A note on distributional equations in discounted risk processes∗

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Abstract
In this paper we give an account of the classical discounted risk processes and their limiting distribution. For the models considered, we set the Markov chains embedded in the continuous-time processes; we also set the distributional equations for the limit distributions. Additionally, we mention some applications of these tools such as finding the moments of the limiting distribution or describing the income rate in a cash flow of money.

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1 Introduction
We consider the following discounted sum in continuous-time
\[
Z^{(δ)} := \{ Z^{(δ)}_t = \sum_{i=1}^{N_t} X_i e^{-δT_i}, \ t ≥ 0 \},
\]
(1)
where \(N\) is a renewal process with interarrival times \(τ_1, τ_2, \ldots\); which are independent identically distributed positive random variables (i.i.d. positive r.v.s.). Process \(N\) is defined through
\[
N_t := \max \left\{ k : \sum_{i=1}^{k} τ_i ≤ t \right\}.
\]
(2)
Also, we have that \(T_i := \sum_{j=1}^{i} τ_j, i = 1, 2, \ldots\), which represents the arrival times. The variables \(X_1, X_2, \ldots\) are i.i.d. positive r.v.s. and \(δ\) is the continuous

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1
time discounting factor. Throughout this paper we assume that the interarrival times and claim sizes are independent, and we denote by \( \tau \) and \( X \) the generic random variables, such that \( \tau \sim \tau_1 \) and \( X \sim X_1 \). To avoid technical problems we assume that \( P(\tau = 0) < 1 \).

Model (1) is well used in the context of insurance: variable \( \tau \) is the time between claims and \( X \) the size of the claims, thus \( Z_\delta(t) \) is the present value of the claims up to time \( t \). One can see that process (1) is an extension of the well-known renewal reward processes (see for instance [2, 30]), and when \( \delta = 0 \) it is a particular instance of the so-called continuous-time random walk (one may find a summary on this type of process in [27]). The renewal reward process is also called the aggregate claim amount in the insurance jargon (see [24]). Processes that resemble (1) have been studied with other names; for example, the Markov shot noise processes in [28, 25], and when the renewal process is Poisson it is called filtered Poisson processes in [15]. This process have renewal properties, feature that classifies it in a more general family called regenerative processes, as labeled in [2]; or it is a particular instance of a semi-Markov process, see e.g. [20]. Specifically, process (1) has been studied previously in [5, 9, 16, 17, 29, 18].

Let \( Z^* \) be the limit of \( Z_\delta(t) \) when \( t \to \infty \), when it is well defined, intuitively one expects the following distributional equation to hold,

\[
Z^* \overset{d}{=} e^{-\delta \tau}(Z^* + X).
\] (3)

In section 2, we give the details to derive previous equation using the so-called embedded Markov chains; this technique has been well used in other papers, see e.g. [11]. Following this idea, in section 3, we do the same for the so-called discounted risk process. Then, we mentioned how one can find the moments recursively (see e.g. [17]). In section 4 it is used the tools developed to give straightforward applications, namely for studying the ruin probability and a perpetual cash flow.

## 2 An embedded Markov chain

The term embedded Markov chain (EMC) refers to the concept of having a continuous-time process that may have a discrete-time process embedded within. An important feature of model (1) is that it renews/regenerates at the very time of an arrival \( T_i \); this helps to identify Markov chains (MCs) embedded in the process. The EMC help to study the limit behaviour of process (1) by studying the corresponding stationary distributions. The type of MCs that arise here can be compared to the so-called perpetuities (see also [19]), and in turn they give rise to distributional equations (DEs), also called stochastic or random equations (some relevant references about DEs are [1, 13, 28]). An important application of the distributional equations is finding properties of stationary distributions, such as moments or even the distribution itself (see e.g. [11, 21]).

To study \( \lim_{t \to \infty} Z_\delta(t) \) we shall proceed in the following way: we prove that \( Z_\delta(t) \) converges in distribution as \( t \to \infty \) when \( E(X) < \infty \) (Proposition 1), then...
we find that process $Z^{(δ)}$ has embedded a MC (Proposition 2), and finally we set a distributional equation (Proposition 5).

**Proposition 1** If $X$ has finite mean, then $Z^{(δ)}_t$ converges in distribution as $t \to \infty$ to a random variable $Z^*$ with finite mean.

