Approximation for the Normal Inverse Gaussian Process using Random Sums*

Carlos G. Pacheco-Gonzalez†

Abstract

We approximate the normal inverse Gaussian (NIG) process with random summations. The random sum we introduce is a random walk subordinated to the first passage time of another independent random walk; the model is interpreted as an internal mechanism at small scale that generates the NIG process. The main result is a functional limit theorem of weak convergence in the Skorohod topology.

Keywords: normal inverse Gaussian model; random summations; hitting times; statistical regularity.

2000 Mathematics Subject Classification: Primary: 60F17;60G50. Secondary: 68U20.

1 Introduction

Barndorff-Nielsen [1] proposed the *normal inverse Gaussian* (NIG) distribution to model the log-returns of asset prices. Since then it has been widely used in financial modelling. Based on random summations, we propose a method to

*This work was finished during a postdoctoral stay at Facultad de Ciencias, UNAM, Mexico City.

†Departamento de Matematicas, CINVESTAV-IPN, A. Postal 14-740, Mexico D.F. 07000, MEXICO. Email: cpacheco@math.cinvestav.mx

approximate the NIG process. The construction also attempts to provide an economic interpretation for such a model.

Works about approximations of the NIG model are available in the literature. A well known reference is Rydberg [2]. Maller et. al. [3] used multinomial models to simulate paths of the NIG process to price American options. Our purpose is also to be able to simulate paths of the NIG process and at the same time provide some insight of the phenomena at small scale, especially in financial series.

The idea of using random summations as a model for "tick-by-tick" financial data has been considered before (see [4] and references therein). The random sums that we propose are deeply connected to the concepts of subordination and hitting times; these concepts are important in financial mathematics and are carefully treated in [5]. The model we consider can be seen as an extension of the so-called *continuous time random walks* (CTRWs). Under conditions, limit theorems for CTRWs can lead to Lévy processes (see [4, 6]). For the model presented here, we study a limit theorem which helps to understand the *statistical regularity*[1] of the high frequency phenomena in financial series.

Briefly we mention that a NIG process arises from the subordination of a Brownian motion to the first passage time of another Brownian motion with positive drift (see [7]). This construction is the key idea for our approximations. Refer to Appendix for a brief review on the NIG model.

The paper is organized as follows. In Section 2 we give the preliminaries, where we introduce the model we use for log-returns; we also provide an economic interpretation of it. Section 3 presents the approximation scheme using the model of Section 2. The convergence is proved in Section 4, and it is also included a simulation study (using SCILAB 4.1.2) for numerical verification. Finally, Section 5 gives conclusions and a discussion on some questions of interest; and the appendix is at the end.

[1] Term taken from [8]

2 Preliminaries

We appeal to the following class of random summations, which shall help to simulate paths from a NIG process:

$$G = \left\{ G_t = \sum_{i=1}^{N_t} X_i, \ t \geq 0 \right\}, \tag{1}$$

where

$$N_t = \min \left\{ k : \sum_{i=1}^{k} \tau_i > t \right\}, \ t \geq 0. \tag{2}$$

Here $X_1, X_2, \ldots$ and $\tau_1, \tau_2, \ldots$ are two independent sequences of independent identically distributed (i.i.d.) real-valued random variables (r.v.s.).

Notice that the identity

$$N_t - 1 = \max \left\{ k : \sum_{i=1}^{k} \tau_i \leq t \right\}$$

holds only when $\tau_1, \tau_2, \ldots$ are positive r.v.s. When this is the case, it corresponds to a renewal process with interarrival times $\tau_1, \tau_2, \ldots$, and the process G can be considered as the so-called continuous time random walk ([4]), or also called *renewal reward process* ([9, 10, 8]). Furthermore, if the interarrival times are exponentially distributed, the CTRW is the well known *Compound Poisson process*. Another case for the CTRW is when the random variables $X_1, X_2, \ldots$ are positive, then in insurance jargon it is termed *Aggregate Claim Amount* ([11]).

In this paper, we do not require the random variables $\tau_1, \tau_2, \ldots$ to be positive. Therefore the process $\{N_t, t \geq 0\}$ defined in (2) is not necessarily a renewal process, instead, it takes the form of the first passage time of a random walk defined with the r.v.s. $\tau_1, \tau_2, \ldots$. We recall this concept.

