Review of finite approximations, Archimedean and non-Archimedean

Trond Digernes

Department of Mathematical Sciences
The Norwegian University of Science and Technology

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A 1-dimensional Hamiltonian operator of the form

\[ H = -\Delta + V \]

where \( \Delta = \frac{d^2}{dx^2} \) is the Laplacian and \( V \) is multiplication by a function \( v \), both regarded as operators on suitable domains in \( L^2(\mathbb{R}) \).

With operators \( P \) and \( Q \) defined as \( (Pf)(x) = \frac{1}{i} f'(x) \) and \( (Qf)(x) = xf(x) \) on suitable domains, we have \( P = \mathcal{F}^{-1} Q \mathcal{F} \) where \( \mathcal{F} \) is the Fourier transform:

\[ \mathcal{F} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-ixy} dy. \]

Notice that \( -\Delta = P^2 = \mathcal{F}^{-1} Q^2 \mathcal{F} \), and that

\[ Q^2 f(x) = x^2 f(x) = |x|^2 f(x), \quad H = P^2 + V. \]
Infinite discrete model over $\mathbb{R}$

$\epsilon > 0, \ G(\epsilon) = \epsilon \mathbb{Z}$.

$(T_h f)(x) = f(x + h), \ h \in G(\epsilon), \ f \text{ any function on } G(\epsilon)$.

$$D(\epsilon)^+ = \epsilon^{-1}(T_\epsilon - I), \quad D(\epsilon)^- = \epsilon^{-1}(I - T_{-\epsilon})$$

$$\Delta(\epsilon) = D(\epsilon)^- D(\epsilon)^+$$

On $L^2(G(\epsilon))$ the $T_h$ are unitary, $D(\epsilon)^- = -(D(\epsilon)^+)^*$ and $\Delta(\epsilon)$ are bounded, and

$$(-\Delta(\epsilon)f, f) = \|D(\epsilon)^+ f\|^2, \quad f \in L^2(G(\epsilon))$$

Hamiltonian $H(\epsilon)$ on $L^2(G(\epsilon))$: $H(\epsilon) = -\Delta(\epsilon) + V_\epsilon$, $V_\epsilon$ acting as multiplication by the restriction of $V$ to $G(\epsilon)$. 
$N^0 = N^0(\epsilon)$: an integer $\geq 1$ depending on $\epsilon$; $\epsilon N^0 \to \infty$ as $\epsilon \to 0$. $N = 2N^0 + 1$. $G(\epsilon)^0 = \{r \epsilon | r = 0, \pm 1, \ldots, \pm N^0\}$.

We get various models depending on boundary conditions. We will describe the following two: periodic model and the "Schwinger model".
Periodic model over $\mathbb{R}$

$$H(\epsilon)^{(p)} = -\Delta(\epsilon)^{(p)} + V_\epsilon.$$  
Identify $G(\epsilon)^0$ with the finite group $G(\epsilon)/(N(\epsilon) \cdot G(\epsilon))$, and hence $L^2(G(\epsilon)^0)$ with the space of functions on $G(\epsilon)$ invariant under $T_h, h \in N(\epsilon) \cdot G(\epsilon)$. $D(\epsilon)^{(\pm)(p)}$ are the restrictions of $D(\epsilon)^{(\pm)}$ to this space.

$$\Delta(\epsilon)^{(p)} = D(\epsilon)^{-(p)} D(\epsilon)^{(p)}+, \quad (-\Delta(\epsilon)^{(p)} f, f) = \|D(\epsilon)^{(p)} f\|^2.$$
The "Schwinger model" \(^1\) over \(\mathbb{R}\)

Also a periodic model, but with different definition of Laplacian and with explicit relation between \(\epsilon\) and \(N\).

