Projective Segre Codes

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Let $a_1, a_2$ be positive integers and let $\mathbb{P}^{a_1-1}, \mathbb{P}^{a_2-1}, \mathbb{P}^{a_1a_2-1}$ be projective spaces over a field $K$.

The **Segre embedding** is given by

$$\psi: \mathbb{P}^{a_1-1} \times \mathbb{P}^{a_2-1} \rightarrow \mathbb{P}^{a_1a_2-1}$$

$$([\alpha_1, \ldots, \alpha_{a_1}], [\beta_1, \ldots, \beta_{a_2}]) \rightarrow [\alpha_i \beta_j],$$

$$[\alpha_i \beta_j] = [\alpha_1 \beta_1, \alpha_1 \beta_2, \ldots, \alpha_1 \beta_{a_2}, \ldots, \alpha_{a_1} \beta_1, \alpha_{a_1} \beta_2, \ldots, \alpha_{a_1} \beta_{a_2}].$$

The map $\psi$ is well-defined and injective.

Given $X_i \subset \mathbb{P}^{a_i-1}, i = 1, 2$, the image of $X_1 \times X_2$ under the map $\psi$, denoted by $X$, is called the **Segre product** of $X_1$ and $X_2$. 
Consider the following polynomial rings over a field $K$ with the standard grading:

$K[x] = K[x_1, \ldots, x_{a_1}] = \bigoplus_{d=0}^{\infty} K[x]_d,$

$K[y] = K[y_1, \ldots, y_{a_2}] = \bigoplus_{d=0}^{\infty} K[y]_d,$

$K[t] = K[t_1,1, \ldots, t_{a_1},a_2] = \bigoplus_{d=0}^{\infty} K[t]_d.$

The **vanishing ideal** of $X_1$ (resp. $X_2$) is the ideal of $K[x]$ (resp. $K[y]$) generated by the homogeneous polynomials that vanish at all points of $X_1$ (resp. $X_2$).

The **vanishing ideal** $I(X)$ of $X$ is a graded ideal of $K[t]$, where the $t_{i,j}$ variables are ordered as $t_{1,1}, \ldots, t_{1,a_2}, \ldots, t_{a_1,1}, \ldots, t_{a_1,a_2}$. 
Notation for Hilbert function and invariants of $K[t]/I(X)$:

- $H_X(d) = \text{Hilbert function of } K[t]/I(X)$,
- $\text{reg } K[t]/I(X) = \text{regularity index}$,
- $\text{deg } K[t]/I(X) = \text{degree}$.
The *coordinate ring* 

\[ K[t]/I(X) \]

and its algebraic invariants can be expressed in terms of the coordinate rings

\[ K[x]/I(X_1) \quad \text{and} \quad K[y]/I(X_2). \]

To see this we need to define the Segre product.
Definition

Let $A = \bigoplus_{d \geq 0} A_d$, $B = \bigoplus_{d \geq 0} B_d$ be two standard algebras over a field $K$. The **Segre product** of $A$ and $B$, denoted by $A \otimes_S B$, is the graded algebra

$$A \otimes_S B := (A_0 \otimes_K B_0) \oplus (A_1 \otimes_K B_1) \oplus \cdots \subset A \otimes_K B,$$

with the normalized grading $(A \otimes_S B)_d := A_d \otimes_K B_d$ for $d \geq 0$. The tensor product algebra $A \otimes_K B$ is graded by

$$(A \otimes_K B)_p := \sum_{i+j=p} A_i \otimes_K B_j.$$
Example

The Segre product of $K[x]$ and $K[y]$ is

$$K[x] \otimes_S K[y] \cong K[\{x_i y_j \mid 1 \leq i \leq a_1, 1 \leq j \leq a_2\}],$$

and the tensor product of $K[x]$ and $K[y]$ is

$$K[x] \otimes_K K[y] \cong K[x, y].$$
The next result is well-known assuming that $X_1$ and $X_2$ are projective algebraic sets.

**Theorem**

*If $X$ is the Segre product of $X_1$ and $X_2$, then:*

(a) $K[x]/I(X_1) \otimes_S K[y]/I(X_2) \simeq K[t]/I(X)$.

(b) $(K[x]/I(X_1))_d \otimes_K (K[y]/I(X_2))_d \simeq (K[t]/I(X))_d$, $d \geq 0$.

