# Problemas directos e inversos: matemáticas y aplicaciones

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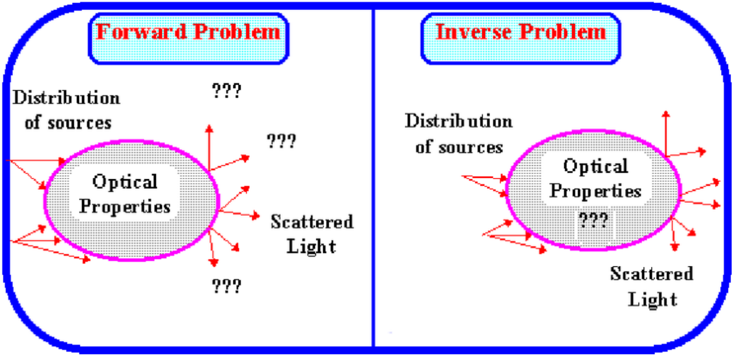
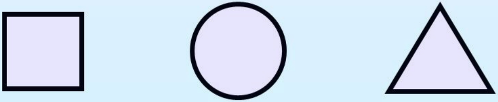

Forward Problem
Distribution of sources
???
???
Optical Properties
Scattered Light
???
Inverse Problem
Distribution of sources
Optical Properties
???
Scattered Light

# 'Can one hear the shape of a drum ?'

- This question was asked by Marc Kac (1966).

Marc Kac (1914-1984)

- Is it possible to have two different drums with the same spectrum (***isospectral drums***) ?




Is IT...
STAR-SHAPED?!
NOPE!
WRONG
AGAIN!

- Ecuación de Schrödinger 1-D

$$-y'' + q(x)y = \lambda y, \quad \lambda \in \mathbb{C} - \text{parámetro espectral}$$

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- **Problemas directos:**

$q(x)$ y algunas condiciones adicionales

$$\Downarrow$$

espectro y algunas características espectrales

Ecuación de Schrödinger 1-D

$$-y'' + q(x)y = \lambda y, \quad \lambda \in \mathbb{C} - \text{parámetro espectral}$$

**Problemas inversos:**

$q(x)$ y algunas condiciones adicionales

$$\Uparrow$$

espectro y algunas características espectrales

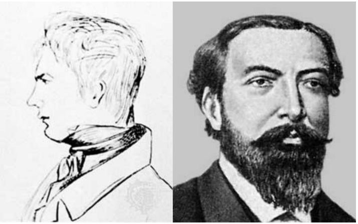
$$-y'' + q(x)y = \lambda y,$$

$$\Uparrow$$

mediante la transformación de Liouville

$$-\left(P(x)Y'(x)\right)' + Q(x)Y(x) = \lambda R(x)Y(x)$$

Ecuación de Sturm-Liouville



Jacques Charles Francois Sturm (1803-1855) y Joseph Liouville (1809-1882)

# Un gran problema

- Consideren

$$-y'' + q(x)y = \lambda y \tag{1}$$

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- Si "se nos olvida" poner una de las dos primas:

$$-y' + q(x)y = \lambda y,$$

la ecuación se resuelve fácilmente en forma cerrada

$$y(x) = Ce^{-\lambda x} e^{\int_a^x q(t)dt}.$$

# Un gran problema

- Si se conoce la solución de la ecuación

$$-y'' + q(x)y = 0$$

¿cómo resolver

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Spectral Parameter Power Series

series de potencias en términos del parámetro espectral

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- V. V. Kravchenko, A representation for solutions of the Sturm-Liouville equation. *Complex Variables and Elliptic Equations* **53** (2008), 775–789.

- V. V. Kravchenko, R. M. Porter, Spectral parameter power series for Sturm-Liouville problems. *Mathematical Methods in the Applied Sciences* **33** (2010), 459–468.

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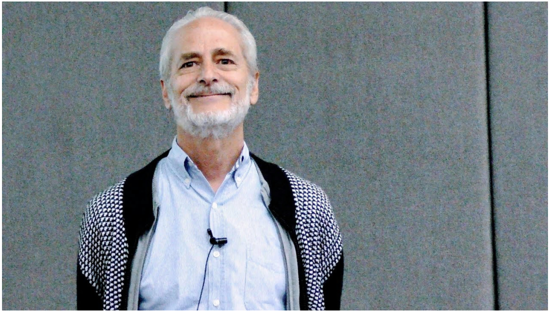
- 

Figure: Mike Porter

- V. V. Kravchenko, R. M. Porter, Spectral parameter power series for Sturm-Liouville problems. *Mathematical Methods in the Applied Sciences* **33** (2010), 459–468.

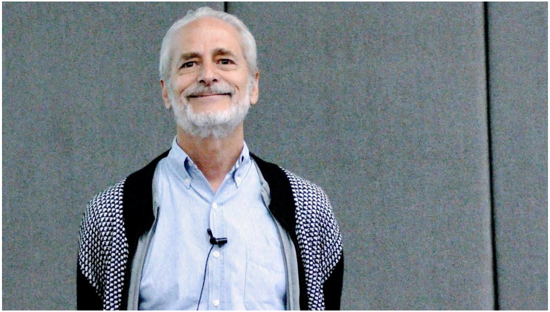
- 

Figure: Mike Porter

- Búsqueda en Wikipedia: **SPPS**

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- Es suficiente para encontrar unas decenas de autovalores, no es la mejor opción cuando se trata de problemas en intervalos infinitos.
- Sin embargo, logramos desarrollar **un método más poderoso** basado en **operadores de transmutación.**