\[
H(\epsilon)^{(s)} = -\Delta(\epsilon)^{(s)} + V_\epsilon.
\]
\[
\epsilon = \epsilon_N = (2\pi/N)^{1/2}.
\]
\(\mathcal{F}_\epsilon\) = the Fourier transform on \(L^2(G(\epsilon)/N \cdot G(\epsilon))\)
\(q_\epsilon\) = multiplication by the coordinate:
\[
(q_\epsilon f)(x) = xf(x), \ x \in G(\epsilon)
\]
\(p_\epsilon = \mathcal{F}_\epsilon^{-1} q_\epsilon \mathcal{F}_\epsilon\)
\[
\Delta(\epsilon)^{(s)} = -p_\epsilon^2
\]

\(^1\)So named because Julian Schwinger used it in his studies of finite quantum mechanics.
Remarkable accuracy of the Schwinger model

For reasons not fully understood the Schwinger model is far superior to the usual finite difference model as far as numerical results are concerned (see the Appendix for numerical results and comparison with the finite difference operator).
In [DVV94] also a third class of finite models were considered; these were given superscript 0. The main result of that paper could then be stated as follows\(^2\):

\(^2\)More precisely, this is the main result of the functional analytic part of the paper. In the probabilistic part a somewhat stronger convergence result was obtained for stochastic Hamiltonians.
Theorem

Let \((\epsilon_n)\) be a sequence tending to 0 and \(* = p, s, 0\). Let
0 \leq h_1 < h_2 < \ldots be the eigenvalues of \(H\) and \(T_j\) the
eigensubspace corresponding to \(h_j\). Then:
(i) if \(J\) is a compact subset of \([0, \infty)\) not containing any
eigenvalues of \(H\), then no eigenvalue of \(H(\epsilon_n)(*)\) belongs to \(J\) if
\(n\) is large enough
(ii) if \(J\) is a compact neighborhood of \(h_j\) not containing any
\(h_i, i \neq j\), all the eigenvalues of \(H(\epsilon_n)(*)\) that belong to \(J\)
converge to \(h_j\); if \(T_{nj}\) is the span of the corresponding
eigenspaces, \(\dim(T_{nj}) = \dim(T_j)\) for \(n\) large enough, and there
is an orthonormal basis of \(T_{nj}\) that converges to an orthonormal
basis of \(T_j\).
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Atomic potential: from the thesis of Erik M. Bakken

The next 13 slides, dealing with atomic potential, are based on Ch. 5 of Erik M. Bakken’s PhD thesis [Bak16].
Hamiltonian $H = -\Delta + V$ acting in $L^2(\mathbb{R}^3)$. $(Vf)(x) = v(x)f(x)$ for some suitable function $v$. Later $v$ will be specialized to the radial function $v(r) = -1/r$.

$\Delta = \text{Laplacian}$. 

**Finite model:**

**Finite grid $G(\varepsilon)$:**

$\{r \varepsilon : r = -n_0, -n_0 + 1, \ldots, n_0 - 1, n_0\}^3$, $n = 2n_0 + 1$.

$\{e_i\}_{i=1}^3$: an orthonormal basis for $\mathbb{R}^3$.

**Discrete Laplacian in the direction $i$:**

$$\Delta_{\varepsilon,i}f(x) = \frac{f(x + \varepsilon e_i) - 2f(x) + f(x - \varepsilon e_i)}{\varepsilon^2}, \quad x \in G(\varepsilon). \quad (1)$$

**Boundary conditions:** $f$ is interpreted to be zero in the above formula at all points which are not in the grid.

**Discrete Laplacian:** $\Delta_{\varepsilon} = \sum_{i=1}^3 \Delta_{\varepsilon,i}$.

**Potential on the grid:** $V_{\varepsilon}f(x) = v_{\varepsilon}(x)f(x)$, $v_{\varepsilon}$: the restriction of $v$ to the grid.
Imbedding

Imbedding $\# : L^2(G(\varepsilon)) \to L^2(\mathbb{R}^3)$ given by

$$f\# = \varepsilon^{-3/2} \sum_{x \in G(\varepsilon)} f(x) \chi_{R(x)},$$

(2)

where $\chi_{R(x)}$ is the characteristic function of the set

$R(x) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : x_i - \varepsilon/2 \leq y_i < x_i + \varepsilon/2, \: i = 1, 2, 3\}.$

