes \Lambda(\psi_i^j; i+j<n).$$

Similarly, for the $(p^n q-2)$-skeleton $B(\binom{n}{2})$ of $BP(n)$,

$$H_*(B(\tbinom{n}{2});Z_p) \cong \Lambda(\psi_i^j; i+j<n).$$

<u>Note</u> 3. If $V(n)$ and $B(\binom{n}{2})$ exist then $V(n)\wedge B(\binom{n}{2})$ may be regarded as $VB(n)$.

Now we can prove the following

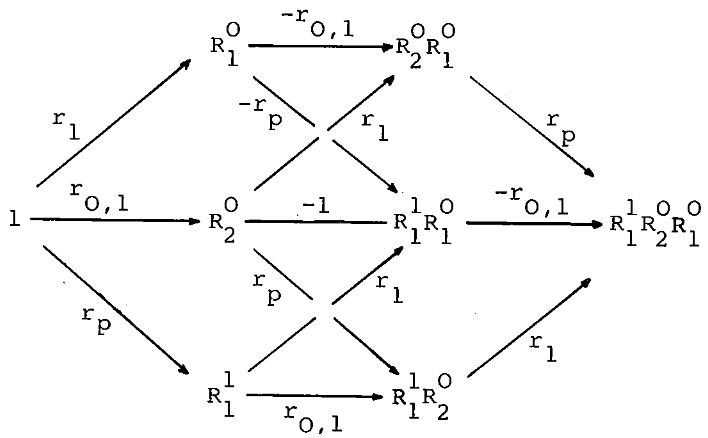
<u>Theorem</u>. For $p\geq 3$, $BP(2)$, $VB(2)$ and $B(3)$ exist. For $p\geq 5$, $BP(3)$, $H_p(3)$ and $VB(3)$ exist.

The main part of the proof is the construction of $C(2)$ and $C(3)$. If $C(n)$ is constructed for $X = BP$, it is also constructed for $X = H_p$ just by changing r_i^j by P_i^j. So we construct $C(2)$ and $C(3)$ for $X = BP$ only.

First consider the case $n = 2$. Since $r_1 = r_1^0$, $r_p = r_1^1$ and $r_{0,1} = r_2^0$ enjoy the relations

$$[r_1, r_p] = r_{0,1} \quad \text{and} \quad [r_1, r_{0,1}] = [r_p, r_{0,1}] = 0,$$

C(2) is defined by the formulas in Problem 2 without taking modulus, that is, it is represented by the following diagram (replacing $x \cdot BP$ by x):



Note 4. BP(1) is mod p equivalent to

$$S^0 \cup_{\beta_1} e^{pq-1} \cup_p e^{pq} \cup_{\alpha_1} e^{(p+1)q} \cup e^{2pq-1} \cup \cdots$$

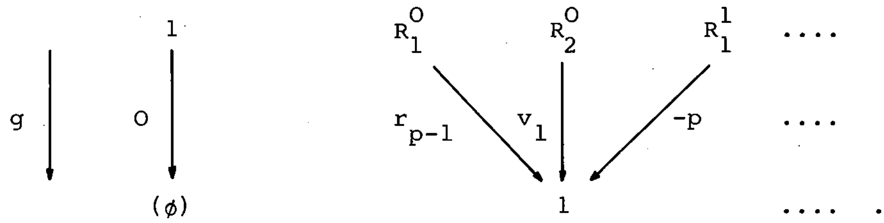
where $P^p(S^0) = e^{pq}$ and $P^1(e^{pq}) = e^{(p+1)q}$ in mod p cohomology.

BP(2) is mod p equivalent to

$$S^0 \cup_{\beta_1} e^{pq-1} \cup_{\alpha_1\beta_1^{p-1}} e^{p^2q-1} \cup_p e^{p^2q} \cup e^{(p^2+p)q-2} \cup e^{(p^2+p)q-1} \cup \cdots$$

where $P^{p^2}(S^0) = e^{p^2 q}$.

Note 5. There exists a chain map $g : C(2) \longrightarrow \Sigma^{pq} C(2)$ of degree 1 such as



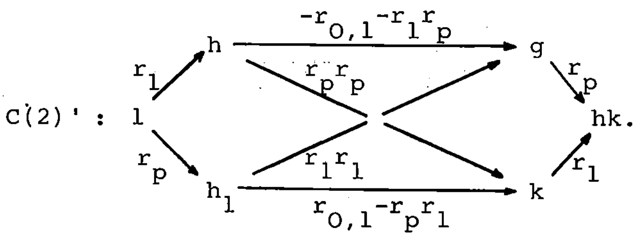
Moreover $gg = 0$. This induces maps $g : BP(2) \longrightarrow \Sigma^{pq-1} BP(2)$ and $g : H_p(2) \longrightarrow \Sigma^{pq-1} H_p(2)$ such that $g_*(\phi^0_1) = 1$.