(c) $H_{X_1}(d)H_{X_2}(d) = H_X(d)$ for $d \geq 0$.

(d) $\text{reg}(K[t]/I(X)) = \max\{\text{reg}(K[x]/I(X_i))\}_{i=1}^2$.

(e) $\deg(K[t]/I(X)) = \deg(K[x]/I(X_1)) \deg(K[y]/I(X_2)) \left(\frac{\rho_1 + \rho_2 - 2}{\rho_1 - 1}\right)$, where $\rho_1 = \dim(K[x]/I(X_1))$, $\rho_2 = \dim(K[y]/I(X_2))$. 
Let $K = \mathbb{F}_q$ be a finite field and let $C$ be a $[s, k]$ linear code of length $s$ and dimension $k$, that is, $C$ is a linear subspace of $K^s$ with $k = \dim_K(C)$.

Given a subcode $D$ of $C$, the support of $D$ is:

$$\chi(D) := \{i \mid \exists (a_1, \ldots, a_s) \in D, \ a_i \neq 0\}.$$

The $r$th generalized Hamming weight of $C$ is:

$$\delta_r(C) := \min\{|\chi(D)| : \ D \text{ is a subcode of } C, \ \dim_K(D) = r\}.$$

If $r = 1$, $\delta_1(C)$ is the minimum distance of $C$. 
Direct product codes

Let \( C_1 \subset K^{s_1} \) and \( C_2 \subset K^{s_2} \) be two linear codes of dimensions \( k_1 \) and \( k_2 \), respectively.

The *direct product* of \( C_1 \) and \( C_2 \), denoted by \( C_1 \otimes C_2 \), is the linear code consisting of all \( s_1 \times s_2 \) matrices in which the rows belong to \( C_2 \) and the columns to \( C_1 \).

**Proposition (Wei and Yang, IEEE Trans. Inform., 1993)**

(a) \( C_1 \otimes C_2 \) has length \( s_1 s_2 \), dimension \( k_1 k_2 \), and minimum distance \( \delta_1(C_1)\delta_1(C_2) \).

(b) \( \delta_2(C) = \min\{\delta_1(C_1)\delta_2(C_2), \delta_2(C_1)\delta_1(C_2)\} \).
Consider the bilinear map $\psi_0$ given by

$$\psi_0 : K^{s_1} \times K^{s_2} \longrightarrow M_{s_1 \times s_2}(K)$$

$$((a_1, \ldots, a_{s_1}), (b_1, \ldots, b_{s_2})) \mapsto \begin{bmatrix}
a_1 b_1 & a_1 b_2 & \ldots & a_1 b_{s_2} \\
a_2 b_1 & a_2 b_2 & \ldots & a_2 b_{s_2} \\
\vdots & \vdots & & \vdots \\
a_{s_1} b_1 & a_{s_1} b_2 & \ldots & a_{s_1} b_{s_2}
\end{bmatrix}$$

Lemma

*There is an isomorphism of $K$-vector spaces*

$$T : C_1 \otimes_K C_2 \rightarrow C_1 \otimes C_2$$

such that $T(a \otimes b) = \psi_0(a, b)$ for $a \in C_1$ and $b \in C_2$. 
Let $K = \mathbb{F}_q$ be a finite field, 

$X = \{[P_1], \ldots, [P_m]\} \subset \mathbb{P}^{s-1}$ with $m = |X|$, 

$S = K[t_1, \ldots, t_s]$ a polynomial ring.

Fix a degree $d \geq 1$. For each $i$ there is $f_i \in S_d$ such that $f_i(P_i) \neq 0$. There is a $K$-linear map given by 

$$ev_d : S_d \rightarrow K^m, \quad f \mapsto \left(\frac{f(P_1)}{f_1(P_1)}, \ldots, \frac{f(P_m)}{f_m(P_m)}\right).$$

The image of $S_d$ under $ev_d$, denoted by $C_X(d)$, is called a projective Reed-Muller-type code of degree $d$ on $X$. 
The **basic parameters** of the linear code $C_X(d)$ are:

(a) *length*: $|X|$,  
(b) *dimension*: $\dim_K C_X(d)$,  
(c) *minimum distance*: $\delta_1(C_X(d))$.

