Why Critical Ideals?

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Joint work with Carlos A. Alfaro and Hugo Corrales.
Outline

1. Critical ideals of a graph
   - Critical groups and characteristic polynomial
2. The algebraic co-rank
3. Critical ideals of graphs with twins
   - Duplication
   - Replication
   - Some Conjectures
   - Perspectives
4. Graphs with a small algebraic co-rank
   - $\gamma = 2$
   - $\gamma = 3$
   - Some Conjectures
5. Trees
   - Some Conjectures
The generalized Laplacian matrix

Let $G$ be a signed multidigraph, $\mathcal{P}$ a Principal Ideal Domain (PID), and

$$X_G = \{x_u \mid u \in V(G)\}$$

the set of variables indexed by the vertices of $G$.

**Definition**

The **generalized Laplacian matrix** of $G$ is given by

$$L(G, X)_{u,v} = \begin{cases} 
  x_u & \text{if } u = v, \\
  -\sigma(uv)m_{uv}1_\mathcal{P} & \text{otherwise},
\end{cases}$$

where $m_{uv}$ is the number of arcs leaving $u$ and entering to $v$ and $1_\mathcal{P}$ is the unity of $\mathcal{P}$. 
The critical ideals of a graph

Furthermore, if $\mathcal{P}[X_G]$ is the polynomial ring over $\mathcal{P}$ on $X_G$.

**Definition**

The *critical ideals* of $G$ are the determinantal ideals given by

$$I_i(G, X) = \langle \text{minors}_i(L(G, X)) \rangle \subseteq \mathcal{P}[X_G] \text{ for all } 1 \leq i \leq n,$$

where $\text{minors}_i(L(G, X))$ are the *minors* of $L(G, X)$ of size $i$.

Clearly

$$\langle 0 \rangle \subsetneq I_n(G) \subsetneq \cdots \subsetneq I_2(G) \subsetneq I_1(G) \subseteq \langle 1 \rangle.$$

Moreover, if $H$ is an induced subdigraph of $G$, then

$$I_k(H) \subseteq I_k(G) \text{ for all } 1 \leq k \leq |V(G)|.$$
Example

If \( P = \mathbb{Z} \), then \( I_i(G, X) = \langle 1 \rangle \) for all \( i \leq 3 \),

\[
l_4(G, X) = \langle x_1 x_2 + x_4 + 1, x_2 x_3 - x_5 - 1, x_3 x_4 + x_1 - 1, x_4 x_5 - x_2 - 1, x_1 x_5 + x_3 + 1 \rangle,
\]

and

\[
l_4(G, X) = \langle x_1 x_2 x_3 x_4 x_5 - x_1 x_2 x_3 + x_2 x_3 x_4 - x_1 x_2 x_5 - x_1 x_4 x_5 + x_3 x_4 x_5 + x_1 - x_2 - x_3 - x_4 - x_5 - 2 \rangle.
\]
Critical ideals of a graph

If \( \mathcal{P} = \mathbb{Z} \) and \( x_i = d_G(v_i) \), then \( L(G, X) \) becomes the usual Laplacian matrix. The critical group of \( G \) is given by
\[
K(G) \oplus \mathbb{Z} = \mathbb{Z}^V / \text{Im } L(G)^t.
\]

Correspondence

Theorem

If \( K(G) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}_{f_i} \) with \( f_1 | \cdots | f_{n-1} \), then
\[
l_i(G, X)_{X=d_G(G)} = \langle \prod_{j=1}^{i} f_j \rangle \text{ for all } 1 \leq i \leq n - 1.
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\]

**Theorem**

Let \( G \) be a graph and \( v \) a vertex of \( G \) such that \( d_G^+(u) = d_G^-(u) \) for all \( u \in V(G) \). If \( K(G) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z} f_i \) with \( f_1 | \cdots | f_{n-1} \), then

\[
l_i(G \setminus v, X)_{X=d_G(u)} = \langle \prod_{j=1}^{i} f_j \rangle \text{ for all } 1 \leq i \leq n - 1.
\]
If \( \mathcal{P} \) is a field, then the critical ideals of \( G \) are principal. That is, there exist \( p_i(t) \in \mathcal{P}[t] \) for all \( 1 \leq i \leq n \) such that

\[
l_i(G, t) = \langle \prod_{j=1}^{i} p_j(t) \rangle \text{ for all } 1 \leq i \leq n.
\]