We can view operators on $L^2(G(\varepsilon))$ as operators on $L^2(\mathbb{R}^3)$ by sending an operator $A$ to $Q_\varepsilon AQ_\varepsilon$ where $Q_\varepsilon$ is the orthogonal projection on the image of $\#$. The projection is given by

$$Q_\varepsilon f = \sum_{x \in G(\varepsilon)} \frac{1}{\varepsilon^3} \int_{R(x)} f(x - y) \, dy \cdot \chi_{R(x)}.$$  

(3)

When one imbeds a Hamiltonian this way, the spectrum of the Hamiltonian is preserved except that 0 becomes an eigenvalue of infinite multiplicity (whether or not 0 was an eigenvalue before the imbedding).
We now set \( v(r) = -1/r \). In this case the spectrum of the Hamiltonian \( H = -\Delta + V \) is known:

Eigenvalues: \( E_k = -\frac{1}{4k^2}, \quad k = 1, 2, 3, \ldots \).

\( E_k \) has multiplicity \( k^2 \).

Continuous spectrum = \([0, \infty)\).
Definition (Coulomb-like Operator)

Let $A$ be a self-adjoint operator which is bounded below, with discrete spectrum below a constant $c$, and with $\sigma_{\text{ess}}(A) = \sigma_{\text{cont}}(A) = [c, \infty)$, where $\sigma_{\text{cont}}(A)$ is the continuous spectrum of $A$. Then we will call $A$ a Coulomb-like operator.
Definition (Convergence of spectra)

Let $A$ be a Coulomb-like operator and let $A_n$, $n = 1, 2, \ldots$ be self-adjoint operators which are bounded below, and let $A_n$ have discrete spectrum. The eigenvalues of $A$ are denoted by $\lambda_1 \leq \lambda_2 \leq \ldots$ and are counted with multiplicity. Assume that:

1. If $J$ is a compact subset of $\mathbb{R}$ containing no eigenvalues of $A$, then no eigenvalues of $A_n$ will be in $J$ for sufficiently large $n$.

2. For every $\lambda \in \sigma(A)$ there exists a sequence $\lambda_n \in \sigma(A_n)$ such that $\lambda_n \to \lambda$. If $J = [a, b]$ is a compact interval with $c < a < b$, then $P_J(A_n)$ converges strongly to $P_J(A)$.

3. If $J$ is a compact neighborhood containing the eigenvalue $\lambda_j$, and no other eigenvalues of $A$ different from $\lambda_j$, then all the eigenvalues of $A_n$ in $J$ converge to $\lambda_j$. Furthermore $\|P_J(A_n) - P_J(A)\| \to 0$ as $n \to \infty$.

We will then say that the spectrum of $A_n$ converges to the spectrum of $A$, and we will denote it by

$$\sigma(A_n) \to \sigma(A).$$
Hamiltonian in a cube

To prove convergence of the finite quantum systems we will first prove convergence inside a finite box. The Hamiltonian in the open cube $T_b = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_i| < b, i = 1, 2, 3\}$ will be denoted by $H_b = -\Delta_b + V_b$, where $\Delta_b$ is the Laplacian, and $V_b$ acts as multiplication by a modified version $v_b$ of the potential function $v$ (in order to deal with the singularity at the origin):

$$v_b(r) = \begin{cases} \frac{-1}{r} & \text{if } r > 1/b \\ -b & \text{if } r \leq 1/b. \end{cases}$$

(5)
Lemma
The Hamiltonian $H_b$ has a compact resolvent, and thus a discrete spectrum.

Lemma
The $k$'th eigenvalue for $H_b$ is always bigger than or equal to the $k$'th eigenvalue of $H$, that is, $\lambda^b_k \geq \lambda_k$.

