The v-number of edge ideals

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Let $S = K[t_1, \ldots, t_s] = \bigoplus_{d=0}^{\infty} S_d$ be a polynomial ring over a field K with the standard grading.

Let G be a *simple graph* with vertex set

$$V(G) = \{t_1, \ldots, t_s\}$$

and edge set $E(G)$.

Some of the results presented here hold for clutters (simple hypergraphs) but for simplicity we focus on graphs.

The *edge ideal* of G, denoted $I(G)$ or simply I, is the ideal of S generated by all squarefree monomials

$$t_e := t_i t_j$$

such that $e = \{t_i, t_j\} \in E(G)$.

A prime ideal $\mathfrak{p}$ of S is an *associated prime* of I if

$$\mathfrak{p} = (I \colon f) := \{g \in S \mid gf \in I\}$$

for some $f \in S_d$.

The v-*number* of I, denoted $\mathrm{v}(I)$, is the following invariant that was introduced by Cooper, Seceleanu, Tohăneanu, Vaz Pinto, –, to study Reed–Muller codes:

$$\mathrm{v}(I) := \min\{d \mid \exists\, f \in S_d,\ \mathfrak{p} \in \mathrm{Ass}(I),\ \text{with } (I\colon f) = \mathfrak{p}\}.$$

If $\operatorname{reg}(S/I)$ is the Castelnuovo–Mumford *regularity* of S/I, then one has

$$\operatorname{reg}(S/I) \leq \dim(S/I).$$

We will show that in some interesting cases one has

$$\operatorname{v}(I) \leq \operatorname{reg}(S/I).$$

A set A of vertices of G is *stable* or *independent* if $e \not\subset A$ for any $e \in E(G)$. A set C of vertices of G is a *vertex cover* if $V(G) \setminus C$ is stable. A *minimal vertex cover* is a vertex cover which is minimal with respect to inclusion.

The *neighbor set* of A, denoted $N_G(A)$, is the set of all vertices of G that are adjacent with at least one vertex of A.

Let $\mathcal{A}_G$ be the family of all stable sets A of G whose neighbor set $N_G(A)$ is a minimal vertex cover of G.

We give a combinatorial formula for the v-number.

Theorem

Let $I = I(G)$ be the edge ideal of a graph G. Then

$$\mathrm{v}(I(G)) = \min\{|A| \colon A \in \mathcal{A}_G\}.$$

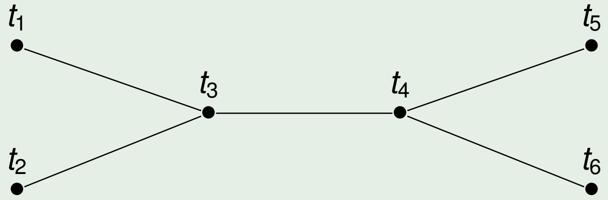
The *independence number* of G, denoted $\beta_0(G)$, is the number of vertices in any largest stable set of vertices of G. The Krull dimension of $S/I(G)$ is equal to $\beta_0(G)$.

Example

Let G be the graph below. Then

$$I = I(G) = (t_1 t_3, t_2 t_3, t_3 t_4, t_4 t_5, t_4 t_6),$$

$$\mathrm{v}(I) = 1, \quad \dim(S/I) = \beta_0(G) = 4, \quad \mathrm{reg}(S/I) = 1.$$



Corollary

$$v(I(G)) \leq \beta_0(G) = \dim(S/I(G)).$$

Proof

Any maximal stable set A is in $\mathcal{A}_G$. Thus, by the previous theorem, one has

$$\mathrm{v}(I(G)) \leq |A| \leq \beta_0(G).$$

The v-number is additive

If $G_1, \ldots, G_r$ are the connected components of G, then

$$\mathrm{v}(I(G)) = \mathrm{v}(I(G_1)) + \cdots + \mathrm{v}(I(G_r)).$$

The simplicial complex Δ_G whose faces are the stable sets of G is called the *independence complex* of G.

Theorem

If Δ_G is vertex decomposable, then

$$\mathrm{v}(I(G)) \leq \mathrm{reg}(S/I(G)).$$

We give an example of a graph G with

$$\mathrm{v}(I(G)) > \mathrm{reg}(S/I(G)).$$

It is an open problem whether $\mathrm{v}(I(G)) \leq \mathrm{reg}(S/I(G)) + 1$ holds for any graph G.

A graph is *well-covered* if every maximal stable set is a maximum stable set.

A graph G belongs to *class* W_2 if $|V(G)| \geq 2$ and any two disjoint stable sets are contained in two disjoint maximum stable sets.

- A complete graph $\mathcal{K}_m$ belongs to W_2 for $m \geq 2$.
- A graph G is in W_2 if and only if G is well-covered, $G \setminus v$ is well-covered for all $v \in V(G)$, and G and has no isolated vertices.

The next result gives an algebraic method to determine if a given graph is in W_2 using *Macaulay2*.

Theorem

Let G be a graph without isolated vertices. Then, G is in W_2 if and only if

$$\mathrm{v}(I) = \dim(S/I).$$

The n-th symbolic power of $I = I(G)$ is given by

$$I^{(n)} := \bigcap_{i=1}^{r} \mathfrak{p}_i^n,$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the associated primes of I.