Example

If \( G \) is the complete graph with six vertices minus a perfect matching and \( \mathcal{P} = \mathbb{Q} \), then

\[
l_i(G, t) = \begin{cases} 
\langle 1 \rangle & \text{if } 1 \leq i \leq 4, \\
\langle t^2(t + 2) \rangle & \text{if } i = 5, \\
\langle t^3(t + 2)^2(t - 4) \rangle & \text{if } i = 6.
\end{cases}
\]

Therefore we get a factorization of the characteristic polynomial of \( G \).
The algebraic co-rank

**Definition**
Given a graph $G$ and $\mathcal{P}$ a PID, let

$$\gamma_{\mathcal{P}}(G) = \max \{ i \mid l_i(G, X) = \langle 1 \rangle \}.$$

**Proposition**
$$\gamma_{\mathcal{P}}(G) = 0 \iff G = K_1.$$

**Theorem (Lorenzini)**
$$\gamma_{\mathcal{P}}(G) = 1 \iff G = K_n.$$

**Theorem**
$$\gamma_{\mathcal{P}}(G) = n - 1 \iff G = P_n.$$

$$0 \leq \gamma_{\mathcal{P}}(G) \leq |V(G)| - 1.$$
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Given a graph $G$ and $\mathcal{P}$ a PID, let

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$$0 \leq \gamma_{\mathcal{P}}(G) \leq |V(G)| - 1.$$
Critical ideals of graphs with twins

Definition

Let $G$ and $v \in V(G)$, the duplication, denoted by $d(G, v)$, is the graph given by $V(d(G, v)) = V(G) \cup \{v^1\}$ and

$$E(d(G, v)) = \{v^1u \mid u \in N_+(v)\} \cup \{uv^1 \mid u \in N_-(v)\}$$

In this case we say that $v$ and $v^1$ are false twins.
Critical ideals of graphs with twins

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Definition

The replication of $v$ on $G$, denoted by $r(G, v)$, is the graph obtained from $d(G, v)$ by adding the arcs $vv^1$ and $v^1v$.

In this case we say that $v$ and $v^1$ are true twins.
Critical ideals of graphs with twins

Example

Let $C_4$ and $d = (-1, 1, 1, 1)$, then $C_n^d$
Theorem

Let $G$ be a graph with $n \geq 2$ vertices and $d \in \mathbb{Z}^n$. Then

$$I_j(G^d, X) \subseteq \langle \{x_v, \ldots, x_vd_v | d_v \geq 1\}, \{x_v + 1, \ldots, x_v-d_v + 1 | d_v \leq -1\} \rangle,$$

Moreover, $I_j(G^d, X) = \langle 1 \rangle \iff I_j(G, X) \{x_v=-1 | d_v \leq -1\} \cup \{x_v=0 | d_v \geq 1\} = \langle 1 \rangle$. 

Corollary

Let $G$ graph with $n$ vertices and $\delta \in \{0, 1, -1\}^n$, then

$$\gamma_P(G^d) = \gamma_P(G^\delta)$$

for all $d \in \mathbb{Z}^n$ with $\text{supp}(d) = \delta$.

Moreover, $\gamma_P(G^d) \leq n$ for all $d \in \mathbb{Z}^n$.

This bound is tight.
Theorem

Let \( G \) be a graph with \( n \geq 2 \) vertices and \( d \in \mathbb{Z}^n \). Then

\[
I_j(G^d, X) \subseteq \langle \{x_v, \ldots, x_{vd_v} \mid d_v \geq 1\}, \{x_v + 1, \ldots, x_{v-d_v} + 1 \mid d_v \leq -1\} \rangle
\]

Moreover,

\[
I_j(G^d, X) = \langle 1 \rangle \iff I_j(G, X)\{x_v = -1 \mid d_v \leq -1\} \cup \{x_v = 0 \mid d_v \geq 1\} = \langle 1 \rangle.
\]

