On spectral perturbation results of compact self-adjoint operators over a Hilbert-like ultrametric space.

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1. The valued fields $\mathcal{R}$ and $\mathcal{C}$

2. The normed space $c_0$

3. Compact self-adjoint operators on $c_0$

4. Spectral Perturbation Theory

5. References
Outline for section 1

1. The valued fields $\mathcal{R}$ and $\mathcal{C}$

2. The normed space $c_0$

3. Compact self-adjoint operators on $c_0$

4. Spectral Perturbation Theory

5. References
The valued fields $\mathcal{R}$ and $\mathcal{C}$

The **Levi-Civita field** $\mathcal{R}$ and the **Complex Levi-Civita field** $\mathcal{C}$ can be considered as formal power series fields:

$$\mathcal{R} = \left\{ \sum_{k=1}^{\infty} a_k d^{t_k} \mid \forall k \in \mathbb{N}, a_k \in \mathbb{R}, t_k \in \mathbb{Q}, t_k < t_{k+1}, \lim t_k = \infty \right\}$$

$$\mathcal{C} = \left\{ \sum_{k=1}^{\infty} a_k d^{t_k} \mid \forall k \in \mathbb{N}, a_k \in \mathbb{C}, t_k \in \mathbb{Q}, t_k < t_{k+1}, \lim t_k = \infty \right\}$$

If $a_1 \neq 0$, then the valuation on $\mathcal{R}$ and $\mathcal{C}$ is defined as

$$\left| \sum_{k=1}^{\infty} a_k d^{t_k} \right| := e^{-t_1} \quad \text{and} \quad |0| = 0.$$
The valued fields $\mathcal{R}$ and $\mathcal{C}$

Notice that $\mathcal{C} = \mathcal{R} + i\mathcal{R}$.

For each nonzero $z = x + iy \in \mathcal{C}$ ($x, y \in \mathcal{R}$) the valuation satisfies:

$$|z| = \max\{|x|, |y|\}$$

The involution $x + iy \mapsto \overline{x + iy} := x - iy$ is an automorphism on $\mathcal{C}$ such that $|z| = |\overline{z}|$ and $z\overline{z} \in \mathcal{R}$ for all $z \in \mathcal{C}$.

- $\mathcal{R}$ is real closed.
- $\mathcal{C}$ is algebraically closed.
The valued fields $\mathcal{R}$ and $\mathcal{C}$

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- $\mathcal{R}$ is real closed.
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The valued fields $\mathcal{R}$ and $\mathcal{C}$

Notice that $\mathcal{C} = \mathcal{R} + i\mathcal{R}$.
For each nonzero $z = x + iy \in \mathcal{C}$ ($x, y \in \mathcal{R}$) the valuation satisfies:

$$|z| = \max\{|x|, |y|\}$$

The involution $x + iy \mapsto \overline{x + iy} := x - iy$ is an automorphism on $\mathcal{C}$ such that $|z| = |\overline{z}|$ and $z\overline{z} \in \mathcal{R}$ for all $z \in \mathcal{C}$.

- $\mathcal{R}$ is real closed.
- $\mathcal{C}$ is algebraically closed.
Notice that $C = \mathcal{R} + i\mathcal{R}$.
For each nonzero $z = x + iy \in C$ ($x, y \in \mathcal{R}$) the valuation satisfies:

$$|z| = \max\{|x|, |y|\}$$

The involution $x + iy \mapsto \overline{x + iy} := x - iy$ is an automorphism on $C$ such that $|z| = |\overline{z}|$ and $z\overline{z} \in \mathcal{R}$ for all $z \in C$.

- $\mathcal{R}$ is real closed.
- $C$ is algebraically closed.
Outline for section 2

1. The valued fields $\mathcal{R}$ and $\mathcal{C}$

2. The normed space $c_0$

3. Compact self-adjoint operators on $c_0$

4. Spectral Perturbation Theory

5. References
The normed space $c_0$

The set

$$c_0 := \left\{ (\lambda_j)_{j \in \mathbb{N}} \mid \forall j \in \mathbb{N}, \lambda_j \in \mathbb{C}, \lim_j \lambda_j = 0 \right\}$$

is a vector space over $\mathbb{C}$.