**Proof.** Since $Z^{(δ)}_t \geq Z^{(δ)}_s$ for $t \geq s$ always, function $P(Z^{(δ)}_t \geq z)$ is increasing in $t$, thus the probabilities are convergent, and we call $Z^*$ the r.v. with such convergent distribution. Moreover, notice that

$$Z^* \overset{d}{=} \lim_{t \to \infty} Z^{(δ)}_t = \sum_{i=1}^{\infty} X_i e^{-δT_i}. \quad (4)$$

By Fatou’s Lemma,

$$E(Z^*) \leq \sum_{i=1}^{\infty} E(X_i) E(e^{-δT_i}). \quad (5)$$

Finally, we know that $E(e^{-δT_i}) = E'(e^{-δτ_i})$ and that $E(e^{-δτ_i}) < 1$, then

$$E(Z^*) \leq E(X) \sum_{i=1}^{\infty} E^i(e^{-δτ_i}) < \infty. \quad (6)$$

**Proposition 2** Let $Y_n$ be the process $Z^{(δ)}$ evaluated at the time of the $n$ arrival, that is, $Y_n := Z^{(δ)}_{T_n}$. Then the following identity holds

$$Y_{n+1} \overset{d}{=} X_ne^{-δτ_n} + e^{-δτ_n}Y_n, \quad n = 0, 1, \ldots, \quad (7)$$

with $Y_0 := 0$. Here $X_n, τ_n$ and $Y_n$ are independent for each $n$.

**Proof.** Using the definition of the process

$$Z^{(δ)}_{T_{n+1}} = \sum_{i=1}^{N_{T_{n+1}}} X_i e^{-δT_i} = \sum_{i=1}^{n+1} X_i e^{-δT_i},$$

$$\overset{d}{=} X e^{-δτ} + \sum_{i=2}^{n+1} X_i e^{-δT_i} \overset{d}{=} X e^{-δτ} + e^{-δτ} \sum_{i=2}^{n+1} X_i e^{-δτ} \overset{d}{=} e^{-δτ} Z^{(δ)}_{T_n}.$$

Hence, equation (7) can be set.

**Remark 3** It is said that process $Y := \{Y_0, Y_1, \ldots\}$ of previous result is an embedded Markov chain of process $Z^{(δ)}$. There are results regarding ergodic properties of stochastic processes through embedded Markov chains (see for instance [6]). Since the paths of $Z^{(δ)}$ are piecewise constant between arrivals, we shall study the limit behaviour of $Z^{(δ)}$ by studying the stationary distribution of $Y$.  

3
Remark 4 Relation (7) is an identity of distributions which itself provides a method for approximating samples of $Z^*$ by running the MC. This is possible due to the fact that the stationary distribution of the MC is the limiting distribution of $Z_t$ as $t \to \infty$. An extensive study on stochastic equations of this type can be found in Vervaat [28].

Proposition 5 Suppose that $Z^* := \lim_{t \to \infty} Z_t$ is well defined. Then we have the following distributional equation

$$Z^* \overset{d}{=} X e^{-\delta \tau} + e^{-\delta \tau} Z^*.$$ (8)

Proof. Notice that $(X_i, \tau_i, Y_i) \to (X, \tau, Z^*)$ as $i \to \infty$. Since $f(x, t, y) := xe^{-\delta t} + e^{-\delta t}y$ is a continuous function, we can apply the continuous mapping theorem (see [4]) to obtain (8) from (7).

Remark 6 Often, the following class of distributional equation arises in insurance applications (see for example [11, 13, 28]):

$$Z_\infty \overset{d}{=} AZ_\infty + B,$$ (9)

where $A, B$ and $Z_\infty$ are random variables, and $Z_\infty$ may or may not be independent of $(A, B)$. The question is to find the distribution of $Z_\infty$ given the distribution of $(A, B)$.

A typical application of the distributional equation (8) is for the calculation of the moments.

Corollary 7 Suppose that $X$ has all its moments finite and that the Laplace transform of $\tau$ exists. Then, the $k$-moment of $Z^*$ satisfies the following recursive formula

$$E((Z^*)^k) = \frac{E(e^{-k\delta \tau})}{1 - E(e^{-k\delta \tau})} \sum_{i=0}^{k-1} \binom{k}{i} E(X^{k-i})E((Z^*)^i),$$ (10)

for $k=1,2, \ldots$.