Corollary

Let \( G \) graph with \( n \) vertices and \( \delta \in \{0, 1, -1\}^n \), then

\[
\gamma_P(G^d) = \gamma_P(G^\delta) \text{ for all } d \in \mathbb{Z}^n \text{ with } \text{supp}(d) = \delta.
\]

Moreover,

\[
\gamma_P(G^d) \leq n \text{ for all } d \in \mathbb{Z}^n.
\]

This bound is tight.
Let $P_{i,k} = \{\prod_{i=1}^{i} x_{v_i}\}$, $P_{0,k} = \{1\}$ and for all $d, d' \geq 0$ let

$$\lambda(d, d') = \begin{cases} 
0 & \text{if } d, d' = 0, \\
1 & \text{otherwise.}
\end{cases}$$
Let $P_{i,k} = \{\prod_{l=1}^{i} x_{v_l}\}$, $P_{0,k} = \{1\}$ and for all $d, d' \geq 0$ let

$$\lambda(d, d') = \begin{cases} 0 & \text{if } d, d' = 0, \\ 1 & \text{otherwise.} \end{cases}$$

**Theorem**

Let $G$ be a graph, $v \in V(G)$, $g = \gamma_{P}(G)$, $g' = \gamma_{P}(d(G, v))$, and $d = g - \gamma_{P}(G \setminus v)$, $d' = g' - g$. If $g \geq 1$, then $0 \leq d + d' \leq 2$ and

$$l_{g'+k}(d^{k+\lambda+i}(G, v), X) = \langle P_{k,k+\lambda+i}, \{ P_{l,k+\lambda+i} \cdot l_{g'+k-l}(G, X)_{x_v=0} \}^{k-1}_{l=0} \rangle$$

for all $k \geq 1$ and $i \geq 0$. 
Replication

Let $\tilde{P}_{i,k} = \{\prod_{l=1}^{i}(x_{v1}^{l} + 1)\}$, $\tilde{P}_{0,k} = \{1\}$.

**Theorem**

Let $G$ be a graph, $v \in V(G)$, $g = \gamma_P(G)$, $g' = \gamma_P(r(G, v))$, and $d = g - \gamma_P(G \setminus v)$, $d' = g' - g$. If $g \geq 1$, then $0 \leq d + d' \leq 2$ and

$$l_{g'+k}(r^{k+\lambda+i}(G, v), X) = \langle \tilde{P}_{k,k+\lambda+i}, \{\tilde{P}_{l,k+\lambda+i} \cdot l_{g'+k-l}(G, X)_{x_v=-1}\}_{l=0}^{k-1}\rangle$$

for all $k \geq 1$ and $i \geq 0$. 
Some Conjectures

**Conjecture**

If $\gamma_P(G \setminus v) = \gamma_P(G)$ for all $v \in V(G)$, then $G$ has at least a pair of twin vertices.
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Conjecture

If $\gamma_P(G) < \frac{n}{2}$ with $n \geq 5$ vertices, then $G$ has at least a pair of twin vertices.
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**Conjecture**

If \( G \) is twin-free, then \( \gamma_P(G) \geq \frac{n}{2} \). 

![Graph representation of some conjectures](image-url)
Some Conjectures

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**Conjecture**

If \( G \) is twin-free, then \( \gamma_P(G) \geq \frac{n}{2} \).

**Remark**

1) \( \Rightarrow \) 2) \( \Leftrightarrow \) 3).
We also have calculated the critical ideals of
1. Complete multipartite graphs,
2. Threshold graphs,
3. Cographs.

We are interested
1. Quasi-threshold graphs,
2. Chordal graphs,
3. Split graphs,
4. Hypercubes,
5. Bipartite graphs.
Graphs with a small algebraic co-rank

Definition

Let $\Gamma \leq k = \{ G \mid G$ is a simple connected graph with $\gamma(G) \leq k \}$.
Graphs with a small algebraic co-rank

Definition

Let $\Gamma_{\leq k} = \{ G \mid G$ is a simple connected graph with $\gamma(G) \leq k \}$.

Definition

A graph is called $\gamma_P$-critical if $\gamma_P(G \setminus v) < \gamma_P(G)$ for all $v \in V(G)$.