Given a stochastic process $A = \{A_t, t \geq 0\}$, the first passage time process R is a non-decreasing stochastic process given by $R_z = \inf\{s : A_s > z\}$, $z \geq 0$. It follows that

$$\inf \left\{ s : \sum_{i=1}^{\lfloor s \rfloor} \tau_i > t \right\} = \min \left\{ k : \sum_{i=1}^{k} \tau_i > t \right\} = N_t, \tag{3}$$

where $\lfloor s \rfloor$ stands for the integer part of s.

Remark 1 *The idea of model (1) follows from the construction of a NIG process. This random summation can be seen as the random walk* $\left\{\sum_{i=1}^{\lfloor t \rfloor} X_i, t \geq 0\right\}$ *subordinated to the first passage time (given by* N_t*) of the random walk* $\left\{\sum_{i=1}^{\lfloor t \rfloor} \tau_i, t \geq 0\right\}$*. In this sense, model (1) resembles a discrete version of a NIG process.*

Economic Interpretation. We give the following interpretation for financial series. Consider G as the increments on asset prices. The process N_t represents the number of transactions up to time t, and the variables $X_1, X_2, \ldots$ represent the changes on each transaction. At time t, the number of transactions N_t is determined when the random walk $H = \left\{\sum_{i=1}^{\lfloor s \rfloor} \tau_i, s \geq 0\right\}$ surpasses a level t. We conceive the random walk H as a hidden factor in the market; as soon as that process surpasses a level, more transactions are triggered.

The random summation (1) is a toy model for the increments on asset prices. It provides with an explanation to the dynamics at small scale in financial markets, and this can be important on its own right. It is well known that financial series experience high frequency. Because of this, we would like to analyze the large scale effect of model (1). Along the concepts in Whitt [8], we want to study the statistical regularity of the process, and we can do this by studying limit theorems.

3 Approximation scheme

In order to perform the approximation of the NIG process, we first construct a sequence of stochastic processes based on the random summations discussed in the previous section.

Consider the following sequence of processes

$$G^{(n)} = \left\{ G_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} X_i^{(n)},\ t \geq 0 \right\},\ n = 1, 2, \ldots \tag{4}$$

where

$$N_t^{(n)} = \min\left\{k : \sum_{i=1}^{k} \tau_i^{(n)} > t\right\},\ t \geq 0.$$

The variables involved are independent and determined as follows. For all integers i and n,

$$P\left(X_i^{(n)} = \frac{\gamma_0}{n} \pm \frac{\Delta_0}{\sqrt{n}}\right) = \frac{1}{2} \text{ and } P\left(\tau_i^{(n)} = \frac{\gamma_1}{n} \pm \frac{\Delta_1}{\sqrt{n}}\right) = \frac{1}{2}, \tag{5}$$

with $\gamma_0, \Delta_0, \Delta_1$ real numbers and $\gamma_1 > 0$. The condition on γ_1 comes from the fact we want the limit of the random walk $\left\{\sum_{i=1}^{\lfloor t \rfloor} \tau_i^{(n)}, t \geq 0\right\}$, when $n \to \infty$, to be a Brownian motion with positive drift. Now we need the coming results.

For the processes

$$X^{(n)} = \left\{X_t^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} X_i^{(n)},\ t \geq 0\right\} \text{ and } Z^{(n)} = \left\{Z_s^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} \tau_i^{(n)},\ s \geq 0\right\}, \tag{6}$$

it is well known from *Donsker's Theorem* that $X^{(n)} \Longrightarrow X$ and $Z^{(n)} \Longrightarrow Z$, where X is a Brownian motion with drift γ_0 and variance Δ_0^2, and Z is another Brownian motion with drift γ_1 and variance Δ_1^2. The processes X and Z are independent. Here $\Longrightarrow$ stands for weak convergence in the Skorohod topology.

Another ingredient we need is the specific parametrization of the subordination using processes X and Z.