**Permite calcular miles de autovalores y autofunciones y resolver problemas directos e inversos en intervalos finitos e infinitos.**

V. V. Kravchenko, L. J. Navarro, S. M. Torba, *Applied Mathematics and Computation*, **314**, issue 1 (2017)

# Problemas de Sturm-Liouville. Intervalo finito

- $$-y'' + q(x)y = \lambda y, \quad 0 < x < L \tag{2}$$
  $$y'(0) - hy(0) = 0, \tag{3}$$
  $$y'(L) + Hy(L) = 0. \tag{4}$$

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  $$y'(L) + Hy(L) = 0. \tag{4}$$
- $q \in L_2(0, L)$
- $h$, $H$ constants

# Problema de Sturm-Liouville directo

- Encontrar todos los valores $\lambda_n$ del parámetro $\lambda$ para los cuales existen soluciones no triviales del problema (los autovalores o eigenvalores)

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- Se sabe que existe una sucesión de los autovalores

$$\{\lambda_n\}_{n=0}^{\infty},$$

$$\lambda_n < \lambda_m, \quad \text{para } n < m$$

$$\lambda_n \to +\infty, \quad \text{cuando } n \to \infty$$

$$\lambda_n \leftrightarrow \varphi_n(x) \text{ única lin. ind.}$$

Se conocen muchas otras propiedades de autovalores y autofunciones del problema de Sturm-Liouville. Sin embargo, métodos analíticos de solución en libros de texto solo para el caso

$$q \equiv 0,$$

y acaso el método de diferencias finitas para la solución numérica.

Sturm-Liouville equation

$$-y''(x)+q(x)y(x)=\rho^2 y(x), \quad x\in(0,L),$$

$q\in L_2(0,L)$ - complex valued.
Fundamental system of solutions: $\varphi(\rho,x)$ and $S(\rho,x)$ satisfying

$$\varphi(\rho,0)=1, \quad \varphi'(\rho,0)=h,$$

$$S(\rho,0)=0, \quad S'(\rho,0)=1,$$

**Theorem [Kravchenko, Navarro, Torba 2017]** $\varphi(\rho,x)$ and $S(\rho,x)$ admit the series representations

$$\varphi(\rho,x)=\cos(\rho x)+\sum_{n=0}^{\infty}(-1)^n g_n(x)\mathbf{j}_{2n}(\rho x),$$
$$S(\rho,x)=\frac{\sin(\rho x)}{\rho}+\frac{1}{\rho}\sum_{n=0}^{\infty}(-1)^n s_n(x)\mathbf{j}_{2n+1}(\rho x),$$

where $\mathbf{j}_k(z)$ - spherical Bessel ($\mathbf{j}_k(z):=\sqrt{\frac{\pi}{2z}}J_{k+\frac{1}{2}}(z)$). The coefficients $g_n(x)$, $s_n(x)$ can be calculated following a recurrent integration procedure, starting with

$$g_0(x)=\varphi(0,x)-1,\quad s_0(x)=3\left(\frac{S(0,x)}{x}-1\right).$$

For every $\rho\in\mathbb{C}$ the series converge pointwise. For every $x\in[0,L]$ the series converge uniformly on any compact set of the complex plane of the variable $\rho$.

Any series of the type

$$\sum_{n=0}^{\infty} a_n J_{\nu+n}(z)$$

is called a *Neumann series*, although in fact Neumann considered* only the special type of series for which $\nu$ is an integer; the investigation of the more general series is due to Gegenbauer†.


G. N. Watson A treatise on the theory of Bessel functions, 1922.

J. E. Wilkins, Neumann series of Bessel functions, Trans. Amer. Math. Soc. 64 (1948), 359–385.

A. Baricz, D. Jankov, T. K. Pogány, Series of Bessel and Kummer-type functions. Lect. Notes in Math., v. 2207. Springer, 2017.

# Origin of the series representations

Transformation operators

$$\varphi(\rho, x) = \cos\rho x + \int_0^x K_c(x,t)\cos\rho t\,dt, \quad \forall\rho\in\mathbb{C}.$$

$$S(\rho, x) = \frac{\sin\rho x}{\rho} + \int_0^x K_s(x,t)\frac{\sin\rho t}{\rho}\,dt, \quad \forall\rho\in\mathbb{C}.$$

[A. Ya. Povzner 1948; B. M. Levitan, *Inverse Sturm-Liouville problems*, VSP, Zeist, 1987; V. A. Marchenko, *Sturm-Liouville Operators and Applications*, AMS Chelsea Publishing, 2011; V. A. Yurko, *Introduction to the theory of inverse spectral problems.* Moscow, Fizmatlit, 2007, 384pp. (Russian); Shishkina E. L. and Sitnik S. M. (2020), *Transmutations, singular and fractional differential equations with applications to mathematical physics*, Elsevier, Amsterdam]
For any fixed $x > 0$ the functions

$$\varphi(\rho, x) - \cos\rho x \quad \text{and} \quad S(\rho, x) - \frac{\sin\rho x}{\rho}$$

are even and belong to the Paley-Wiener space $PW_x$.