Notice that $c_0 = c_0(\mathbb{R}) \oplus ic_0(\mathbb{R})$, i.e. for each $z = (z_k) \in c_0$, there are unique $x = (x_k)$ and $y = (y_k)$ in $c_0(\mathbb{R})$ such that $z = x + iy$ and the norm on $c_0$ satisfies:

$$||z|| := \max_k |z_k| = \max_k \max \{|x_k|, |y_k|\} = \max \{|x||, |y||\}.$$

The space $(c_0, || \cdot ||)$ is Banach.
The normed space $c_0$

The set

$$c_0 := \left\{ (\lambda_j)_{j \in \mathbb{N}} \mid \forall j \in \mathbb{N}, \lambda_j \in \mathbb{C}, \lim_j \lambda_j = 0 \right\}$$

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Notice that $c_0 = c_0(\mathbb{R}) \oplus ic_0(\mathbb{R})$, i.e. for each $z = (z_k) \in c_0$, there are unique $x = (x_k)$ and $y = (y_k)$ in $c_0(\mathbb{R})$ such that $z = x + iy$ and the norm on $c_0$ satisfies:

$$||z|| := \max_{k \in \mathbb{N}} |z_k| = \max_{k \in \mathbb{N}} \max\{|x_k|, |y_k|\} = \max\{||x||, ||y||\}.$$ 

The space $(c_0, || \cdot ||)$ is Banach.
The inner product on $c_0$

**Theorem (Narici & Beckenstein (2005) [2, 6.1])**

Consider the form $\langle \cdot, \cdot \rangle : c_0 \times c_0 \to \mathbb{C}$, $\langle z, w \rangle = \sum_{k=1}^{\infty} z_k \overline{w_k}$. The statements below hold for all $z, z', w \in c_0$ and $\alpha, \beta \in \mathbb{C}$.

1. $\langle \cdot, \cdot \rangle$ is well-defined.
2. $\langle z, z \rangle = 0 \iff z = 0$
3. $\langle \alpha z + \beta z', w \rangle = \alpha \langle z, w \rangle + \beta \langle z', w \rangle$
4. $\langle z, w \rangle = \langle w, z \rangle$
5. $|\langle z, w \rangle| \leq ||z|| ||w||$
6. $\langle z, w \rangle = 0, \forall w \in c_0 \Rightarrow z = 0$.
7. $||z|| = \sqrt{|\langle z, z \rangle|}$
$c_0$ is not a Hilbert space

- The proper subspace $D := \{(z_k) \in c_0 : \sum_{k=1}^{\infty} z_k = 0\}$ is closed in $c_0$ such that $D^\perp = \{0\}$.

- $(c_0)' = \ell^\infty$.

- The Hanh Banach theorem does not hold on $c_0$. 
Definition

Consider the standard Schauder basis \( \{e_1, e_2, \ldots \} \) of \( c_0 \). The projection map \( e_j' : c_0 \to \mathbb{C} \) defined by

\[
e_j'(x) := \langle x, e_j \rangle
\]

is a member of \( c'_0 \) for all \( j \in \mathbb{N} \).

Furthermore, \( e_j'(e_i) = \delta_{ij} \), \( ||e_i'|| = 1 \) for all \( i \in \mathbb{N} \), \( x = \sum_{i=1}^{\infty} e_i'(x)e_i \) and \( ||x|| = \max_{i \in \mathbb{N}} |e_i'(x)| \) for all \( x \in c_0 \).
Consider the standard Schauder basis \( \{e_1, e_2, \ldots \} \) of \( c_0 \). The projection map \( e'_j : c_0 \to \mathbb{C} \) defined by
\[
e'_j(x) := \langle x, e_j \rangle
\]
is a member of \( c'_0 \) for all \( j \in \mathbb{N} \).
Furthermore, \( e'_j(e_i) = \delta_{ij} \), \( ||e'_i|| = 1 \) for all \( i \in \mathbb{N} \), \( x = \sum_{i=1}^{\infty} e'_i(x)e_i \) and \( ||x|| = \max_{i \in \mathbb{N}} |e'_i(x)| \) for all \( x \in c_0 \).
Outline for section 3

1. The valued fields $\mathcal{R}$ and $\mathcal{C}$

2. The normed space $c_0$

3. Compact self-adjoint operators on $c_0$

4. Spectral Perturbation Theory

5. References
Continuous linear operators on $c_0$

**Definition**

$L(c_0) := \{ T : c_0 \to c_0 : T \text{ is continuous and linear} \}$ is a Banach space under the norm $\|T\| := \sup \left\{ \frac{\|T(x)\|}{\|x\|} : x \neq 0 \right\}$