Proposition 2 *Let $X = \{X_t,\ t \geq 0\}$ and $Z = \{Z_t,\ t \geq 0\}$ be two independent Brownian motions with drifts γ_0, γ_1 and variances Δ_0^2, Δ_1^2, respectively. Let $Y = \{Y_t,\ t \geq 0\}$ be the first passage time of Z. Then, the subordination $L = \{L_t = X_{Y_t},\ t \geq 0\}$ is a NIG process with the parameters*

$$\left(\sqrt{\left(\frac{\gamma_1}{\Delta_1 \Delta_0}\right)^2 + \left(\frac{\gamma_0}{\Delta_0^2}\right)^2},\ \frac{\gamma_0}{\Delta_0^2},\ \frac{\Delta_0}{\Delta_1},\ 0\right). \tag{7}$$

Proof. By means of the Wiener-Hopf factorization (see Theorem 46.3 and Example 46.6 in [12]) we know that

$$E\left[e^{-\theta Y_t}\right] = \exp\left[-t\frac{1}{\Delta_1^2}\left(\sqrt{\gamma_1^2 + 2\Delta_1^2\theta} - \gamma_1\right)\right] = \exp\left[\frac{t\gamma_1}{\Delta_1^2}\left(1 - \sqrt{1 - \frac{2\Delta_1^2\theta}{\gamma_1^2}}\right)\right];$$

using (18) we verify that this corresponds to the inverse Gaussian r.v. $IG\left(\frac{t\gamma_1}{\Delta_1^2},\ \frac{\gamma_1^2}{\Delta_1^2}\right)$.

Then, we have that L_t, conditioned to Y_t, is the normal r.v. $N\left(\gamma_0 Y_t,\ \Delta_0^2 Y_t\right)$, which is the same as $N\left(\frac{\gamma_0}{\Delta_0^2}\left(\Delta_0^2 Y_t\right),\ \Delta_0^2 Y_t\right)$, or also $N\left(\frac{\gamma_0}{\Delta_0^2} Y_t^*,\ Y_t^*\right)$, with $Y_t^* \sim IG\left(\frac{t\gamma_1}{\Delta_1^2},\ \frac{\gamma_1^2}{\Delta_1^2\Delta_0^2}\right)$ (see again (18)).

Then, it follows from bullet 4 in Appendix (equations (16)) that

$$\beta = \frac{\gamma_0}{\Delta_0^2},\ \delta\gamma^2 = \frac{t\gamma_1}{\Delta_1^2}, \gamma = \frac{\gamma_1^2}{\Delta_1^2\Delta_0^2} \text{ and } \mu = 0, \tag{8}$$

and the parameters (7) can be derived. ∎

It is known that the NIG process comes from a subordination of the Brownian motion (see bullet 4 in Appendix), however we are constructing specifically the form we need.

Remark 3 *To this point, notice that given Δ_0, for instance, we can find Δ_1, γ_0 and γ_1 using equations (8). Hence, for some parameters of the NIG process with $\mu = 0$ there are infinitely many choices to build the process from a subordination.*

4 Weak convergence

From Proposition 2 we have the process L as a candidate for the limit of $G^{(n)}$, when $n \to \infty$. Indeed, this is the next result.

Theorem 4 *As $n \to \infty$, the sequence of processes $G^{(n)}$ converges weakly to L (of Proposition 2) in the Skorohod topology.*

Proof. As mentioned before, we know that $X^{(n)} \Longrightarrow X$ and $Z^{(n)} \Longrightarrow Z$.

First, for the process $N^{(n)} = \left\{N_t^{(n)}, t \geq 0\right\}$, we need to prove that $\frac{N^{(n)}}{n} \overset{f.d.d.}{\to} Y$, where $\overset{f.d.d.}{\to}$ stands for convergence of the finite dimensional distributions.