$\{\mathbf{j}_n(\rho x)\}_{n=0}^{\infty}$ - basis in $PW_x$.

$$\varphi(\rho, x) = \cos(\rho x) + \sum_{n=0}^{\infty} (-1)^n g_n(x) \mathbf{j}_{2n}(\rho x),$$

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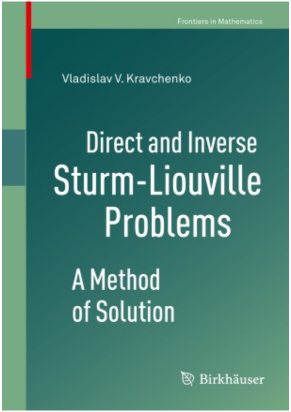
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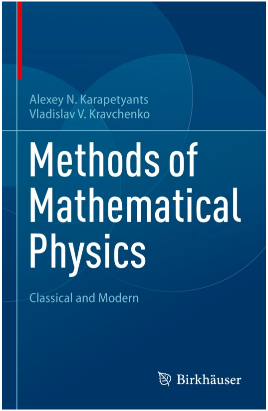
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Vladislav V. Kravchenko
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Physics
Classical and Modern
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# Two important features

1. Since

$$g_0(x) = \varphi(0,x) - 1, \quad s_0(x) = 3\left(\frac{S(0,x)}{x} - 1\right),$$

$q(x)$ can be recovered from the first coefficients of the series:

$$q(x) = \frac{g_0''(x)}{g_0(x) + 1}$$

and

$$q(x) = \frac{(xs_0(x))''}{xs_0(x) + 3x}.$$

# Two important features

2. Consider the partial sum

$$S_N(\rho, x) := \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho}\sum_{n=0}^{N}(-1)^n s_n(x)\mathbf{j}_{2n+1}(\rho x).$$

The estimate holds

$$|S(\rho, x) - S_N(\rho, x)| < \varepsilon_N(x) \quad \text{for all } \rho \in \mathbb{R},$$

where $\varepsilon_N(x) > 0$ and $\varepsilon_N(x) \to 0$, $N \to \infty$.
Same for $\varphi(\rho, x)$:

$$|\varphi(\rho, x) - \varphi_N(\rho, x)| < \varepsilon_N(x).$$

Problema de prueba ([J. D. Pryce, *Numerical solution of Sturm-Liouville problems*, Oxford: Clarendon Press, 1993])

$$\begin{cases} -u'' + e^x u = \lambda u, & 0 \le x \le \pi, \\ u(0,\lambda) = 0, & u(\pi,\lambda) = 0. \end{cases}$$

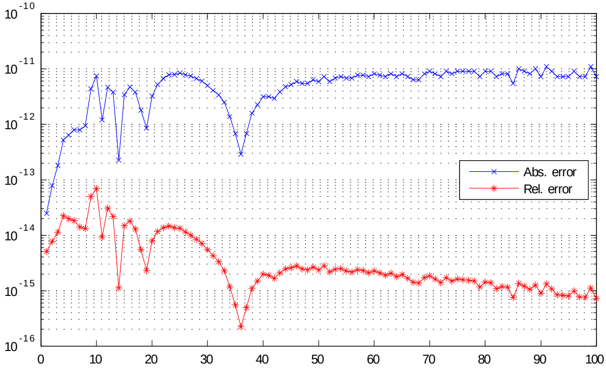
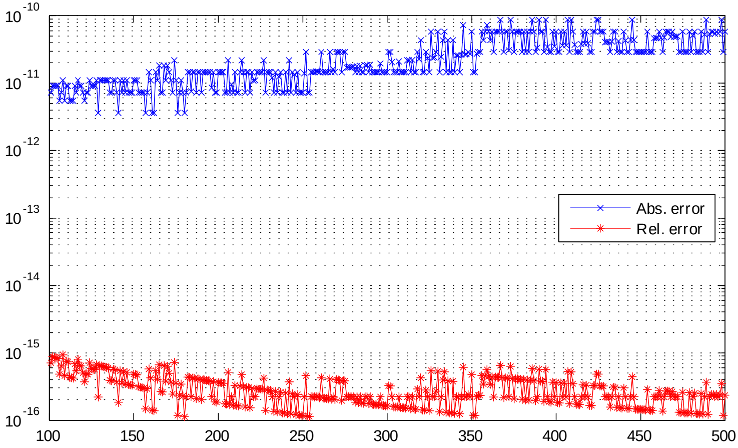



Figure: N=29. Error abs. (azul); error rel. (rojo) de los primeros 100 autovalores.

# Los mismos datos pero para los siguientes 400 eigenvalores



Con precisión de 200 digitos en Wolfram Mathematica los primeros 10000 autovalores y autofunciones se obtienen con el error absoluto de orden $10^{-105}$.

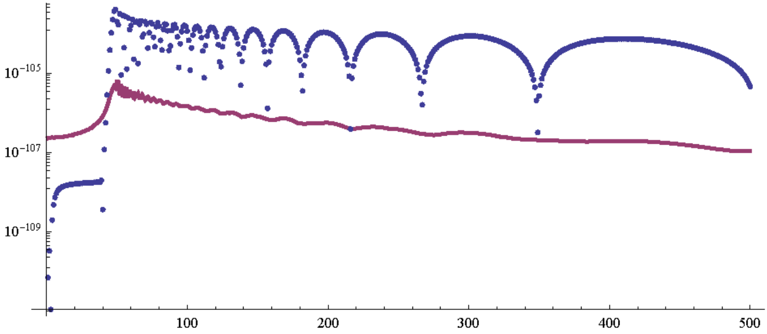



Figure: Abs. error of the first 500 eigenvalues (blue) and eigenfunctions (violet).