**Definition**

A linear operator $T : c_0 \to c_0$ is said to be **self-adjoint** if

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

for all $x, y \in c_0$. 
Definition

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for all \( x, y \in c_0 \).
Definition (van Rooij (1978) [3, 4.39, 4.40])

Let $E$ and $F$ be two normed spaces over a non-Archimedean valued field. An operator $T \in L(E, F)$ is said to be compact when satisfies any of the following equivalent conditions:

1. $T(B_E)$ is compactoid, where $B_E = \{ x \in E : ||x|| \leq 1 \}$,
2. for each $\varepsilon > 0$, there exists $S \in L(E, F)$ of finite-dimensional range such that $||T - S|| < \varepsilon$,
3. there are vectors $a_1, a_2, \cdots \in F$, and functionals $g_1, g_2, \cdots \in E'$ such that $\lim_k ||g_k|| ||a_k|| = 0$ and $T = \sum_{k=1}^{\infty} g_k a_k$, i.e. the sequence $(\sum_{k=1}^{n} g_k(\cdot)a_k)_{k \in \mathbb{N}}$ converges uniformly to $T$. 
Definition

If \((v_i)_{i \in \mathbb{N}}\) is an orthonormal basis of \(C^n\) (or \(c_0\)), then we say that \(T \in L(C^n)\) (or \(T \in L(c_0)\)) is diagonalizable if \(T = \sum_i \alpha_i \langle \cdot, v_i \rangle v_i\).

Notation: \(E_i := \langle \cdot, v_i \rangle v_i\).
The proof of the spectral theorem for compact self-adjoint operators in the Archimedean case cannot be adapted to the non-Archimedean case because in the classical case the proof is based on the following facts:

1. The spectrum of a compact self-adjoint operator is non-empty, which is proved by using Liouville’s Theorem. In the non-Archimedean case Liouville’s Theorem holds for functions $f : K \to K$ that admit a power series expansion. But it is unknown whether a function $f : K \to K$ that is differentiable has a power series expansion. In the classical case this is proved by using the Cauchy’s Theorem which heavily depends on the connectedness of $\mathbb{C}$. In our case, any non-Archimedean valued field is totally disconnected.

2. $\sup \left\{ \frac{\langle Tx, x \rangle}{\langle x, x \rangle} : x \neq 0 \right\}$ and $\inf \left\{ \frac{\langle Tx, x \rangle}{\langle x, x \rangle} : x \neq 0 \right\}$ are eigenvalues for $T$, when $T$ is a compact self-adjoint operator on a Hilbert space. In the non-Archimedean context, an upper bounded set of scalars may not have a supremum. Similarly with infimum.
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2. $\sup \left\{ \langle Tx, x \rangle / \langle x, x \rangle : x \neq 0 \right\}$ and $\inf \left\{ \langle Tx, x \rangle / \langle x, x \rangle : x \neq 0 \right\}$ are eigenvalues for $T$, when $T$ is a compact self-adjoint operator on a Hilbert space. In the non-Archimedean context, an upper bounded set of scalars may not have a supremum. Similarly with infimum.
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Theorem (B. Diarra (2009) [1])

Every $T \in L(c_0)$ has a matrix representation $[T] = (\alpha_{ij})$ in the sense of $Tx = [T]x$, with $\|T\| = \sup\{|\alpha_{ij}| : i, j \in \mathbb{N}\}$. An infinite matrix $[T]$ represents a compact self-adjoint operator $T$ if and only if the row and column vectors of $[T]$ form a null sequence in $c_0$, and $[T] = [T]^t$ i.e.

$$[T] = \begin{pmatrix} \|c_1\| & \|c_2\| & \|c_3\| & \rightarrow 0 \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \rightarrow 0 & \|r_1\| \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \rightarrow 0 & \|r_2\| \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \rightarrow 0 & \|r_3\| \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \end{pmatrix}$$

where $\alpha_{ij} = e_i'(Te_j)$ for all $i, j \in \mathbb{N}$, $r_i = (\alpha_{i1}, \alpha_{i2}, \ldots) \in c_0$ is the $i$-th row vector of $[T]$, and $c_j = (\alpha_{1j}, \alpha_{2j}, \ldots) \in c_0$ is the $j$-th column vector of $[T]$, and $r_i = \overline{c_i}$. 
Theorem (B. Diarra (2009) [1])

Every \( T \in L(c_0) \) has a matrix representation \([T] = (\alpha_{ij})\) in the sense of \(Tx = [T]x\), with \(\|T\| = \sup\{|\alpha_{ij}| : i, j \in \mathbb{N}\}\). An infinite matrix \([T]\) represents a compact self-adjoint operator \(T\) if and only if the row and column vectors of \([T]\) form a null sequence in \(c_0\), and \([T] = [T]^t\) i.e.