Set any two finite collection of positive real numbers $\{t_1, t_2, \ldots, t_m\}$ and $\{r_1, r_2, \ldots, r_m\}$ with $0 < t_1 < t_2 < \ldots < t_m$. We have the following set of equalities,

$$P\left(\frac{N_{t_i}^{(n)}}{n} \leq r_i; i = 1, \ldots, m\right) \tag{9}$$

$$= P\left(\min\left\{k : \sum_{i=1}^{k} \tau_i^{(n)} > t_i\right\} \leq nr_i; i = 1, \ldots, m\right)$$

$$= P\left(\sum_{i=0}^{k} \tau_i^{(n)} > t_i,\ \text{for some } k = 1, 2, \ldots, \lfloor nr_i \rfloor; i = 1, \ldots, m\right)$$

Taking limits we have

$$\lim_{n\to\infty} P\left(\sum_{i=0}^{k} \tau_i^{(n)} > t_i,\ \text{for some } k = 1, 2, \ldots, \lfloor nr_i \rfloor; i = 1, \ldots, m\right)$$

$$= \lim_{n\to\infty} P\left(\sum_{i=0}^{\lfloor ns \rfloor} \tau_i^{(n)} > t_i,\ \text{for some } s \in [0, r_i]; i = 1, \ldots, m\right)$$

$$= P\left(Z_s > t_i,\ \text{for some } s \in [0, r_i]; i = 1, \ldots, m\right) = P\left(Y_{t_i} \leq r_i; i = 1, \ldots, m\right),$$

and so $\frac{N^{(n)}}{n} \overset{f.d.d.}{\to} Y$.

Since the process $N^{(n)}$ is càdlàg (i.e. it has sample paths which are right continous, with left limits), so is $\frac{N^{(n)}}{n}$, then we can use Theorem 3 in [13] to conclude that $\frac{N^{(n)}}{n} \Longrightarrow Y$ as $n \to \infty$.

Notice that $G^{(n)}$ can be seen as a composition of $X^{(n)}$ and $\frac{N^{(n)}}{n}$, i.e.

$$G_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} X_i^{(n)} = \sum_{i=1}^{\left\lfloor n\frac{N_t^{(n)}}{n} \right\rfloor} X_i^{(n)}.$$

Since X has continuous-paths and Y is increasing and càdlàg, we appeal to Theorem 13.2.2 in [8] to conclude that this composition is a continuous map in the Skorohod topology. By the Continuous-mapping Theorem,

$$G^{(n)} \Longrightarrow L \text{ as } n \to \infty.$$

■

4.1 Simulations

In this section, we develop some simulations (programs written in SCILAB 4.1.2) to show the behaviour of the random sums.

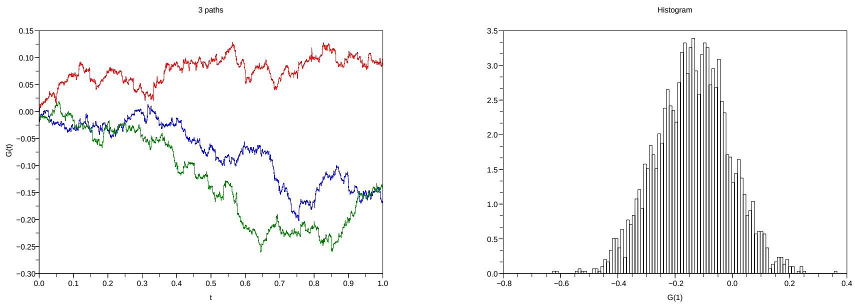

Figure 1: Simulation of model.

First of all, we may choose a NIG process we ought to study. Let us take values for $\alpha, \beta, \delta, \mu$, previously calculated in Prause [14] for a NIG distribution; data from the New York Stock Exchange Composite Index from the period January 2, 1990 to November 29, 1996. Those parameters are:

$$\alpha = 136.6,\ \ \beta = -8.95,\ \ \delta = .0059,\ \ \mu = .000791. \tag{10}$$

Since they are estimated on a daily basis, we transform them to an annual basis, and the only change to do that is taking $\delta = .0059 \times 365$ (see bullet 3 in Appendix). We also neglect the value of μ, which is relatively small in all the estimations in Prause [14]. Thus we consider a NIG process with parameters

$$\alpha = 136.6, \beta = -8.95, \delta = 2.1535, \mu = 0. \tag{11}$$

Now we take $\Delta_0 = 0.1$, and using equations (8), we find $\Delta_1 = 0.0464$, $\gamma_0 = -0.0895$ and $\gamma_1 = 0.632$. According to Theorem 4, using variables (5), the process $G^{(n)}$ defined as model (4) converges to a NIG process with parameters (11), as $n \to \infty$. Let us take $n = 100000$ and a time interval $[0, 1]$; we then generate three sample paths (Figure 1).