Let $\text{For}(\Gamma_{\leq k}) = \{ G \mid \gamma_P$-critical with $\gamma_P(G) = k + 1 \}$.
**Graphs with a small algebraic co-rank**

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>Let ( \Gamma_{\leq k} = { G \mid G \text{ is a simple connected graph with } \gamma(G) \leq k } ).</td>
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Let \( \text{For}(\Gamma_{\leq k}) = \{ G \mid \gamma_P \text{-critical with } \gamma_P(G) = k + 1 \} \).

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<td>( G \in \Gamma_{\leq k} \text{ if and only if } G \text{ is For}<em>k(\Gamma</em>{\leq k}) )-free.</td>
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</table>

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Theorem

\[ \text{Forb}(\Gamma \leq 2) \text{ is equal to} \]

\[ \begin{array}{c}
\text{\includegraphics[width=2cm]{example1.png}} & \text{\includegraphics[width=2cm]{example2.png}} & \text{\includegraphics[width=2cm]{example3.png}} & \text{\includegraphics[width=2cm]{example4.png}} & \text{\includegraphics[width=2cm]{example5.png}} \end{array} \]
Theorem

Forb(Γ_{≤2}) is equal to

\[
\begin{align*}
&K_{n_1,n_2,n_3} \\
&L_{n_1,n_2,n_3}
\end{align*}
\]

Theorem

Let G be a simple connected graph. Then, \( G \in \Gamma_{≤2} \) if and only if G is an induced subgraph of one of the following graphs:

\[
\begin{align*}
&K_{n_1,n_2,n_3} \\
&L_{n_1,n_2,n_3}
\end{align*}
\]
Figure: Family $\mathcal{F}_3$
Some Conjectures

Conjecture

For all $k \in \mathbb{N}$ the set $\text{Forb}(\Gamma \leq k)$ is finite.
Some Conjectures

Conjecture

For all \( k \in \mathbb{N} \) the set \( \text{Forb}(\Gamma \leq k) \) is finite.

Conjecture

There exists a finite set

\[
\mathcal{H} = \{(G, \delta) \mid G \text{ is a graph and } \delta \in \{0, 1\}^n\}
\]

such that \( H \in \Gamma \leq k \) if and only if \( H = G^d \) with \( \text{supp}(d) = \delta \) for some pair \( (G, \delta) \in \mathcal{H} \).
Trees

**Definition**

A set of edges $M$ is a 2-matching if $d_M(v) \leq 2$. 

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Trees

Definition
A set of edges $M$ is a **2-matching** if $d_M(v) \leq 2$.

Definition
A **2-matching** $M$ of $G^{\text{loop}}$ is **minimal** if not exists a 2-matching $M'$ such that
- $|M| = |M'|$,
- $\text{loops}(M') \subset \text{loops}(M)$.
Minimal 2-matchings

Theorem

If $T$ be a tree with $n$ vertices, then

$$l_i(T, x) = \langle \{ \det(L(M, X)) \mid M \in \mathcal{V}^i_2(T) \} \rangle$$

for all $1 \leq i \leq n$.

where $\mathcal{V}^i_2(T)$ is the set of minimal 2-matching of $T^{\text{loop}}$ of size $i$. 
Minimal 2-matchings

**Theorem**

If $T$ be a tree with $n$ vertices, then

$$l_i(T, x) = \langle \{ \det(L(M, X)) \mid M \in \mathcal{V}_2^i(T) \} \rangle \text{ for all } 1 \leq i \leq n,$$

where $\mathcal{V}_2^i(T)$ is the set of minimal 2-matching of $T^{\text{loop}}$ of size $i$.

**Corollary**

Let $T$ be a tree, then $\gamma(T) = \nu_2(T)$.
Some Conjectures

**Conjecture**

*If T be a tree with n vertices, then*

\[ B = \{ \det(L(M, X)) \mid M \in V_2^i(T) \} \]

*is a reduced Gröbner basis of \( I_i(T, x) \) for all \( 1 \leq i \leq n \).*

Until now we proved this conjecture for \( i \) equal to \( n - 1 \) and \( \nu_2(T) + 1 \).
Thank You

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