Proof.
This follows from a Max-Min theorem, by regarding $C_c^\infty(T_b)$ as a subspace of $C_c^\infty(\mathbb{R}^3)$ (by setting functions in $C_c^\infty(T_b)$ equal to zero outside $T_b$).
Since $H$ is bounded from below, it can be made positive by adding $\alpha I$ to it, for some $\alpha > 0$. Also, if $H$ is positive, so is $H_b$, hence the $\alpha$ which makes $H$ positive, makes $H_b$ positive, too. For this reason we will from now on assume $H$ and $H_b$ to be positive (although we are actually working with $H + \alpha I$ and $H_b + \alpha I$).
The rest of the proof goes in two steps. First we show convergence of the finite models inside a finite cube, and by using bigger and bigger cubes we will in the second part show convergence of the finite models in $\mathbb{R}^3$. It will be convenient to use a different set of parameters than $n$ and $\varepsilon$ to describe the grid when we prove convergence inside a fixed cube. So for the open cube $T_b = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_i| < b, i = 1, 2, 3\}$, let

$$
\varepsilon = \frac{2b}{n + 1}.
$$

(6)

Then for the grid $G(\varepsilon)$ described above we will rather use the notation $G_{b,n}$. 
We will also need to imbed $L^2(G_{b,n})$ into $L^2(T_b)$:

$$f\# = \varepsilon^{-3/2} \sum_{x \in G_{b,n}} f(x) \chi_{R(x)},$$  \hspace{1cm} (7)

where $\chi_{R(x)}$ is the characteristic function of the set $R(x) = \{(y_1, y_2, y_3) \in T_b : x_i - \varepsilon/2 \leq y_i < x_i + \varepsilon/2, i = 1, 2, 3\}$. Note that with this imbedding, $f\#(x)$ will be zero if any of the coordinates $x_i$ is less than $\varepsilon/2$ away from the boundary.

Imbedding the Hamiltonian has the same effect on the spectrum as the imbedding into $L^2(\mathbb{R}^3)$: it preserves the spectrum, except that 0 becomes an eigenvalue of infinite multiplicity. We will now change notation for all operators on the grid: Replace $H_\varepsilon$ by $H_{b,n}$, $V_\varepsilon$ by $V_{b,n}$, $\Delta_\varepsilon,i$ by $\Delta_{b,n,i}$, $\Delta_\varepsilon$ by $\Delta_{b,n}$, and $Q_\varepsilon$ by $Q_{b,n}$. We will also use $v_{b,n}$ instead of $v_\varepsilon$, but $v_{b,n}$ is now the restriction of the above defined $v_b$ (instead of $v$) to the grid. As we did for $H$ and $H_b$, we will write $H_{b,n}$ for $H_{b,n} + \alpha I$. 

Lemma
$H_{b,n}$ converges strongly to $H_b$ on $C_0^\infty(\bar{T}_b)$ as $n \to \infty$.

Lemma
Let $K_{b,n} = (I + H_{b,n})^{-1}$, and let $g_{b,n} \in L^2(G_{b,n})$ with $\|g_{b,n}\|_{L^2(G_{b,n})} \leq 1$. Also let $f_{b,n} = K_{b,n}g_{b,n}$. Then the sequence $(f_{b,n})_n$ is relatively compact in $L^2(T_b)$.

Using the two previous lemmas, the next theorem follows from Lemma 2.3 and Theorem 2.4 in [DVV94].

Theorem
With $H_b$ and $H_{b,n}$ defined as above, $\sigma(H_{b,n}) \to \sigma(H_b)$ as $n \to \infty$, where the spectral convergence is in the sense of the definition on slide 18.
Now we are ready to start proving that there is a sequence of finite Hamiltonians $H_{b,n(b)}$ such that $\sigma(H_{b,n(b)}) \to \sigma(H)$ as \( b \to \infty \).

**Lemma**

*Let* \( n(b)/b \to \infty \) *as* \( b \to \infty \). *Then the operators* \( H_{b,n(b)} \) *converge strongly to* \( H \) *on the common core* \( C^\infty_c(\mathbb{R}^3) \) *as* \( b \to \infty \).

**Lemma**

*For any sequence* \( \delta_b > 0 \) *such that* \( \delta_b \to 0 \) *as* \( b \to \infty \), *there exists a sequence* \( n(b) \) *such that* \( n(b)/b \to \infty \) *and* \( \lambda_{b,n(b)}^b \geq \lambda_k - \delta_b \) *for* \( k \leq b \).