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[T] = \begin{pmatrix}
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\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & & & 0
\end{pmatrix}
\]

where \(\alpha_{ij} = e'_i(Te_j)\) for all \(i, j \in \mathbb{N}\), \(r_i = (\alpha_{i1}, \alpha_{i2}, \ldots) \in c_0\) is the \(i\)-th row vector of \([T]\), and \(c_j = (\alpha_{1j}, \alpha_{2j}, \ldots) \in c_0\) is the \(j\)-th column vector of \([T]\), and \(r_i = \bar{c}_i\).
In other words, an infinite matrix $[T]$ represents a compact self-adjoint operator $T$ if and only if

$$[T] = \sum_{k=1}^{\infty} d^{t_k} A_k$$

where, for every $k \in \mathbb{N}$, $t_k \in \mathbb{Q}$, $t_k < t_{k+1}$, $\lim t_k = \infty$, $A_k$ is an infinite matrix with entries in $\mathbb{C}$ such that $A_k = \overline{A_k}^t$, and its nonzero entries are in the first $n_k$ rows and columns for some $n_k$. 

$$A_k = \begin{bmatrix}
0 & 0 & \cdots \\
A_k & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$
Corollary

Consider a compact self-adjoint operator $T \in L(c_0)$ with $\|T\| = 1$. If we define

$$B_k := A_0 + \sum_{i=1}^{k} d^t A_i$$

and $t_k \in \mathbb{Q}$ such that $0 < t_k < t_{k+1}$ for $k = 0, 1, \ldots$, then

$$\|T - B_k\| = |d^{t_{k+1}}| \to 0 \quad \text{and} \quad T = \lim_{k \to \infty} B_k.$$
Outline for section 4

1. The valued fields $\mathcal{R}$ and $\mathcal{C}$

2. The normed space $c_0$

3. Compact self-adjoint operators on $c_0$

4. Spectral Perturbation Theory

5. References
Definition (Again)

If \((v_i)_{i \in \mathbb{N}}\) is an orthonormal basis of \(\mathbb{C}^n\) (or \(c_0\)), then we say that \(T \in L(\mathbb{C}^n) \) (or \(T \in L(c_0)\)) is diagonalizable if \(T = \sum_i \alpha_i \langle \cdot, v_i \rangle v_i\).

Notation: \(E_i := \langle \cdot, v_i \rangle v_i\).

Lemma

If \(\lambda \in \mathbb{C}\) is not an eigenvalue for \(G \in L(\mathbb{C}^n)\) and \(G = \sum_i \lambda_i E_i\), then

\[
(\lambda I_n - G)^{-1} = \sum_i \frac{1}{\lambda - \lambda_i} E_i
\]
Definition (Again)

If \((v_i)_{i \in \mathbb{N}}\) is an orthonormal basis of \(\mathbb{C}^n\) (or \(c_0\)), then we say that \(T \in L(\mathbb{C}^n)\) (or \(T \in L(c_0)\)) is diagonalizable if \(T = \sum_i \alpha_i \langle \cdot, v_i \rangle v_i\).

Notation: \(E_i := \langle \cdot, v_i \rangle v_i\).

