Puentes de Bessel y densidades de frontera

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Asset Management

Actinver Casa de Bolsa

Motivación

Proceso de Bessel
- Puente de Bessel
- Puente de Bessel $n = 3$
- Movimiento Browniano con deriva lineal

Difusiones controladas

Procesos y Puentes de Bessel

- El Browniano ha dominado las matemáticas financieras debido a que es utilizado para modelar el precio de activos financieros
- En el caso del proceso de Bessel, este proceso está relacionado con los modelos de Cox, Ingersoll y Ross (1985), con el modelos de volatilidad estocástica de Heston (1993), y el modelo CEV de Cox (1976), por mencionar sólo algunos.

- (Lévy 1948) Désiré André usa el principio de reflexión para determinar la densidad del primer cruce de frontera a un nivel constante, i.e. $T_a := \inf\{t \geq 0 | B_t = a\}$

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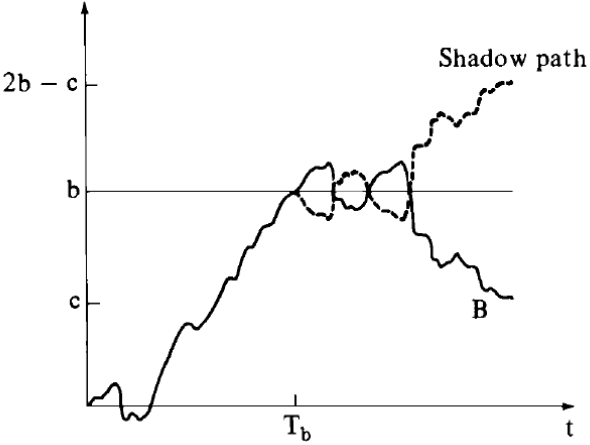
$$\mathbb{P}(T_a < t) = 2P(B_t > a) = \int_0^t \frac{|a|}{\sqrt{2\pi u^3}} e^{-\frac{a}{2u}} du.$$

Sea $h(t, a) := \mathbb{P}(T_a \in dt)$.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- **Theorem 6.2.** The density φ of the first passage time is related to a solution of the heat equation ω as follows

$$\varphi(t,a) = \omega\left(t, a + \int_0^t f'(u)du\right)$$

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and, given that $\{X_u\}_{0\leq u\leq t}$ is a 3D Bessel bridge, it is bounded by

$$h(t,a)e^{-af'(0)-\frac{1}{2}\int_0^t (f'(u))^2du - \int_0^t f''(u)\mathbb{E}[X_u]du}$$
$$\leq \omega\left(t, a + \int_0^t f'(u)du\right)$$
$$\leq h(t,a)e^{-af'(0)-\frac{1}{2}\int_0^t (f'(u))^2du}.$$

- El proceso de Bessel X de orden n se puede caracterizar como

$$X_t = \|B_t\|$$

donde $\|\cdot\|$ es la norma Euclideana en $\mathbb{R}^n$ y W es un Browniano en $\mathbb{R}^n$.

- El proceso de Bessel X de orden n es solución de la siguiente ecuación diferencial estocástica

$$dX_t = \frac{n-1}{2}\frac{1}{X_t}dt + dB_t$$

en este caso B es un proceso de Bessel en $\mathbb{R}$ y $X_0 = 0$.

- El puente de Bessel de orden δ, que comienza en $X_0 = a$ y alcanza $c = 0$ en $t = T$ es solución a

$$dX_t = \left(\frac{\delta - 1}{2} \frac{1}{X_t} - \frac{X_t}{T - t} \right) dt + dB_t, \qquad t < T. \qquad (1)$$

- Tenemos que para $n = 3$ y usando (1), la dinámica de X es

$$dX_t = \left(\frac{1}{X_t} - \frac{X_t}{T - t} \right) dt + dB_t, \qquad t < T. \tag{2}$$

Así mismo dado que (note que h es solución a la ecuación de calor trasera)

$$h(t, x) := \frac{x}{\sqrt{2\pi (T - t)^3}} e^{-\frac{x^2}{2(T-t)}}$$

podemos expresar (3) como

$$dX_t = \frac{h_x(t, X_t)}{h(t, X_t)} dt + dB_t, \qquad t < T.$$

- La dinámica de un Bronwiano con deriva lineal b es

$$dX_t = bdt + dB_t, \qquad t < T. \tag{3}$$

Así mismo dado que (note que h es solución a la ecuación de calor trasera)

$$h(t, x) := e^{bx - \frac{1}{2}b^2 t}$$

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\end{aligned}$$

$$(1) \qquad dX_t = \mu(t)dt + \sqrt{D}dW_t \qquad 0 \le t \le s$$

Let

$$\mathcal{L}_D(\mu) = -\frac{1}{2D}\mu^2$$

$$J(t,x,\mu) := \mathbb{E}_{t,x}\int_t^s \mathcal{L}_D(\mu(s))ds + \psi(s, X(s)),$$

$$V(t,x) = \inf_\mu [J(t,x;\mu)]$$

$$\frac{\partial V}{\partial t}(t,x) + \inf_\mu \left[\mathcal{L}_D(\mu) + \frac{D}{2}\frac{\partial^2 V}{\partial x^2}(t,x) + \mu\frac{\partial V}{\partial x}(t,x)\right] = 0$$