There are algebraic characterizations of the Cohen–Macaulay property of $I(G)^{(2)}$ given by T. Hoang, N. C. Minh and T. N. Trung but there is still no characterization of this property in terms of the graph G.

An edge in a graph is *critical* if its removal increases the independence number. An *edge-critical graph* is a graph with only critical edges.

Every connected edge-critical graph G is a block, that is, $G \setminus v$ is connected for all $v \in V(G)$.

Using the next result—together with the tables of edge-critical graphs given by M. D. Plummer and B. Small—we give the list of all connected graphs G with fewer than 10 vertices such that $I(G)^{(2)}$ is Cohen–Macaulay over a field of characteristic 0.

Theorem

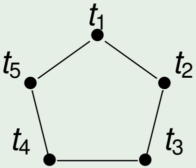
If $I(G)^{(2)}$ is Cohen–Macaulay, then G is edge-critical.

T. Hoang, N. C. Minh, T. N. Trung

If $I(G)^{(2)}$ is Cohen–Macaulay, then G is W_2.

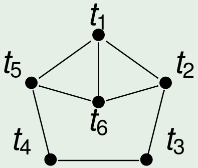
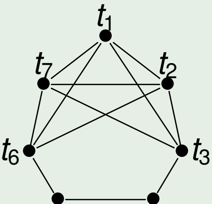
Examples

Table: Connected graphs with $I(G)^{(2)}$ Cohen–Macaulay

$\lvert V(G)\rvert$	Graph G	$I = I(G)$
2	$\mathcal{K}_2$	(t_1t_2)
3	$\mathcal{K}_3$	(t_1t_2, t_1t_3, t_2t_3)
4	$\mathcal{K}_4$	$(t_1t_2, t_1t_3, t_1t_4, t_2t_3, t_2t_4, t_3t_4)$
5		$(t_1t_2, t_2t_3, t_3t_4, t_4t_5, t_1t_5)$
5	$\mathcal{K}_5$	$I(\mathcal{K}_5)$
6	$\mathcal{K}_6$	$I(\mathcal{K}_6)$

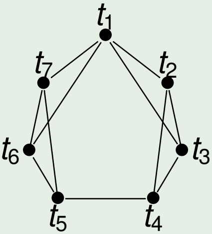
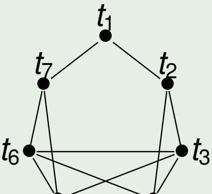
Examples

Table: Connected graphs with $I(G)^{(2)}$ Cohen–Macaulay

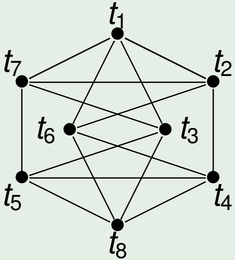
$\|V(G)\|$	Graph G	$I = I(G)$
6		$(t_1t_2, t_1t_5, t_1t_6, t_2t_3, t_2t_6, t_3t_4, t_4t_5, t_5t_6)$
7	$\mathcal{K}_7$	$I(\mathcal{K}_7)$
7		$(t_1t_2, t_1t_3, t_1t_6, t_1t_7, t_2t_3, t_2t_6, t_3t_4, t_4t_5, t_5t_6, t_3t_7, t_6t_7, t_2t_7)$

Examples

Table: Connected graphs with $I(G)^{(2)}$ Cohen–Macaulay

$\lvert V(G) \rvert$	Graph G	$I = I(G)$
7		$(t_1t_2, t_1t_3, t_1t_6, t_1t_7, t_2t_3, t_2t_4, t_3t_4, t_4t_5, t_5t_6, t_5t_7, t_6t_7)$
7		$(t_1t_2, t_5t_3, t_4t_6, t_1t_7, t_2t_3, t_2t_4, t_3t_4, t_4t_5, t_5t_6, t_5t_7, t_6t_7, t_3t_6)$

Examples

$\|V(G)\|$	Graph G	$I = I(G)$
8	$\mathcal{K}_8$	$I(\mathcal{K}_8)$
8		$(t_1t_3, t_1t_2, t_1t_6, t_1t_7, t_2t_7, t_2t_6, t_2t_4, t_3t_7, t_3t_5, t_3t_8, t_4t_5, t_4t_6, t_4t_8, t_5t_7, t_5t_8, t_6t_8)$

Theorem

Let G be a graph. If $\beta_0(G) = 2$, then $I(G)^{(2)}$ is Cohen–Macaulay if and only if G is edge-critical.

If G is an edge-critical graph without isolated vertices and $\beta_0(G) = 2$, we prove that G is in W_2, then using that any 1-dimensional connected complex is vertex decomposable (Provan and Billera), we prove that

$$\mathrm{v}(I(G)) = \mathrm{reg}(S/I(G)) = 2$$

For graphs G with independence number at least 3 and second symbolic power Cohen–Macaulay, it is an open problem whether $\mathrm{v}(I(G)) = \mathrm{reg}(S/I(G))$.

THE END