Additionally, with the same parameters, we generate a histogram of the process at time 1; we do this by simulating 3000 trajectories and taking the last value of each path (here we take $n = 10000$).

Remark 5 *Notice that this approximation is made for the NIG process with the* *parameter* $\mu = 0$. *Nevertheless, it is possible to simulate from the distribution* $NIG(\alpha, \beta, \delta, \mu)$ *when* $\mu \neq 0$, *by limiting* $G_1^{(n)} + \mu$.

5 Conclusions and open questions

In order to approximate the NIG process, we have proposed a model for log-returns using random summations; when doing that, we provide an insight of the behaviour in financial markets at small scale. One of the main characteristics of this method is that it gives a way to obtain paths-simulations for the NIG process in a fairly simple way.

The NIG model has been extended to the so-called *generalized hyperbolic* (GH) model, which has proved to be successful to model financial series (see [14]). A natural question to ask concerns the existence of random summations, which may be somewhat simple, to approximate the GH model. Notably, in [15] there is proposed a method to simulate from a GH random variable using random sums, but the scheme differs in spirit to the idea presented here.

Another important model for financial modelling is the *variance gamma* (VG) process. This process is a Brownian motion subordinated to a gamma process; this model was first introduced by Madam and Seneta [16]. A rather intriguing fact is the connection between the NIG process and the VG process. Without going into details the following result is proved in [5].

Let X be a variance gamma process, and B a Brownian motion; both processes independent. Denote by W the *variation* of X, i.e.

$$W_t = \int_0^t |dX_t| \,. \tag{12}$$

Also, consider the first time when the drifted Brownian motion $\left\{B_t + \sqrt{2}t, t \geq 0\right\}$ hits the variation W, that is

$$T_t = \inf\left\{s : B_s + \sqrt{2}s > W_t\right\}, \quad t \geq 0. \tag{13}$$

Then, the following identity holds

$$X_t \stackrel{d}{=} B_{T_t} \text{ for each } t \geq 0. \tag{14}$$

The result links two important stochastic processes that are well applied for financial modelling. This gives us depeer understanding of the relation between models, which at first glance seem different. It can be expected, or argued, that the same internal mechanism of financial series shows different statistical regularity. In lieu of this insight, it would be of interest to provide a more generic model that explains the local behaviour at small scale and at the same time that has capability to give rise to models such as GH, NIG and VG.

Another important issue is associated to the theory of financial valuation. It is known that the *Cox-Ross-Rubinstein*(CRR) model can be used to approximate prices for the *Black-Scholes*(BS) model, yet the former provides a fairly simple but powerful way to do that; this fact is due to the simplicity of the binomial model. We also wonder about the ability of the random summations presented here to price using a sequence of *Martingale measures*, just as it is done for the BS model using the CRR model. This issue is pursued in [17], however, still in an heuristic way.

6 Appendix section

1) A random variable X has NIG distribution, denoted $X \sim NIG(\alpha, \beta, \delta, \mu)$, if the density is given by

$$f_X(x) = \frac{\delta\alpha \exp(\delta\gamma + \beta(x-\mu))}{\pi\sqrt{\delta^2 + (x-\mu)^2}} K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2}),$$

where $K_1(w) = \frac{1}{2}\int_0^\infty \exp(\frac{-1}{2}w(t+\frac{1}{t}))dt$ (modified Bessel function of the third kind), and the parameters satisfying: $0 \leq |\beta| \leq \alpha$ and $\delta > 0$ (see also www.wikipedia.org).