# Inverse two-spectra problem

Consider the Sturm-Liouville equation

$$-y'' + q(x)y = \rho^2 y, \quad x \in (0, L), \tag{5}$$

with the boundary conditions

$$y'(0) - hy(0) = 0, \quad y'(L) + Hy(L) = 0, \tag{6}$$

where $h$ and $H$ are arbitrary constants.
Denote for $\lambda = \rho^2$

$$\Delta(\lambda) := \varphi'(\rho, L) + H\varphi(\rho, L). \tag{7}$$

Obviously, the function $\varphi(\rho, x)$ fulfills the first condition in (6), $\varphi'(\rho, 0) - h\varphi(\rho, 0) = 0$, and thus, the spectrum of problem (5), (6) is a sequence $\{\lambda_n = \rho_n^2\}_{n=0}^{\infty}$ such that

$$\Delta(\lambda_n) = 0.$$

Consider

$$-y'' + q(x)y = \rho^2 y, \quad x \in (0, L)$$

subject to another pair of boundary conditions

$$y'(0) - hy(0) = 0, \quad y(L) = 0. \tag{8}$$

The spectrum of this problem is a sequence $\left\{\nu_n = \mu_n^2\right\}_{n=0}^{\infty}$ such that

$$\varphi(\mu_n, L) = 0.$$

**Problem (Two-spectra inverse problem)** Given two sequences of numbers $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$, find $q \in L_2(0, L)$, and the constants $h$, $H$, such that $\{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of problem (5), (6) and $\{\nu_n\}_{n=0}^{\infty}$ the spectrum of problem (5), (8).

- La teoría de este problema se ha desarrollado en trabajos de V. A. Ambartsumyan (1929), G. Borg (1946), V. A. Marchenko, B. M. Levitan, I. M. Gelfand (1950's),...

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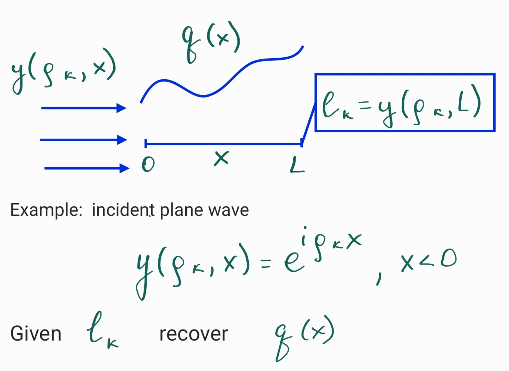
- The results on the characterization of the spectra are well known: [V. A. Yurko, Introduction to the theory of inverse spectral problems, Fizmatlit, Moscow, 2007 (in Russian)]
  For a more general situation:
  [A. M. Savchuk, A. A. Shkalikov, Inverse problem for Sturm–Liouville operators with distribution potentials: reconstruction from two spectra, Russ. J. Math. Phys. 12 (2005), 507–514]

Inverse coefficient problems for
$-y''(x)+q(x)y(x)=\rho^2 y(x), \quad x\in(0,L)$,




Special cases include:

inverse two-spectra problem;
recovery of the potential $q(x)$ from a Weyl function;
recovery of a cross-sectional area of a human vocal tract.

Cauchy data at the origin are given

$$a(\rho_k) = y(\rho_k, 0) \quad \text{and} \quad b(\rho_k) = y'(\rho_k, 0),$$

together with the values $l_k = y(\rho_k, L)$.
**Problem A** Given input data

$$\{\rho_k,\ a(\rho_k),\ b(\rho_k),\ l_k\}, \tag{9}$$

recover $q(x)$.

**Example** $y(\rho, x)$ can be a plane wave: $y(\rho, x) = e^{i\rho x}$ for $x \in (-\infty, 0)$. Then, of course, $a(\rho_k) = 1$ and $b(\rho_k) = i\rho_k$.

# Some existing methods for inverse S-L problems

W. Rundell, P. E. Sacks, *Reconstruction techniques for classical inverse Sturm–Liouville problems.* Math. Comput. 58 (1992), 161–83.

M. Ignatiev, V. Yurko, *Numerical methods for solving inverse Sturm-Liouville problems.* Results Math. 52 (2008), no. 1-2, 63–74.

A. Kammanee, C. Böckmann, *Boundary value method for inverse Sturm-Liouville problems.* Applied Mathematics and Computation, 214 (2009) 342-352.

B. M. Brown, V. S. Samko, I. W. Knowles, M. Marletta, *Inverse spectral problem for the Sturm–Liouville equation*. Inverse Problems 19 (2003) 235–252.

B. D. Lowe, M. Pilant, W. Rundell, *The recovery of potentials from finite spectral data*. SIAM J. Math. Anal. 23 (1992), no. 2, 482–504.

All are iterative.

No one solves Problem A.

No one deals with complex potentials.

# Application of NSBF representations in inverse problems


V. Kr., *On a method for solving the inverse Sturm–Liouville problem*, J. Inverse Ill-posed Probl. 27 (2019), 401–407.

V. Kr., **Direct and inverse Sturm-Liouville problems: A method of solution**, Birkhäuser, Cham, 2020.

V. Kr., S. M. Torba, *A direct method for solving inverse Sturm-Liouville problems*, Inverse Problems 37 (2021), 015015.

V. Kr., S. M. Torba, *A practical method for recovering Sturm-Liouville problems from the Weyl function*. Inverse Problems 37 (2021), 065011.

V. Kr., K. V. Khmelnytskaya, F. A. Çetinkaya, *Recovery of inhomogeneity from output boundary data*. Mathematics, 10 (2022).