**Lemma**

*Let* \( A_n \) *be a sequence of self-adjoint operators with discrete spectrum converging strongly to a Coulomb-like operator* \( A \) *on a common core. If there exists a sequence* \( \delta_n \to 0 \) *such that* \( \lambda_k^n \geq \lambda_k - \delta_n \) *for* \( k \leq n \), *then* \( \sigma(A_n) \to \sigma(A) \).
Now we can use this lemma to get the main theorem. Notice that the imbedded versions of the finite-dimensional operators do not have purely discrete spectrum because 0 is in the essential spectrum, but this does not matter since it comes after the discrete spectrum of $H$ and is a part of the essential spectrum of $H$.

**Theorem (Theorem 4.3 in [Bak16])**

Let as before $H$ be the Coulomb Hamiltonian, and $H_{b,n(b)}$ the finite-dimensional Hamiltonians associated to $H$. Let $\delta_b$ be a sequence converging to 0 and let $n(b)/b$ go to infinity as $b \to \infty$ such that $\lambda_k^{b,n(b)} \geq \lambda_k - \delta_b$ for $k \leq b$. Then $\sigma(H_{b,n(b)}) \to \sigma(H)$ as $b \to \infty$. 
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Local fields I

A local field is a non-discrete, locally compact field. The only connected local fields are $\mathbb{R}$ and $\mathbb{C}$. Disconnected local fields are, in fact, totally disconnected.

Every local field comes equipped with a canonical absolute value which defines its topology. It is induced by the Haar measure, and is Archimedean in the case of $\mathbb{R}$ and $\mathbb{C}$, and non-Archimedean in all other cases. It coincides with the usual absolute values for the fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}_p$.

For a general local field $K$ we will denote the canonical absolute value by $|\cdot|$ (for $\mathbb{Q}_p$ we will still denote it by $|\cdot|_p$).
Convention

From now on the term ’local field’ means a (totally) disconnected, non-discrete, locally compact field.

Let $K$ be a local field with canonical absolute value $|\cdot|$. Define

$$O = \{ x \in K : |x| \leq 1 \}, \quad P = \{ x \in K : |x| < 1 \}, \quad U = O \setminus P.$$  

$O$: a compact subring of $K$ (the ring of integers). It is a discrete valuation ring, i.e., a principal ideal domain with a unique maximal ideal.

$P$: the unique non-zero maximal ideal of $O$, called the prime ideal. Any element $\beta \in P$ such that $P = \beta O$ is called a uniformizer (or a prime element) of $K$. For $\mathbb{Q}_p$ one can choose $\beta = p$. 

Local fields III

$U = O \setminus P$: The group of units of $O$.

$O/P$: a finite field with $q = p^f$ elements ($p$: a prime number, $f$: a natural number).

If $\beta$ is a uniformizer, then $|\beta| = 1/q$. Range($|\cdot|$) = \( \{ q^N : N \in \mathbb{Z} \} \).

If $S$ is a complete set of representatives for the residue classes in $O/P$, every non-zero element $x \in K$ can be written uniquely in the form:

$$x = \beta^{-m}(x_0 + x_1 \beta + x_2 \beta^2 + \cdots),$$

where $m \in \mathbb{Z}$, $x_j \in S$, $x_0 \not\in P$. With $x$ written in this form, we have $|x| = q^m$. 
Theorem (Classification of local fields)

Characteristic zero. Every local field of characteristic zero is a finite extension of $\mathbb{Q}_p$ for some $p$.