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\[
(\lambda I_n - G)^{-1} = \sum_i \frac{1}{\lambda - \lambda_i} E_i
\]
Lemma

Let $F, G \in L(C^n)$ and let $\lambda \in C$ an eigenvalue of $F$. Then either $\lambda$ is an eigenvalue for $G$ or $1 \leq \|(\lambda I_n - G)^{-1}\| \|F - G\|$. 
Theorem

Let \( F, G \in L(C^n) \) and let \( \lambda \in \mathbb{C} \) be an eigenvalue of \( F \). If \( \lambda \) is not an eigenvalue of \( G \) and \( G = \sum_i \lambda_i E_i \), then

\[
\min_i |\lambda - \lambda_i| \leq \|F - G\|.
\]
Corollary (Again)

Consider a compact self-adjoint operator $T \in L(c_0)$ with $||T|| = 1$. If we define

$$B_k := A_0 + \sum_{i=1}^{k} d^t A_i$$

and $t_k \in \mathbb{Q}$ such that $0 < t_k < t_{k+1}$ for $k = 0, 1, \ldots$, then

$$||T - B_k|| = |d^{t_{k+1}}| \to 0 \quad \text{and} \quad T = \lim_{k \to \infty} B_k.$$

Corollary

If $\lambda \in \mathbb{R}$ is an eigenvalue of $B_k$, then there exists an eigenvalue $\tau \in \mathbb{R}$ of $B_{k+1}$ such that

$$|\lambda - \tau| \leq |d^{t_{k+1}}|.$$
Corollary

If $\lambda \in \mathbb{R}$ is an eigenvalue of $B_k$, then there exists an eigenvalue $\tau \in \mathbb{R}$ of $B_{k+1}$ such that

$$|\lambda - \tau| \leq |d^{tk+1}|.$$
Corollary (Again)

Consider a compact self-adjoint operator \( T \in L(c_0) \) with \( \|T\| = 1 \). If we define

\[
B_k := A_0 + \sum_{i=1}^k d^{t_i} A_i
\]

and \( t_k \in \mathbb{Q} \) such that \( 0 < t_k < t_{k+1} \) for \( k = 0, 1, \ldots \), then

\[
\|T - B_k\| = |d^{t_k+1}| \to 0 \quad \text{and} \quad T = \lim_{k \to \infty} B_k.
\]

Theorem

For every eigenvalue \( \lambda_0 \in \mathcal{R} \) of \( A_0 \), there exists a convergent sequence \( (\lambda_k)_{k \in \mathbb{N}} \) in \( \mathcal{R} \), where \( \lambda_k \) is an eigenvalue of \( B_k \). Moreover, its limit is an approximate eigenvalue of \( T \).
Theorem

For every eigenvalue $\lambda_0 \in \mathcal{R}$ of $A_0$, there exists a convergent sequence $(\lambda_k)_{k \in \mathbb{N}}$ in $\mathcal{R}$, where $\lambda_k$ is an eigenvalue of $B_k$. Moreover, its limit is an approximate eigenvalue of $T$. 
Theorem

Consider $B_1 = A_0 + d^{t_1} A_1$ where $A_0, A_1$ are nonzero Hermitian complex matrices and $t_1 \in \mathbb{Q}, t_1 > 0$. Let $\lambda_0 \in \mathbb{R}$ and $\lambda_1 \in \mathbb{R}$ be eigenvalues of $A_0$ and $B_1$ respectively such that $|\lambda_0 - \lambda_1| \leq |d^{t_1}|$. If $u \in \ker(B_1 - \lambda_1 I)$, then there exists $v \in \ker(A_0 - \lambda_0 I)$ such that $||u - v|| \leq |d^{t_1}|$. 
Example of Compact self-adjoint operator

Example (A. Barria Comicheo, 2018)

Let \( T \in L(c_0) \) be defined by

\[
T(x_1, x_2, x_3, \ldots) = \left( \sum_{i=2}^{\infty} x_i d^{i-2}, x_1, dx_1, d^2 x_1, \ldots \right)
\]

The matrix that defines this operator relative to the canonical basis of \( c_0 \) is:

\[
[T] = \begin{pmatrix}
0 & 1 & d & d^2 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
d & 0 & 0 & 0 & \cdots \\
d^2 & 0 & 0 & 0 & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]
Example of Compact self-adjoint operator

There exists a base for $c_0$ such that $T$ has the following matrix representation:

$$[T] = \begin{pmatrix}
\sigma & 0 & 0 & 0 & 0 & \cdots \\
0 & -\sigma & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}$$

where $\sigma := \sqrt{\sum_{i=0}^{\infty} d^{2i}}$. 
Thank you!
Outline for section 5

1. The valued fields $\mathcal{R}$ and $\mathcal{C}$

2. The normed space $c_0$

3. Compact self-adjoint operators on $c_0$

4. Spectral Perturbation Theory

5. References
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