The following gives the well-known relation between projective Reed-Muller-type codes and the theory of Hilbert functions:

(a) $\deg(S/I(X)) = |X|$.  
(b) $H_X(d') = \dim_K C_X(d)$ for $d \geq 0$.  
(c) $\delta_1(C_X(d)) = 1$ for $d \geq \text{reg}(S/I(X))$. 
The basic parameters of projective Reed-Muller-type codes have been computed in a number of cases:

- If $X = \mathbb{P}^{s-1}$, $C_X(d)$ this is the classical projective Reed–Muller code and formulas for its basic parameters were given by [Sørensen, IEEE Trans. Inform. Theory, 1991].

- If $X$ is a projective torus, $C_X(d)$ is the generalized projective Reed–Solomon code and formulas for its basic parameters were given by [Sarmiento, Vaz Pinto, -, Appl. Algebra Engrg. Comm. Comput., 2011].

$X$ is a projective torus if $X$ is the image of $(K^*)^s$, under the map $(K^*)^s \to \mathbb{P}^{s-1}$, $x \to [x]$, where $K^* = K \setminus \{0\}$. 
In what follows $K = \mathbb{F}_q$ is a finite field

**Definition**

If $X$ is the Segre product of $X_1$ and $X_2$, we say that the projective Reed-Muller-type code $C_X(d)$ is a *projective Segre code* of degree $d$. 
We come to our main result:

**Theorem (Tochimani, Vaz Pinto, -, 2014)**

Let $\mathbf{X}$ be the Segre product of $\mathbf{X}_1$ and $\mathbf{X}_2$. If $d \geq 1$, then:

(a) $|\mathbf{X}| = |\mathbf{X}_1||\mathbf{X}_2|$, 

(b) $\dim_K(C_{\mathbf{X}}(d)) = \dim_K(C_{\mathbf{X}_1}(d)) \dim_K(C_{\mathbf{X}_2}(d))$, 

(c) $\delta_1(C_{\mathbf{X}}(d)) = \delta_1(C_{\mathbf{X}_1}(d)) \delta_1(C_{\mathbf{X}_2}(d))$, 

(d) $C_{\mathbf{X}}(d)$ is the direct product $C_{\mathbf{X}_1}(d) \otimes C_{\mathbf{X}_2}(d)$, 

(e) $\delta_2(C_{\mathbf{X}}(d)) = \min\{\delta_1(C_{\mathbf{X}_1}(d)) \delta_2(C_{\mathbf{X}_2}(d)), \delta_2(C_{\mathbf{X}_1}(d)) \delta_1(C_{\mathbf{X}_2}(d))\}$, 

(f) $\delta_1(C_{\mathbf{X}}(d)) = 1$ for $d \geq \max\{\text{reg}(K[\mathbf{x}]/l(\mathbf{X}_1)), \text{reg}(K[\mathbf{y}]/l(\mathbf{X}_2))\}$. 

This result tells us that the direct product of projective Reed-Muller-type codes is again a projective Reed-Muller-type code.
Our main theorem gives generalizations of some results:

(a1) If $X_1 = \mathbb{P}^{a_1-1}$ and $X_2 = \mathbb{P}^{a_2-1}$, we recover the formula for the minimum distance of $C_X(d)$ given by [González, Rentería, Tapia-Recillas, Finite Fields Appl., 2002].

(a2) If $X_i$ is a projective torus for $i = 1, 2$, we recover the formula for the minimum distance of $C_X(d)$ given by [González, et. al., Congr. Numer., 2003].
We also recover the following result:

**Corollary (González, et. al., Int. J. Contemp. Math. Sci., 2009)**

Let $X$ be the Segre product of two projective torus $X_1$ and $X_2$. Then $\delta_2(C_X(d))$ is equal to

$$\min\{\delta_1(C_{X_1}(d))\delta_2(C_{X_2}(d)), \delta_2(C_{X_1}(d))\delta_1(C_{X_2}(d))\}.$$
Definition

If $X_i$ is parameterized by monomials $z^{v_1}, \ldots, z^{v_s}$, we say that $C_{X_i}(d)$ is a parameterized projective code.

Corollary

If $C_{X_i}(d)$ is a parameterized projective code for $i = 1, 2$, then so is the corresponding projective Segre code $C_X(d)$. 
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