Positive characteristic. Every local field of positive characteristic $p$ is isomorphic to the field $\mathbb{F}_q((t))$ of Laurent series over a finite field $\mathbb{F}_q$, where $q = p^f$ for some integer $f \geq 1$. 
Model over $K$

We use the complex Hilbert space $L^2(K)$ with respect to Haar measure on $K$. The operators $P$ and $Q$ cannot be defined as in the real case since in general we cannot multiply a complex number by an element of $K$. However, in the real case the Laplacian can also be expressed as $\Delta = -\mathcal{F}^{-1}Q^2\mathcal{F}$. In the local field case it is therefore natural to define the Laplacian as:

$$\Delta = -\mathcal{F}^{-1}Q^2\mathcal{F}$$

where now

$$Q^2f(x) = |x|_p^2f(x).$$
With \( P = \mathcal{F}^{-1} Q \mathcal{F} \) we have \( \Delta = -P^2 \), and for the Hamiltonian we take
\[
H = -\Delta + V = P^2 + V,
\]
where the potential \( V \) is given by \( (Vf)(x) = v(x)f(x) \) for a suitable function \( v \) on \( K \) \( (f \in L^2(K), \ x \in K) \).

In the local field setting it is customary to study \( H = P^\alpha + V \) for any \( \alpha > 0 \), since the qualitative behavior of \( P^\alpha \) and \( H \) does not change with \( \alpha > 0 \). So we finally take
\[
H = P^\alpha + V
\]
as our object of study.
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Finite model over $K$

Let $n$ be a natural number, set $B_n = H_{-n} = \beta^{-n}O$ (ball of radius $q^n$) and $G_n = H_{-n}/H_n$.

$G_n$ is a finite cyclic group with $q^{2n}$ elements and with generator $\beta^{-n} + H_n$.

Since $H_n$ is an open subset of $K$, we obtain a Haar measure $\mu_n$ on $G_n = H_{-n}/H_n$ from the Haar measure $\mu$ of $K$ by setting $\mu_n(x + H_n) = \mu(x + H_n) = \mu(H_n) = q^{-n}$, for $x + H_n \in G_n = H_{-n}/H_n$.

So each point of $G_n$ has mass $= q^{-n}$, and the total mass of $G_n$ is $q^{2n} \cdot q^{-n} = q^n$.

Each element of $G_n$ has a unique representative of the form $a_{-n}\beta^{-n} + a_{-n+1}\beta^{-n+1} + \cdots + a_{-1}\beta^{-1} + a_0 + a_1\beta + \cdots + a_{n-2}\beta^{n-2} + a_{n-1}\beta^{n-1}$, $a_i \in S$. We denote this set by $X_n$, and call it the canonical set of representatives for $G_n$; we also give it the group structure coming from its natural identification with $G_n$. 


With the above choice of Haar measure on $G_n$ the mapping which sends the characteristic function of the point $x + H_n$ in $G_n$ to the characteristic function of the subset $x + H_n$ in $K$, becomes an isometric imbedding of $L^2(G_n)$ into $L^2(K)$.

**Important subspaces of $L^2(K)$:**

- $\mathcal{C}_n = \{ f \in L^2(K) | \text{supp}(f) \subset B_n \}$
- $\mathcal{S}_n = \{ f \in L^2(K) | f \text{ is locally constant of order } \leq p^{-n} \}$
- $\mathcal{D}_n = \mathcal{C}_n \cap \mathcal{S}_n$

We have: $\mathcal{F}\mathcal{C}_n = \mathcal{S}_n$ and $\mathcal{F}\mathcal{S}_n = \mathcal{C}_n$, so $\mathcal{F}\mathcal{D}_n = \mathcal{D}_n$. 

**Imbedding**
Commutation rules

Let $C_n$ and $S_n$ be the orthogonal projections on $C_n$ and $S_n$, respectively:

$$C_nf = f \cdot 1_{B_n}, \quad S_nf(x) = q^n \int_{H_n} f(x + y)d_n(y).$$

$C_n$ and $S_n$ commute, so $D_n = C_nS_n$ is the orthogonal projection on $D_n$.