Proof. This is done by a direct use of the Newton’s binomial theorem to the distributional equation (9). First, expanding the binomial, taking expectations and then solving for the $k$-moment.

The procedure described in Corollary 7 for finding moments through distributional equations is quite common in the literature, see for example [19, p. 465] or [21, 28]. Notice that using the distributional equations one may also attempt to find the characteristic function or the moment generating function.

3 The present value distribution

Using process (1), a popular model in insurance is the so-called total surplus (also termed the discounted risk process) given by

$$U_t = \eta + r(t) - Z_t^{(5)}, t \geq 0,$$ (11)
where \( r(t) := \int_0^t \rho e^{-\delta s} ds \) is the present value of the incomes received by the company, which is determined with the premium rate \( \rho \). Variable \( \eta \) is the initial capital of the company. In [29] model (11) is called the renewal risk process. Process \( U = \{ U_t, t \geq 0 \} \) in (11) represents the present value of the total surplus (earnings or losses) of the company up to time \( t \).

When \( t \to \infty \), \( U_t \) may converge to a random variable, which is interpreted as the total earnings or losses of the perpetuity, that is to say, the total outcome of the business. The interested reader can find a more extensive discussion of this concept in [23]. A natural question is to find the so-called present value distribution, which is defined as the distribution of

\[
U^* := \lim_{t \to \infty} U_t. \tag{12}
\]

Finding the present value distribution has been done for a general class of models based on the Poisson process; important references are [10, 11, 13, 14]. Specially in [11] one finds a good account.

One can see that if \( Z_1^{(\delta)} \) converges in distribution as \( t \to \infty \), so does \( U_t \). To this end we have the following.

**Theorem 8** Let \( Z \) be the process specified in (1). When \( \lim_{t \to \infty} Z_1^{(\delta)} \) exists in distribution, we have the following distributional equation

\[
U^* \overset{d}{=} \alpha - e^{-\delta \tau} (X + \alpha - U^*), \tag{13}
\]

where \( \alpha = \eta + \frac{\rho}{\delta} \). Moreover, the distribution of \( U^* \) coincides with the stationary distribution of the following MC

\[
W_{n+1} = \alpha - e^{-\delta \tau_{n+1}} (X_{n+1} + \alpha - W_n), \tag{14}
\]

with \( W_0 \in \mathbb{R} \).

**Proof.** First, we will exploit the renewal property to obtain an identity in distribution. Then, we appeal to the continuous mapping theorem to set equation (13), which itself gives rise to the MC (14).

By using the definition of \( T_n, n = 1, 2, \ldots \), we have

\[
U_{T_n} = \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \sum_{i=0}^{N_{T_n}} X_i e^{-\delta T_i},
\]

\[
= \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left( X_1 e^{-\delta \tau_1} + \sum_{i=2}^{N_{T_n}} X_i e^{-\delta T_i} \right),
\]

\[
\overset{d}{=} \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left( X e^{-\delta \tau_1} e^{-\delta \tau} + e^{-\delta \tau} \sum_{i=2}^{n-1} X_i e^{-\delta T_i} \right) \]

\[
= \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left( X e^{-\delta \tau_1} e^{-\delta \tau} + \sum_{i=1}^{n-1} X_i e^{-\delta T_i} \pm \eta \pm \int_0^{T_{n-1}} \rho e^{-\delta s} ds \right),
\]

\[
= \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left( X e^{-\delta \tau_1} e^{-\delta \tau} + \sum_{i=1}^{n-1} X_i e^{-\delta T_i} \right) \]

\[
= \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left( X e^{-\delta \tau_1} e^{-\delta \tau} + \sum_{i=1}^{n-1} X_i e^{-\delta T_i} \right) \]

5
Thus, when taking limits we have distributional equation (13).

**Remark 9** Notice that the moments of $U^*$ may be computed using equation (13) as in Corollary 7, however this approach does not give a formula as friendly as recurrence (10). It is more convenient to use the fact that

$$U^* = \alpha - Z^*,$$

with $\alpha = \eta + \frac{\rho}{\delta}$, (15)

which yields

$$E\left((U^*)^k\right) = (-\alpha)^k \sum_{i=0}^{k} \binom{k}{i} (-1)^i E\left((Z^*)^{k-i}\right), k = 1, 2, \ldots.$$ (16)

Corollary 7 and Remark 9 find the moments of $Z^*$ and $U^*$, respectively. An interesting point is to find the actual limit distribution, i.e. finding the solutions (8) and (13). Next, we use the Markov chains to perform numerical approximations.