V. Kr., *Spectrum completion and inverse Sturm-Liouville problems*. Math. Meth. Appl. Sci., 46 (2023), 5821–5835.

V. Kr. *Reconstruction techniques for complex potentials.* Submitted.

**Auxiliary problem**

$$\text{Given } \varphi(\rho, L) \text{ and } S(\rho, L), \text{ find } q(x).$$

**Naturally arises when solving, e.g., two-spectra inverse problems**

W. Rundell, P. E. Sacks, *Reconstruction techniques for classical inverse Sturm–Liouville problems.* Math. Comput. 58 (1992), 161–83.

Kh. Chadan, D. Colton, L. Päivärinta, W. Rundell, *An introduction to inverse scattering and inverse spectral problems*. SIAM, Philadelphia, 1997.

V. V. Kravchenko, S. M. Torba, *A direct method for solving inverse Sturm-Liouville problems*, Inverse Problems 37 (2021), 015015.

**or recovering from a Weyl function**

V. V. Kravchenko, S. M. Torba, *A practical method for recovering Sturm-Liouville problems from the Weyl function*. Inverse Problems 37 (2021), 065011.

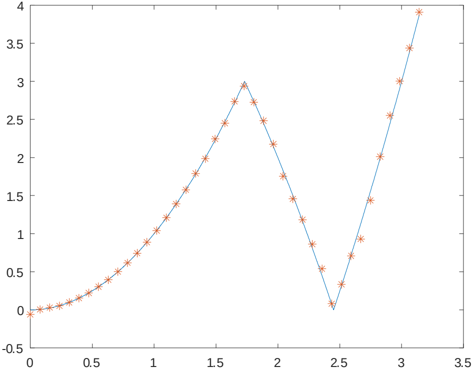



Figure: El potencial $q(x) = \left|3 - \left|x^2 - 3\right|\right|$ recuperado a partir de 200 autovalores.


V. V. Kravchenko *On a method for solving the inverse Sturm-Liouville problem.* J of Inverse and Ill-Posed Problems v. 27 (2019), no. 3, 401–407.

V. V. Kravchenko **Direct and inverse Sturm-Liouville problems: A method of solution**, Birkhäuser (2020)

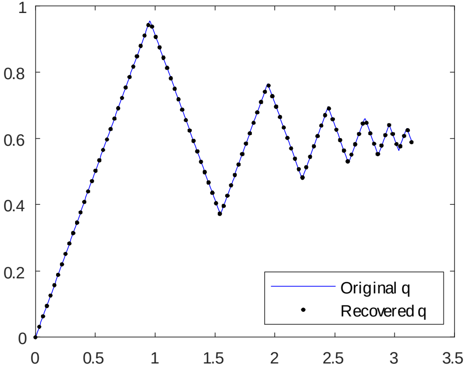



Figure: Potencial recuperado (200 autoval.)


V. V. Kravchenko, S. M. Torba V. V. Kravchenko, S. M. Torba A direct method for solving inverse Sturm-Liouville problems. Inverse Problems v. 37, 2021, # 015015.

V. V. Kravchenko, S. M. Torba A practical method for recovering Sturm-Liouville problems from the Weyl function. Inverse Problems 2021, 37(6), 065011.

## Example

$$q(x) = \frac{10\cos(13x)}{(x+0.1)^2} + \pi e^x \sin(20.23x)\, i, \quad x \in [0,1]$$

with the input data: 101 points $\rho_k$ are chosen uniformly distributed on the segment $[0.1, 100]$, and $a(\rho_k) = \sin \rho_k$, $b(\rho_k) = \cos \rho_k$.

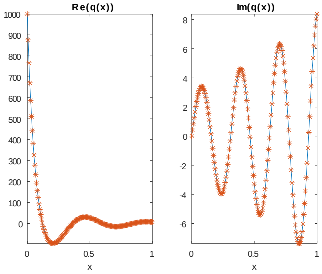



Figure: Max. abs. error $0.8 \cdot 10^{-3}$.

V. Kr. *Reconstruction techniques for complex potentials*. Submitted.

## Example

$$q(x) = \frac{1}{4}\left((6x-\pi)^6 - 8(6x-\pi)^4 + (10.8x-\pi)^2\right) + 20.23 + i\Gamma(x+\pi),$$

$x \in [0,1]$. Same choice of all the sets of points and parameters involved.

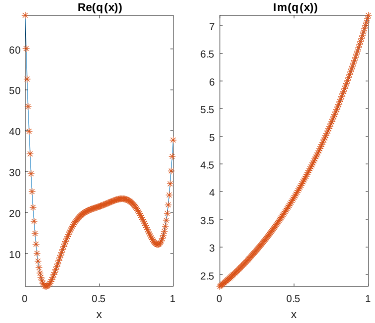



Figure: Maximum abs. error $1.8 \cdot 10^{-4}$.

$$-y'' + q(x)y = \lambda y, \quad x \in (-\infty, \infty),$$

$$\int_{-\infty}^{\infty} (1 + |x|)\, |q(x)|\, dx < \infty.$$

La ecuación posee las únicas soluciones de Jost $y = e(\rho, x)$, $y = g(\rho, x)$ tales que

$$e(\rho, x) = e^{i\rho x} (1 + o(1)), \quad x \to +\infty$$

$$g(\rho, x) = e^{-i\rho x} (1 + o(1)), \quad x \to -\infty$$

[K. Chadan, P. C. Sabatier *Inverse problems in quantum scattering theory*. Springer, NY, 1989]

1. **Espectro discreto:** La ecuación puede tener un número finito de autovalores, tales $\lambda_n < 0$, para los cuales $\exists\ y(x) \in L_2(-\infty, \infty)$. En tal caso $y(x) = c_1 e(\rho, x) = c_2 g(\rho, x)$, y junto con los autovalores se introducen las constantes de normalización

$$\alpha_k := \left( \int_{-\infty}^{\infty} e^2(\rho_k, x) dx \right)^{-1}.$$

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2. **Coeficiente de reflexión:**