Furthermore, $D_n$ commutes with the Fourier transform $\mathcal{F}$ on $K$, and $\mathcal{F}D_n = D_n\mathcal{F}$ coincides with the finite Fourier transform $\mathcal{F}_n$ on $G_n$. 
Dynamical operators for the finite model

Functions on the finite grid $G_n$ can be identified with functions on $K$ which have support in $B_n$ and are invariant under translation by elements of $H_n$. For the position operator $Q_n$ on $L^2(G_n)$ we take the restriction to these functions of the position operator $Q$ on $L^2(K)$: $(Qf)(x) = |x|_p f(x)$. The momentum operator $P_n = \mathcal{F}_n^{-1} Q_n \mathcal{F}_n$ then becomes the restriction of $P = \mathcal{F}^{-1} Q \mathcal{F}$ to $L^2(G_n)$. For our finite Hamiltonian we take

$$H_n = P_n^\alpha + V_n$$

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\(^3\)See slide 33 for a comment on the exponent $\alpha$. 
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Definition (Uniform compactness)

A sequence of bounded operators \((M_n)\) on a Hilbert space \(\mathcal{H}\) is said to satisfy a condition of uniform compactness \(^4\) if the following conditions hold:

1. The sequence \((M_n)\) is uniformly bounded.
2. There are subspaces \(L_n\) with \(L_n\) invariant under \(M_n\) such that for every sequence \((g_n)\) with \(g_n \in L_n\) and \(\|g_n\| \leq 1\), the sequence \((M_n g_n)\) is relatively compact in \(\mathcal{H}\).

\(^4\)Notice that the individual operators \(M_n\) are not required to be compact on the whole space \(\mathcal{H}\) (and in our applications they will not be). Still, if the above conditions are fulfilled, we will say that the sequence \((M_n)\) is uniformly compact, even if each \(M_n\) is compact only on the subspace \(M_n\).
For our purposes the usefulness of uniform compactness lies in the following two results. They give a strong connection between the spectral data of the approximating operators \((M_n)\) and their strong limit \(M\).

**Lemma**

Let \(M_n, L_n\) be as in the definition above, and assume that the sequence \(M_n\) converges strongly to a bounded operator \(M\). Assume further that there are eigenvectors \(g_n\) and corresponding eigenvalues \(\lambda_n\) such that \(g_n \in L_n, \|g_n\| = 1\) and \(M_n g_n = \lambda_n g_n\). Then any non-zero cluster point \(\lambda_0\) of the sequence \((\lambda_n)\) is an eigenvalue of \(M\), and there is a subsequence of \((g_n)\) which converges to a vector \(g\) such that \(Mg = \lambda_0 g\).
Proposition (Cfr. Lemma 3 in [DVV94])

Keep the notation and assumptions of the previous lemma. In addition, assume the following: (i) The operators $M_n, M$ are self-adjoint, and $0 \leq M, M_n \leq I$, (ii) $M$ is compact on $\mathcal{H}$, and $M_n$ is compact on $L_n$. Then the following hold:

1. If $J$ is a compact subset of $(0, 1]$ with $J \cap \sigma_p(M) = \emptyset$, then $J \cap \sigma_p(M_n) = \emptyset$ for large $n$.

2. If $\lambda \in \sigma_p(M)$, there exists a sequence $(\lambda_n)$ with $\lambda_n \in \sigma(M_n)$ such that $\lambda_n \to \lambda$. Further, if $J$ is a compact neighborhood of an eigenvalue $\lambda \in \sigma_p(M)$, not containing any other eigenvalues of $M$, then any sequence $(\lambda_n)$ with $\lambda_n \in \sigma_p(M_n) \cap J$ converges to $\lambda$.

3. Let $\lambda$ and $J$ be as in (2). Then $\dim P^M_n(J) = \dim P^M(J)$ for large $n$, and for each orthonormal basis $\{e_1, \ldots, e_m\}$ for $r(P^M(J))$ there is, for each $n$, an orthonormal basis $\{e^n_1, \ldots, e^n_m\}$ for $r(P^M_n(J))$ such that $\lim_{n \to \infty} e^n_i = e_i$, $i = 1, \ldots, m$. 
Proposition

With $M_n = (I + H_n)^{-1}$, $L_n = D_n \simeq L^2(G_n)$, and $\mathcal{H} = L^2(K)$, the resolvents $(I + H_n)^{-1}$ are uniformly compact in the sense of Definition 14.
The analog of the main convergence theorem in [DVV94] can now be established for a general local field in [BD15]:
Main convergence theorem\textsuperscript{5}
Analog of Theorem 4 in [DVV94]

Theorem

1. If $J$ is a compact subset of $[0, \infty)$ with $J \cap \sigma(H) = \emptyset$, then $J \cap \sigma(H_n) = \emptyset$ for large $n$.