**Example 10** In figure 1 we show the approximations of $Z^*$ and $U^*$ for a model where the claim sizes and the interarrival times are exponential, both with parameters $1$; and we have taken $\delta = 0.1$, $\eta = 5$ and $\rho = 0.3$. We have run $10^6$ times the corresponding Markov chains (7) and (14), and obtained numerically the histograms with partition 200.

4 Applications

Now we present some applications of the embedded Markov chains and the distributional equations. First, we find a bound for the ruin probability. Then, we discuss about the probability of ending negative in perpetual cash flow. Finally, we give an example to show that the income rate may not be set too high or too low.
4.1 The ruin probability

Calculating the ruin probability has been a great deal in risk theory for discounted and non-discounted sums. Since the celebrated works of Lundberg and Cramer, many articles and books have been published to address this problem; few references are [3, 12, 19, 24], and a concise summary can be found in [7].

Consider model (11). The ruin probability is defined as follows: Given the initial capital $\eta$, the variable $\chi_\eta$ is the first time when $U_t$ goes below 0 (when the company goes bankrupt or ruined). It is expressed as $P(\chi_\eta < \infty)$ where

$$\chi_\eta = \inf \{ s : U_s < 0 \}. \quad (17)$$

The ruin probability has been studied extensively with model (1) when the interarrival times are exponential r.v.s and $\delta = 0$ (see e.g. [12]). Here we take $\delta > 0$ but still assume that $\tau$ is exponentially distributed. Under these assumptions, Harrison [14] gives bounds for the ruin probability in terms of the present value distribution.

**Proposition 11** Using model (11) with $\tau$ being an exponential random variable, suppose that $Z^\ast \triangleq \lim_{t \to \infty} Z^{(\delta)}_t$ is well defined. Then the following upper bound for the ruin probability holds

$$P(\chi_\eta < \infty) \leq \frac{P(Z^\ast > \eta + \rho/\delta)}{P(Z^\ast > \rho/\delta)}. \quad (18)$$

**Proof.** By Corollary 2.4 in [14] we have that

$$P(\chi_\eta < \infty) \leq \frac{H(-\eta)}{H(0)}, \quad (19)$$

where $H$ is the distribution function of $-\eta + \lim_{t \to \infty} U_t$. Recall that $\lim_{t \to \infty} r(t) = \rho/\delta$. ■

Previous sections give grounds for finding numerically the bound for the ruin probability. This is easily achieved by approximating $P(Z^\ast > z)$ using the embedded Markov chain of process $Z^{(\delta)}$, which is explained in Remark 4 and carried out in Example 10.

4.2 The business ruin probability

We now turn to the study of the long time behaviour of $U_t$ as $t \to \infty$, which is the perpetual cash flow (such as in pension schemes); we may think of this as the “total/final outcome of the business”. In this paper we study the probability of ending up loosing at the infinite horizon:

$$P(U^* < 0). \quad (20)$$

We call quantity (20) the business-ruin probability.
In the case of the insurance company, it is not realistic to take $t \to \infty$, because the insurer would not continue operating in case of bankruptcy (i.e. when $U_t < 0$).

Compare to the classical concept of ruin probability, the business-ruin probability is a less stringent version. This, due to the fact that the process $U$ can go below $0$ but may end up positive as $t \to \infty$. Therefore, the event of going business-ruin implies that the process $U$ went negative at some point, thus

$$P(U^* < 0) \leq P(\chi_\eta < \infty),$$

(21)

where $\chi_\eta = \inf \{s : U_s < 0\}$.

A natural question is to find an income rate $\rho$ that guarantees certain level of total earnings. Moreover, we may find $\rho$ that helps to achieve low probability of ending up loosing or a high probability of ending up earning. Notice that $U^*$ depends on the rate $\rho$, and we can write $U^*(\rho)$ to emphasize this. The following definition gives criteria to choose an income rate.

**Definition 12** For $\epsilon \in (0, 1)$, we call the quantity $\rho_\epsilon$ the $\epsilon$-percentile income rate if it is such that

$$P(U^*(\rho_\epsilon) \in A) = \epsilon,$$

(22)

for some Borel set $A$.