Note 6. By considering a fibre of the sequence

$$H_p(2) \xrightarrow{g} \Sigma^{pq-1} H_p(2) \xrightarrow{g} \Sigma^{2(pq-1)} H_p(2) \longrightarrow \dots ,$$

we can obtain $V(2)$ for $p > 3$, since g kills ϕ^0_1. But this breaks for $p = 3$ and we obtain $V(1\frac{1}{2})$ only.

Note 7. By a chain equivalence R^0_2 and $R^1_1 R^0_1$ are cancelled, and we obtain an equivalent chain complex



Moreover, combining with the reduced form of g, as in Note 6, we obtain a BP-resolution of S^0 up to degree p^2q-2 which is essentially same as "BP-relation" of Thomas-Zahler used in proving the non-triviality of some γ_i.

Finally we consider the case $n = 3$. The operations $\{r_i^j; i+j \leq 3\}$ are no more closed under $[\ ,\ ]$:

$$[r_1, r_{p^2}] = r_{p^2-p} r_{0,1},\ [r_p, r_{p^2}] = r_{0,p} + f r_{0,1},\ [r_{0,1}, r_{p^2}] = r_{0,0,1},$$

$$[r_1, r_{0,p}] = r_{0,0,1},\ [r_p, r_{0,p}] = r_{p-1} r_{0,0,1},\ [r_{0,1}, r_{0,p}] = 0,$$

$$[r_{p^2}, r_{0,p}] = (r_{p^2-1} - r_{0,p-1}) r_{0,0,1} \quad \text{and} \quad [r_i^j, r_{0,0,1}] = 0 \quad \text{for}$$

$i + j < 3$,

where f is uniquely determined by the second equality (explicitly: $f = -r_{p^2-1} + \sum_{i=0}^{p-2} \frac{(-1)^i}{i+1} r_{p-i-1} r_{p^2-pi-p} r_{0,i}$).

However we can construct $C(3)$ by taking $f_{x,y} = -1$ or r_i^j for the first two cases in Problem 2 and modigying $f_{x,y}$ for the third case as follows. If $g r_m^n$ is a term of above $[r_i^j, r_k^\ell]$ then

$$f_{x,y} = -g \quad \text{for} \quad x = zR_m^n,\ y = zR_k^\ell R_i^j \quad (R_i^j, R_k^\ell, R_m^n \notin z).$$

In order to complete the definition of ∂ in $C(3)$, we must add two more extra cases:

$$f_{x,y} = \begin{cases} -r_{p^2-p1} & \text{for } x = zR_3^0R_2^0,\ y = zR_1^2R_2^1R_1^0 \quad (x = 1, R_1^1) \\ -f' & \text{for } x = zR_3^0R_2^0,\ y = zR_1^2R_2^1R_1^1 \quad (x = 1, R_1^0), \end{cases}$$

where f' is determined by

$$f'r_{0,1} = [r_{p-1}, r_{p^2}] - [r_p, r_{p^2-1}].$$

To check the condition $\partial^2 = 0$, we need various relations in $[\ ,\ ]$. For example, between $R_3^0R_2^0$ and $R_1^2R_2^1R_1^1R_1^0$ there are 10 monomials connected by non-trivial maps. They are cancelled by $[1,f] = [1,r_{p^2-1}] = 0$ and $[f',r_1] = [r_{p-1}, r_{p^2-p}] + [r_p, r_{p^2-p-1}]$.

Consequently we can construct $C(3)$, and then $H_p(3)$ and $BP(3)$ for $p > 3$ by applying the first lemma. For $p = 3$ and $X = BP$, $C(3)$ is realized except the last term, then $B(3)$ is obtained as a skeleton.