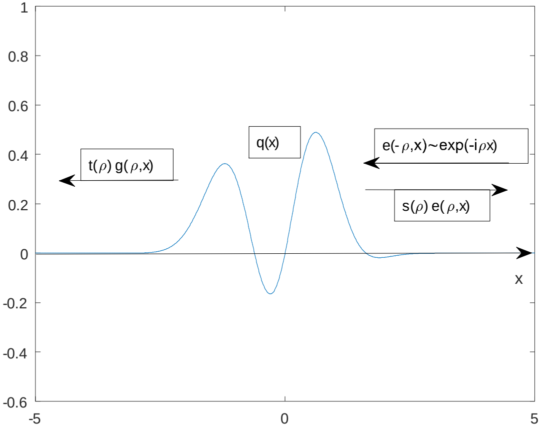


Figure: Schematic representation of the scattering model.

- **Problema directo:** Dado $q(x)$, encontrar los datos de dispersión

$$s(\rho),\ \rho \in \mathbf{R}, \quad \lambda_k,\ \alpha_k,\ k = \overline{1, N}$$

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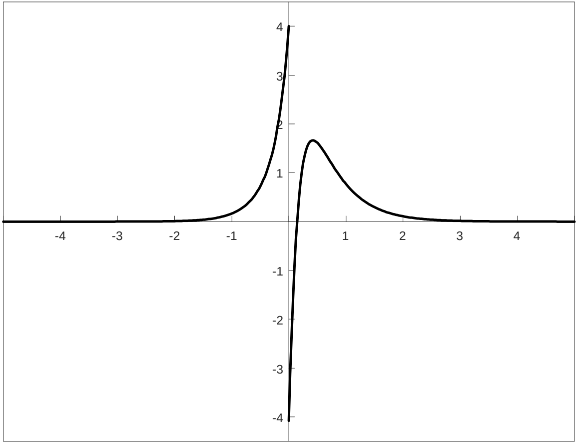
- K. Chadan, P. C. Sabatier *Inverse problems in quantum scattering theory*. Springer, 1989
  V. A. Marchenko, *Sturm-Liouville operators and applications: revised edition*. AMS Chelsea Publishing, 2011
  M. J. Ablowitz, *Nonlinear dispersive waves: asymptotic analysis and solitons.* Cambridge University Press, 2011.

# Example

de [T. Aktosun, P. Sacks, *Mathematical Methods in the Applied Sciences* **25**, (2002), 347-355.]



$$q(x) = \begin{cases} q_1(x), & x < 0 \\ q_2(x), & x > 0 \end{cases}$$

donde

$$q_1(x) = \frac{16\left(\sqrt{2}+1\right)^2 e^{-2\sqrt{2}x}}{\left(\left(\sqrt{2}+1\right)^2 e^{-2\sqrt{2}x} - 1\right)^2}$$

y

$$q_2(x) = \frac{96e^{2x}\left(81e^{8x} - 144e^{6x} + 54e^{4x} - 9e^{2x} + 1\right)}{\left(36e^{6x} - 27e^{4x} + 12e^{2x} - 1\right)^2}.$$

# El coeficiente de reflexión correspondiente

$$s(\rho) = \frac{(\rho + i)(\rho + 2i)(101\rho^2 - 3i\rho - 400)}{(\rho - i)(\rho - 2i)(50\rho^4 + 280i\rho^3 - 609\rho^2 - 653i\rho + 400)}.$$

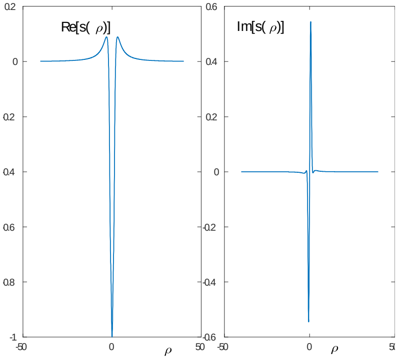



Figure: $s(\rho)$ calculado. Error absoluto $< 10^{-12}$.

# Solución del problema inverso de dispersión

Dados los datos de dispersión

$$s(\rho),\, \rho \in \mathbf{R}, \quad \lambda_k,\, \alpha_k,\, k = \overline{1, N},$$

encontrar $q(x)$.
[V. V. Kravchenko, On a method for solving the inverse scattering problem on the line. Math Meth Appl Sci. 42 (2019), 1321-1327]

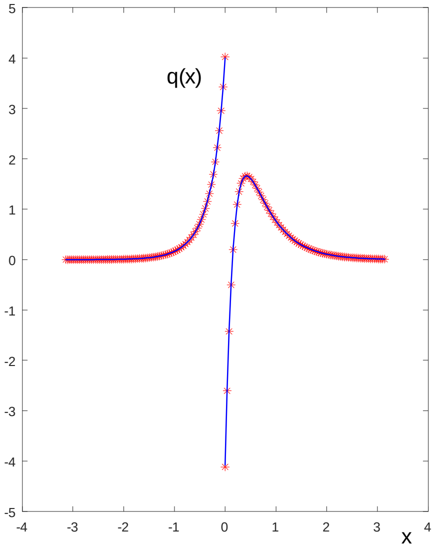



Figure: Potencial $q(x)$ recuperado con 25 ecuaciones.