solving we get that

$$(2) \qquad u = DV_x(t,x)$$

or

$$\frac{\partial V}{\partial t}(t,x) + \frac{D}{2}\frac{\partial^2 V}{\partial x^2}(t,x) + \frac{D}{2}\frac{\partial V^2}{\partial x}(t,x) = 0$$

if $h(t,x) = V_x(t,x)$ then

$$\frac{\partial h}{\partial t}(t,x) + \frac{D}{2}\frac{\partial^2 h}{\partial x^2}(t,x) + Dh(t,x)\frac{\partial h}{\partial x}(t,x) = 0$$

which is to be sovled with the final boundary condition $h(s,x) = V_x(s,x)$. From (1) and (2) and u a solution to the heat equation we have that

$$\begin{aligned} dX_t &= DV_x(t,x)dt + \sqrt{D}dW_t \\ &= Dh(t,x)dt + \sqrt{D}dW_t \\ &= D\frac{u_x(t,x)}{u(t,x)}dt + \sqrt{D}dW_t. \end{aligned}$$

The last line follows from the Cole-Hopf transform. Some algebra leads to $V(t,x) = \ln(u(t,x))$.

Theorem 3.1. *Let $S \subset \mathbb{R}$ be an interval and let $\alpha : S \to \mathbb{R}$ be a function such that the following SDE has a unique strong solution, see* [17],

$$dY_t = \alpha(Y_t)dt + dW_t, \ Y_0 = a \in S, \ t \in [0, \tau_0), \tag{9}$$

where $\tau_0 \leq \infty$ is a stopping time with respect to Y, and which can take any value in $[0, \infty)$ with positive probability. Also, let $T > 0$ be fixed, and assume that there exists a positive solution $h : [0, T] \times S \to \mathbb{R}$ of the PDE

$$-h_t(s, y) = \frac{1}{2}h_{xx}(s, y) + \alpha(y)h_x(s, y), \ y \in S, \ s \in [0, T].$$

Then, if $Z_t := h(t, Y_t)/h(0, a)$ with $t < \tau_0$, the following defines a probability measure

$$Q(A) := E[Z_t I_A] \text{ for all } A \in \mathcal{F}_{\tau_0}. \tag{10}$$

And under Q the process Y is solution of the SDE

$$dX_t = \left[\alpha(X_t) + \frac{h_x(t, X_t)}{h(t, X_t)}\right] dt + dW_t, \ X_0 = a, \ t \in [0, \tau_0). \tag{11}$$

To be more explicit, under Q in (10), the process Y is denoted X. We will write E_P or E_Q to emphasize under which measure one is calculating an expectation. Below we will give an example that fits into this theorem.

Example 3.2. We can corroborate that the Bessel bridge fits into the context of Theorem 3.1. Indeed if

$$\alpha(x) := \frac{\delta - 1}{2x},$$

then the function

$$h(x,t) := \frac{T}{(T-t)^{\delta/2}} \exp\left\{-\frac{x^2}{2(T-t)}\right\} \tag{3.4}$$

is the desired solution to the parabolic PDE

$$-h_t(t,x) = \frac{1}{2} h_{xx}(t,x) + \alpha(x) h_x(t,x), \;\; x \in [0,\infty), t \in [0,T].$$

Moreover, one can check that the Bessel bridge X is recovered from the Bessel process Y; in symbols:

$$(P) \;\; dY_t = \alpha\left(Y_t\right) dt + dW_t,$$

$$(Q) \;\; dX_t = \left(\alpha\left(X_t\right) + \frac{h_x(t,X_t)}{h(t,X_t)}\right) dt + dW_t,$$

$$Q = \frac{h(t,Y_t)}{h(0,a)} P \;\; \text{on } \mathcal{F}_t.$$

In this case h_x/h simplifies considerably.

Theorem 3.4. *Under the conditions of Theorem 3.1. Define* $\tau := \inf\{s > 0 : X_s = b\}$ *and suppose that this is such that the condition of Corollary 3.3 holds. Then*

$$Q(\tau < t) = \int_0^t \frac{h(s,b)}{h(0,a)} P(\tau \in ds), \ t < T. \tag{3.6}$$

Proof. Using Corollary 3.3, we have that

$$\begin{aligned} Q(\tau < t) &= E_Q\left[I_{\{\tau<t\}}\right] \\ &= E_P\left[\frac{h(t,Y_t)}{h(0,a)} I_{\{\tau<t\}}\right] \\ &= \int_0^\infty E_P\left[\frac{h(t,Y_t)}{h(0,a)} I_{\{\tau<t\}} | \tau = s\right] P(\tau \in ds) \\ &\quad \text{(applying the optional sampling theorem)} \\ &= \int_0^t \frac{h(s,b)}{h(0,a)} P(\tau \in ds), \end{aligned}$$

where we have used the fact that $\tau = u$ implies $Y_u = b$. □

- Hernández-del-Valle, G. (2005) First passage time densities of Brownian motion and applications to credit risk, Ph.D. thesis, Columbia University.
- Hernández-del-Valle, G. (2013) On hitting times, Bessel bridges and Schrödingers equation, *Bernoulli* **19** (5A), 1559–1575.
- Hernández-del-Valle, G. and Pacheco, C.G. (2015) Hitting times for Bessel processes, *Comm. Stoch. Analy.*, **9** (1), 79–92.