We call $\rho_\beta$ with $\beta > 0$ the $\beta$-mean income rate if is such that

$$E(U^*(\rho_\beta)) = \beta.$$  

(23)

Generally, we would be interested in an income rate that allows us to minimize the potential loss, maximize profit, or simply such that we reach a minimum level of profit; previous definition takes into account these ideas.

The calculation of $\rho_\beta$ is direct from equation (13). Equating the expectation $E(U^*)$ to $\beta$, and solving for $\rho_\beta$ we have that

**Proposition 13** The $\beta$-mean income rate is given by

$$\rho_\beta = \frac{\delta^2 (\beta - \eta) (1 - E(e^{-\delta T})) + E(X)E(e^{-\delta T})}{1 - E(e^{-\delta T})}.$$  

(24)

**Remark 14** For the $\epsilon$-percentile income rate, one needs to be more specific. For instance, if we are interested on minimizing loss, a natural choice for $A$ is $(-\infty, \theta]$, where $\theta \geq 0$ is a minimum level of tolerance. Likewise, we can set $A = [\theta, \infty)$ if we want to achieve certain level of profit. In any case, the calculation of $\rho_\epsilon$ requires the knowledge of the distribution of $U^*$ (the present value distribution).

**Example 15** In Example 10, for $\rho = 0.3$, we have that $P(U^* < 0) \approx 0.71$. Thus the $\epsilon$-percentile premium rate is $\rho_\epsilon = 0.3$ with $\epsilon \approx 0.71$. This is an unfavorable scenario for the company, because with a high probability the business will end up loosing.
4.3 An income rate sensitive to the market

The income rate $\rho$ is a quantity that depends on the price per contract. That is, the rate of income is a factor that can be determined by how cheap or expensive the actual price of the contract is.

Let $p$ be the price per contract. Price $p$ may be so expensive that no one would be able to afford it (and thus no income would be obtained); or, the price could be so cheap that even though many would buy it, the income rate would not be enough to pay the potential losses. The value of $p$ would affect the income rate $\rho$ and the frequency of arrivals, defined by $\tau$. Thus, the company does not want to set a very expensive or very cheap price per contract, rather it needs to find an optimal price.

Consider the distributional equation (13). If we take expectation of both sides of (13), and solve for $E(U^*)$ we obtain

$$E(U^*) = \frac{\eta + \frac{\rho}{\delta} - E(e^{-\delta \tau})(E(X) + \eta + \frac{\rho}{\delta})}{1 - E(e^{-\delta \tau})}.$$  \hspace{1cm} (25)

Here, $\rho$ and $E(e^{-\delta \tau})$ are functions of $p$. Then, an optimal price $p$ can be found by maximizing (25).

Now, we give a model that is specified by $p$.

Suppose that $\tau$ is an exponential r.v. with mean $\frac{1}{\lambda(p)}$, where

$$\lambda(p) = \frac{ap}{b}, \text{ for } a > 0, b > 0. \hspace{1cm} (26)$$

Moreover, suppose that $\rho(p)$ is given by

$$\rho(p) = p^ce^{-dp}, \text{ for } c > 0, d \geq 0. \hspace{1cm} (27)$$

If $p$ is small (cheap), more people would buy a contract, and thus more potential losses might arrive in the future. If $p$ is large (expensive), less people would buy the insurance, thus the company would have less losses. These phenomena are reflected in (26) and (27).

Now we have that

$$E(e^{-\delta \tau}) = \frac{\lambda(p)}{\lambda(p) + \delta}.$$  

Furthermore, if $X \sim \exp(\mu)$, formula (25) becomes

$$E(U^*) = \left( \eta + \frac{\rho(p)}{\delta} - \frac{\lambda(p)}{\lambda(p) + \delta} \left( \mu + \eta + \frac{\rho(p)}{\delta} \right) \right) \frac{\lambda(p) + \delta}{\delta}. \hspace{1cm} (28)$$

Using Example 10 and setting $a = 0.05, b = 0.1, c = 0.5$ and $d = 0.05$ to define functions (26) and (27), in Figure 4.3 we plot $E(U^*)$ as function of $p$. We can see that $E(U^*)$ attains a maximum: the optimal price per contract. Related to this application, see [26] for control problems in insurance.
Figure 2: $E(U^*)$ as function of $p$

References