<u>Note</u> 8. Let $B(2)$ be the $((p^2+p)q-2)$-skeleton of $B(3)$ then there are cofiberings

$$S^0 \longrightarrow B(1) \longrightarrow S^{pq-1}$$

$$B(1) \longrightarrow B(2) \longrightarrow \Sigma^{p^2q-1}B(1),$$

$$B(2) \longrightarrow B(3) \longrightarrow \Sigma^{(p^2+p)q-1}B(2)$$

and

$$B(3) \simeq_p S^0 \cup_{\beta_1} e^{pq-1} \cup_{\alpha_1\beta_1^{p-1}} e^{p^2q-1} \cup_{\beta_1} e^{(p^2+p)q-2} \cup e^{(p^2+p)q-1} \cup \cdots .$$

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ON STABLE HOMOTOPY GROUPS OF SPHERES AND SPECTRA

Hirosi Toda

We shall use the terminologies of the previous note "Constructing some spectra" and discuss stable homotopy groups of sphere spectrum S and spectra $B(n)$, $BP(n)$, $V(n)$ up to degree near p^3q, $p > 3$. In particular, the 8-cell complex $B(3)$ exists. Applications of the spectrum $B(3)$ will be considered in the following three cases:

(I) detect "good" elements

$$x \in \pi_*(S), \text{ i.e., } j_*(x) \neq 0 \text{ (e.g., } x = \alpha_i, \beta_i, \gamma_i, \varepsilon_i),$$

(II) detect "bad" elements

$$x \in \pi_*(S), \text{ i.e., } j_*(x) = 0 \text{ (e.g., } x = \beta_1^r, \beta_i\beta_1^r, \varepsilon'\beta_1^r),$$

(III) compute $\pi_*(B(n))$ for $n = 3,2,1$ and then $n = 0 (B(0) = S)$,

where $j : S \longrightarrow B(3)$ is the inclusion to the bottom sphere.

Up to degree p^3q-2, we can identify $B(3)$ with $BP(3)$. Consider the spectral sequence:

$$H_*(C(3) \bigcirc \pi_*(V(m) \wedge BP)) \Longrightarrow \pi_*(V(m) \wedge BP(3)) = V(m)_*(BP(3))$$

which collapses for $p>3$. The chain complex $C(3)$ has 64 bases. It is convenient to use a reduced chain complex $C'(3)$ of 24 bases:

$$\{1;\ h_0,h_1,h_2;\ g_0,k_0,h_0k_0,g_1,k_1;\ h_0k_0,\ell_1,\ell_2,\ell_3,h_1k_1,\ell_4;\ h_1\ell_1,$$
$$h_2\ell_1,m_1,h_0\ell_4,h_1\ell_4;\ h_0m_1,g_0\ell_4,k_0\ell_4;\ h_0k_0\ell_4\},$$

where

$\deg h_i = p^iq-1$, $\deg g_i = p^i(p+2)q-2$, $\deg k_i = p^i(2p+1)-2$,

$\deg \ell_1 = (p^2+2p+3)-3$, $\deg \ell_2 = (p^2+3p+1)q-3$, $\deg \ell_3 = (2p^2+p+2)q-3$,

$\deg \ell_4 = (3p^2+2p+1)q-3$, $\deg m_1 = (2p^2+4p+2)q-4$.

Since $\pi_*(V(m)\wedge BP) = Z_p[v_1,v_2,\ldots]/(v_1,\ldots,v_m) = Z_p[v_{m+1},v_{m+2},\ldots]$, we have easily

$$\pi_*(V(3)\wedge BP(3)) \cong C'(3)\otimes Z_p \qquad \text{for} \quad *<(p^3+p^2+p+1)q$$

and

$$\pi_*(V(2)\wedge BP(3)) \cong C'(3)\otimes Z_p[v_3] \quad \text{for} \quad *<(p^3+p^2+p+1)q-1.$$

For $m=1$, $\pi_*(V(1)\wedge BP(3)) \cong H(C'(3)\otimes Z_p[v_2,v_3,\ldots])$ and the boundary homomorphism ∂ is given by

$$\partial = \begin{cases} r_1 : x \longrightarrow h_0x & \text{for} \quad x = 1,h_2,k_0,\ell_4,m_1,k_0\ell_4, \\ r_1r_1 : y \longrightarrow z & \text{for} \quad (y,z) = (h_1,g_0),(k_1,\ell_3),(\ell_2,h_1\ell_1), \\ & \qquad (h_1\ell_4,g_0\ell_4), \\ r_1r_1r_1 : u \longrightarrow w & \text{for} \quad (u,v) = (g_1,\ell_1),(h_1k_1,h_2\ell_1), \end{cases}$$