Realización del método de la transformada inversa de dispersión

S. M. Grudsky, V. V. Kravchenko, S. M. Torba *Realization of the inverse scattering transform method for the Korteweg-de Vries equation.* Mathematical Methods in the Applied Sciences, 2023, v. 46, issue 8, 9217-9251.

Problema de Sturm-Liouville sobre un semieje

[B. B. Delgado, K. V. Khmelnytskaya, V. V. Kravchenko, *The transmutation operator method for efficient solution of the inverse Sturm-Liouville problem on a half-line.* Mathematical Methods in the Applied Sciences, 42, (2019), 7359–7366.

[B. B. Delgado, K. V. Khmelnytskaya, V. V. Kravchenko, *A representation for Jost solutions and an efficient method for solving the spectral problem on the half line.* Mathematical Methods in the Applied Sciences, v. 43 (2020) 9304–9319.

Problema de dispersión sobre un semieje

[A. N. Karapetyants, K. V. Khmelnytskaya, V. V. Kravchenko, *A practical method for solving the inverse quantum scattering problem on a half line*, J. Phys.: Conf. Ser. 1540 (2020), 012007]

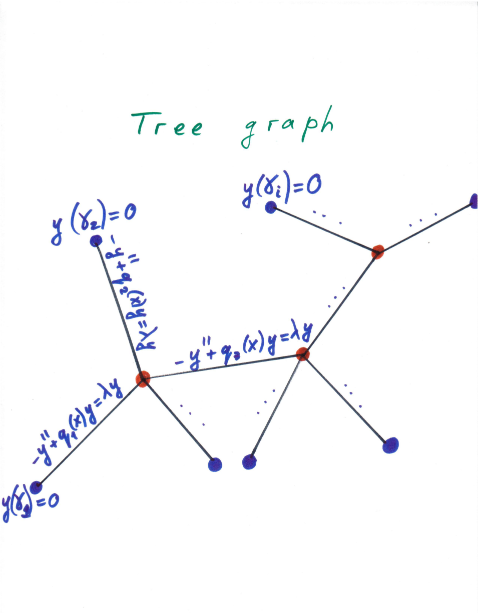
Problema de dispersión cuántica para la ecuación de Bessel perturbada

$$Lu := -u'' + \left(\frac{\ell(\ell+1)}{x^2} + q(x)\right) u = \rho^2 u, \quad x > 0$$

[V. V. Kravchenko, S. M. Torba, R. Castillo-Pérez A Neumann series of Bessel functions representation for solutions of perturbed Bessel equations. Applicable Analysis, 2018, v. 97, issue 5, 677-704]
[V. V. Kravchenko, E. L. Shishkina, S. M. Torba, *A transmutation operator method for solving the inverse quantum scattering problem.* Inverse Problems v. 36 (2020) 125007 (23pp)]
[V. V. Kravchenko, S. M. Torba *Transmutation operators and a new representation for solutions of perturbed Bessel equations*. Mathematical Methods in the Applied Sciences v. 44 (2021) 6344–6375]

# Problemas espectrales inversos en grafos cuánticos

en conjunto con Sergei A. Avdonin, Universidad de Alaska, EUA y Kira V. Khmelnytskaya, Universidad Autónoma de Querétaro


Tree graph
y(γi)=0
y(γ2)=0
-y''+q3(x)y=λy
-y''+q1(x)y=λy
y(γ1)=0

Let $\Omega$ be a tree graph with $m$ leaves, $q \in L_2(\Omega)$. A function $w$ is a solution of

$$-w''(x) + q(x)w(x) = \rho^2 w(x), \tag{10}$$

if additionally to satisfying

$$-w_i''(x) + q_i(x)w_i(x) = \rho^2 w_i(x), \quad x \in (0, L_i), \tag{11}$$

on each edge, on every interior vertex it satisfies:

$$w \in C(\Omega), \tag{12}$$

$$\sum_{j=1}^{M} \partial w_j(v) = 0, \quad \text{Kirchhoff-Neumann condition} \tag{13}$$

# Weyl matrix

**Definition** Let $w_i$ be a solution of (10) such that

$$w_i(\gamma_i) = 1 \quad \text{and} \quad w_i(\gamma_j) = 0 \text{ for all } j \neq i.$$

Then $w_i$ is called the **Weyl solution** associated with the leaf $\gamma_i$.
**Definition** The $m \times m$ matrix-function $\mathbf{M}(\lambda)$, $\lambda \notin \mathbb{R}$, consisting of the elements $\mathbf{M}_{ij}(\lambda) = \partial w_i(\gamma_j)$, $i, j = 1, \ldots, m$ is called the **Weyl matrix**.

Weyl matrix = Dirichlet-to-Neumann map

If $u$ is a solution of (10) satisfying the Dirichlet condition at the boundary vertices $u(\gamma, \lambda) = f(\lambda)$, then

$$\partial u(\gamma, \lambda) = \mathbf{M}^T(\lambda) f(\lambda), \quad \lambda \notin \mathbb{R}.$$

**Problem**

*Given $\Omega$ and the Weyl matrix at a finite number of points $\lambda_k$, $k = 1, \ldots, K$, find the potential $q(x)$ approximately.*

We combine the leaf peeling method (Avdonin & Kurasov 2008) with the approach explained.