2) The mean (m) and variance (v) are respectively given by

$$m = \mu + \frac{\delta\beta}{\gamma}, \quad v = \frac{\delta\alpha^2}{\gamma^3}. \tag{15}$$

3) A NIG process, namely $L = \{L_t, t \geq 0\}$, with parameters $(\alpha, \beta, \delta, \mu)$ is a Lévy process such that

$$L_{t-s} \sim NIG(\alpha, \beta, \delta(t-s), \mu(t-s)), \text{ for } s < t.$$

4) The NIG distribution arises from the following normal variance-mean mixture with the *inverse Gaussian* distribution,

$$X \mid Y \sim N(\mu + \beta Y, Y), \text{ with } Y \sim IG\left(\delta\gamma, \gamma^2\right), \tag{16}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$, and $IG(\eta, \lambda)$ is the inverse Gaussian distribution. A random variable Y has distribution $IG(\eta, \lambda)$ if the density is given by

$$f_Y(y) = \frac{\eta}{\sqrt{2\pi\lambda}} y^{-\frac{3}{2}} e^{\frac{(\eta-\lambda y)^2}{2\lambda y}}. \tag{17}$$

5) For the IG distribution the following is known

$$E\left(e^{\theta Y}\right) = \exp\left[\eta\left(1 - \sqrt{1 - \frac{2\theta}{\lambda}}\right)\right], \quad \text{and} \quad aY \sim IG\left(\eta, \frac{\lambda}{a}\right). \tag{18}$$

Acknowledgments

I would like to express my gratitude to my former PhD advisor, Jordan Stoyanov, for his supervision. I also thank Onésimo Hernández-Lerma for his general advice, and Andreas Kyprianou for some useful comments on this work.

References

[1] Barndorff-Nielsen, O.E., Processes of normal inverse Gaussian type. *Finance Stoch. 1998*, **2**, 41–68.

[2] Rydberg, T.H., The normal inverse Gaussian Lévy process: simulation and approximation. *Comm. Statist. Stochastic Models 1997*, **13(4)**, 887–910.

[3] Maller, R. A.; Solomon, D.H.; Szimayer, A., A multinomial approximation for American option prices in Lévy process models. *Math. Finance 2006*, **16(4)**, 613–640.

[4] Scalas, E., Five Years of Continuous-time Random Walks in Econophysics. In: *The Complex Network of Economic Interactions*, Lectures Notes in Economics and Mathematical Systems 567, A. Namatame, T. Kaizouji, and Y. Aruka (Eds.); Springer, 3–16.

[5] Geman, H.; Madan, D.B.; Yor, M., Stochastic volatility, jumps and hidden time changes. *Finance Stoch. 2002,* **6**, 63–90.

[6] Becker-Kern, P.; Meerschaert, M. M.; Scheffler, H.P., Limit theorems for continuous-time random walks with two times scales. *J. Appl. Probab. 2004*, **41**, 455–466.

[7] Barndorff-Nielsen, O.E., Normal inverse Gaussian distributions and stochastic volatility modelling. *Scand. J. Statist. 1997,* **24**, 1–13.

[8] Whitt, W., *Stochastic-Processes Limits*, Springer, New York, 2002.

[9] Asmussen, S., *Ruin Probability*, World Scientific, Singapore, 2002.

[10] Grimmett, G.; Stirzaker, D., *Probability and Random Processes*, Oxford University Press, New York, 2001.

[11] Rolski, T.; Schmidli, H.; Schmidt, V.; Teugels, J., *Stochastic Processes for Insurance and Finance*, John Wiley, & Sons, Chichester, 1999.

[12] Sato, K.I., *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, 1999.

[13] Bingham, N.H., Limit Theorems for Occupation Times of Markov Processes. *Z. Wahrscheinlichkeitstheorie verw. Geb. 1971*, **17**, 1–22.

[14] Prause, K., *The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measure*. PhD Thesis, Universität Freiburg, 1999.

[15] Eberlein, E.; Hammerstein, E. A. v., Generalized Hyperbolic and Inverse Gaussian distributions: Limiting cases and Approximation of Processes. In: *Seminar on Stochastic Analysis, Random Fields and Applications IV*, Progress in Probability 58, R.C.Dalang, M. Dozzi, F. Russo (Eds.), Birkhäser Verlag ,221–264, 2004.

[16] Madam, D.B. ;Seneta, E., The variance gamma model for share market returns. *J. Business 1990*, **63**, 511–524.

[17] Pacheco-González, C.G., *Stability of Random Dynamical Systems and Applications.* PhD Thesis, University of Newcastle upon Tyne, 2006.