where r_1 is derivative and $r_1(v_3) = -v_2^p, r_1(v_2) = 0$. Thus for the above elements $\{x,y,z,u,w\}$ we have $(* < (p^3+p^2+p)q)$

$$\pi_*(V(1) \wedge BP(3)) \cong \{x,y,yv_3,u,uv_3,uv_3^2\} \otimes Z_p[v_2] \oplus$$

$$\oplus (\{h_0 x\} \otimes Z_p[v_2]/(v_2^p) \oplus \{z\} \otimes Z_p[v_2]/(v_2^{2p}) \oplus$$

$$\oplus \{w\} \otimes Z_p[v_2]/(v_2^{3p})) \otimes Z_p[v_3].$$

To compute $\pi_*(V(0) \wedge BP(3))$, we use the following exact sequence induced by the cofibering $\Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{\pi_1} \Sigma^{q+1} V(0)$:

$$\cdots \longrightarrow \pi_{*-q}(BP(3);Z_p) \xrightarrow{(\alpha \wedge 1)_*} \pi_*(BP(3);Z_p)$$

$$\xrightarrow{(i_1 \wedge 1)_*} \pi_*(V(1) BP(3)) \xrightarrow{(\pi_1 \wedge 1)_*} \cdots .$$

This sequence coincides with the homology exact sequence of $C'(3)$ induced by the short exact sequence of π_* of the above cofibering:

$$0 \longrightarrow Z_p[v_1,v_2,\ldots] \xrightarrow{v_1=\alpha_*} Z_p[v_1 v_2,\ldots] \xrightarrow{i_{1*}} Z_p[v_2,v_3,\ldots] \longrightarrow 0.$$

Then $\pi_*(BP(3))$ is computed from $\pi_*(BP(3);Z_p) = \pi_*(V(0) \wedge BP(3))$ by the universal coefficient theorem. The following table shows the generators of $\pi_{rq-\varepsilon}(BP(3))$ in which

Remark that $\beta_1^p \varepsilon'(n) = 0$ since $\{\beta_1^p, \alpha_1, \beta_1^p\} = 0$. Then similar to the above one we have

Theorem 4. $\varepsilon'(n)\beta_1^{p-1} \neq 0$ for $0 \leq n < p-2$ and $\varepsilon'(n)\beta_1^{p-2} \neq 0$ for $n = p-1$ and $n = p-1$.

By an argument of extended power construction we can prove

Theorem 5. $\alpha_1 \varepsilon'(p-2) = \beta_1^{p^2-2p+1}$ thus $\beta_1^{p^2-p+1} = 0$.

Note that $\beta_1^6 = 0$ for $p = 3$.

Finally we consider applications of type (II). The theorems 3,4 are also examples of type (II) if we use $B(3)$. We shall show another example.

Theorem 6. For $1 \leq s < p-2$, there exist non-zero elements κ_s in the p-component of $\pi_{(s+p+1)(p+1)q-5}(S)$.

This is proved as follows. Let $f : S^{(p^2+p)q-2} \longrightarrow B(2)$ be the attaching map of the cell $e^{(p^2+p)q-1}$ of $B(3)$, $j_1 : B(1) \longrightarrow B(2)$ inclusion and let $\pi_1 : B(1) \longrightarrow S^{pq-1}$ be the natural map. From the non-triviality of γ_1, we have, up to non-zero coefficient,

$$\{\alpha_1\beta_1^{p-1}, \beta_1, p, \beta_{s+1}\} = \beta_{p+s}\alpha_1.$$

Thus we have

Lemma. There exists $\xi_s \in \pi_*(B(1))$, $1 \leq s < p-2$, such that

$$f_*(\beta_{s+1}) = j_{1*}(\xi_s) \quad \text{and} \quad \pi_{1*}(\xi_s) = \beta_{p+s}\alpha_1.$$

Next we check that $\pi_{(s+p+1)(p+1)q-4}(B(3)) = 0$. It follows then $f_*(\beta_{s+1}\alpha_1) = j_{1*}(\xi_s\alpha_1) \neq 0$ in $\pi_*(B(2))$. Since $\beta_{p+s}\alpha_1\alpha_1 = 0$, $\xi_s\alpha_1$ is the image of an element κ_s, which is the required element. Of course κ_s vanishes in $B(3)$.

The elements $\{\kappa_s\beta_1^r\}$ are the next family after Oka's computation.

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Número 2

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R. O'Malley
R. Sacker
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