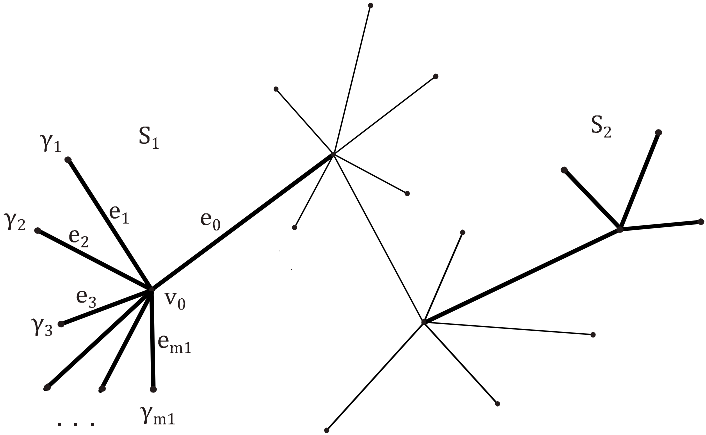



Figure: An example of a tree graph with two sheaves $S_1$ and $S_2$ highlighted in bold lines. Here $v_0$ is the abscission vertex and $e_0$ is the stem edge of $S_1$.

# Sequence of steps

Given a tree graph $\Omega$ and the Weyl matrix $\mathbf{M}(\rho^2)$ for a number of values $\rho_k^2$, $k = 1, \ldots, K$.

1. For a sheaf $S_1$ we have the (continuity) equalities

$$\varphi_i(\rho_k, L_i) + \mathbf{M}_{ii}(\rho_k^2) S_i(\rho_k, L_i) = \mathbf{M}_{ij}(\rho_k^2) S_j(\rho_k, L_j),$$

$$\mathbf{M}_{ij}(\rho_k^2) S_j(\rho_k, L_j) = \mathbf{M}_{il}(\rho_k^2) S_l(\rho_k, L_l), \tag{14}$$

valid for all $j, l \neq i$, $1 \leq j, l \leq m_1$. Substituting NSBF we obtain

$$\varphi(\rho, L_i) \quad \text{and} \quad S(\rho, L_i)$$

for all $1 \leq i \leq m_1$.

2. Applying the method we compute $q_i(x)$.

3. Next, the leaf edges of $S_1$ are removed, and the Weyl matrix $\widetilde{\mathbf{M}}(\rho^2)$ for the new smaller tree graph $\widetilde{\Omega}$ is calculated following the leaf peeling method.

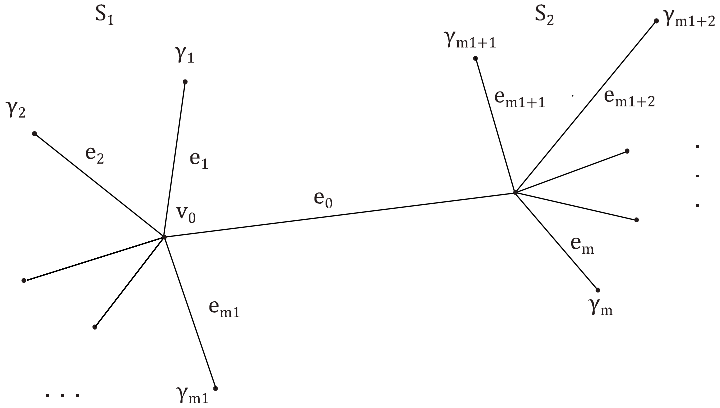



Figure: The tree graph considered in numerical tests.

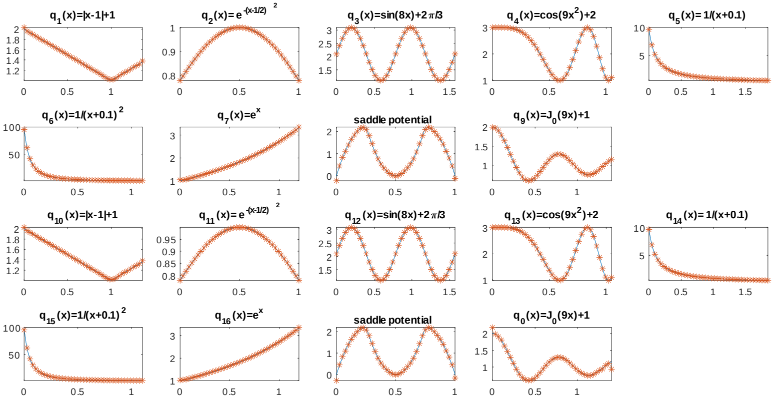
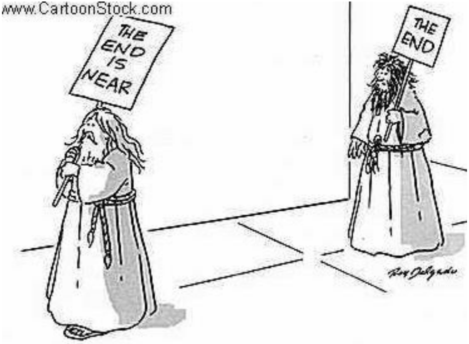



Figure: The potential of the quantum graph, recovered from the Weyl matrix given at 180 points, with $N = 9$.

S. A. Avdonin, V. V. Kravchenko (2023), *Method for solving inverse spectral problems on quantum star graphs.* Journal of Inverse and Ill-Posed Problems, 2023, v. 31, issue 1, 31-42.

S. A. Avdonin, K. V. Khmelnytskaya, V. V. Kravchenko *Recovery of a potential on a quantum star graph from Weyl's matrix.* Inverse Problems and Imaging (to appear).

S. A. Avdonin, K. V. Khmelnytskaya, V. V. Kravchenko *Reconstruction techniques for quantum trees.* Submitted, available at arXiv: 2302.05970.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