Centro de Investigación y de Estudios Avanzados del IPN
Departament of Mathematics

Admissions Examination for the Master's Program

30 June 2014

Name: ______________________
Area: ______________________
Advisor: ______________________

Instrucciones: Solve all problems of sections 1 and 2 and as many as possible from section 3. All solutions must be justified appropriately. The examination will last for three hours.

1. **Linear algebra**

1.1 Determine for which real values a, b the following matrix is diagonalizable over $\mathbb{R}$:

$$A := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

1.2 Let V be the vector space of all polynomials $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots a_n t^n,\ \forall n \in \mathbb{N}$ with coefficients $a_0, a_1, \cdots, a_n$ en $\mathbb{R}$.

(a) Prove that $B = \{1, t, t^2, t^3, \ldots\}$ forms a basis for V.

(b) Find a linear transformation $T\colon V \to V$ which is onto but not bijective.

1.3 Let $A \in M_{n\times n}(\mathbb{C})$. We say that A is Hermitian if $A = (\overline{A})^T$, i.e., $[A]_{ij} = \overline{[A_{ji}]}$. Prove:

(a) A is Hermitian if and only if $\langle A\,\alpha, \beta\rangle = \langle \alpha, A\beta\rangle\ \ \forall \alpha, \beta \in \mathbb{C}^n$.

(b) If A is Hermitian, then its spectrum $S_{\mathbb{C}}(A)$ is a subset of $\mathbb{R}$.

2. **Cálculo**

2.1 Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$. Prove that if f is continuous at 0, then f is continuous in all of $\mathbb{R}$.

2.2 Calculate the derivative of the function

$$F(x) = \int_a^{\left(\int_a^x \frac{1}{1+\sin^2 t} dt\right)} \frac{1}{1+\sin^2 t} dt.$$

2.3 Find the following limits:

(a) $\lim\limits_{n\to\infty} \frac{n!}{n^n}$ (b) $\lim\limits_{n\to\infty} \sqrt[n]{a}$, $a > 0$.

Hint for (a): $n! = n(n+1)\ldots k!$ for $k < n$, in particular for $k < \frac{n}{2}$.

3. **Optional problems**

3.1 Prove that if in a group G every element is its own inverse, then G is abelian.

3.2 Calculate the integral $\int_\gamma e^z dz$ where γ is the arc of the unit circle joining 1 to i.

3.3 Let X be a topological space and let $f : X \to \mathbb{R}$ be continuous. Prove that the set $Z_f := \{x \in X | f(x) = 0\}$ is closed.

3.4 Give an example of a sequence of functions $\{f_n\}_n \in L_2(\mathbb{R})$ which converge to 0 pointwise but which to not converge to 0 in L_2.

3.5 Disign a Türing machine to enumerate the language $\{0^n 1^n | n \geq 0\}$.