2. If $\lambda \in \sigma(H)$, there exists a sequence $(\lambda_n)$ with $\lambda_n \in \sigma(H_n)$ such that $\lambda_n \to \lambda$. Further, if $J$ is a compact neighborhood of an eigenvalue $\lambda \in \sigma(H)$, not containing any other eigenvalues of $H$, then any sequence $\lambda_n$ with $\lambda_n \in \sigma(H_n) \cap J$ converges to $\lambda$.

3. Let $\lambda$ and $J$ be as in (2). Then $\dim P^H_n(J) = \dim P^H(J)$ for large $n$, and for each orthonormal basis $\{e_1, \ldots, e_m\}$ for $\text{Ran}(P^H(J))$ there is, for each $n$, an orthonormal basis $\{e^n_1, \ldots, e^n_m\}$ for $\text{Ran}(P^H_n(J))$ such that $\lim_{n \to \infty} e^n_i = e_i$, $i = 1, \ldots, m$.

\textsuperscript{5}\sigma(A) = \text{spectrum of } A. \ P^A = \text{spectral measure of } A. \ \text{Ran}(A) = \text{range of } A.
Outline

1 Finite approximations over the reals
   Discrete spectrum
   Mixed spectrum: Atomic potential (from the thesis of Erik M. Bakken)
     Hamiltonian inside a finite box
     Convergence theorems

2 Finite approximations over local fields (discrete spectrum)
   Preliminaries
   Finite models
   Standard methods
     Stochastic methods
       Stochastics at the finite level

3 References

4 Appendix: Numerical results
   Numerical results over $\mathbb{R}$
   Numerical results over $K$
In [DVV94] two proofs of the main theorem were given: a functional analytic one and a stochastic one. The stochastic proof gave a stronger convergence result for the eigenfunctions (uniform convergence on compacta). The proof in [BD15] used functional analytic methods. It is of interest to give a stochastic proof also in the non-Archimedean case, and this has been accomplished in [BDW17]. We now give a review of the results in [BDW17].
Brownian motion and the heat equation over $\mathbb{R}$

Brownian motion is described by a family of Wiener measures $(\mathcal{W}_x)_{x \in \mathbb{R}}$, which in turn are generated by the probability densities $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/4t}$ in the following sense:

$$
\int_{C([0,\infty):\mathbb{R})} f(\omega(t)) d\mathcal{W}_x(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) p_t(x - y) dy
$$

for all "observables" $f$ belonging to a suitable class of functions on $\mathbb{R}$. The function $u(x, t) = p_t(x)$ is a fundamental solution of the heat equation

$$
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t)
$$
Brownian motion and the heat equation over \( \mathbb{R}^n \)

which by Fourier transform becomes

\[
\frac{\partial \hat{u}}{\partial t}(\xi, t) = -\xi^2 \hat{u}(\xi, t)
\]  

(9)

and so

\[
\hat{p}_t(\xi) = \hat{u}(\xi, t) = e^{-t\xi^2},
\]  

(10)

taking into account that \( p_t(x) \) is a fundamental solution. The \((p_t)_{t>0}\) form a semi-group under convolution, and thus give rise to a semi-group of operators \((T_t)_{t>0}\) by \( T_t f = p_t \ast f \). The infinitesimal generator of \((T_t)_{t>0}\) is the Laplacian \( \Delta \) (on a suitable domain), so we can also write \( e^{t\Delta} f = p_t \ast f \).
Brownian motion and the heat equation over $K$

Over a local field $K$ we still let $t$ be a positive real parameter, but the role of the Laplacian $\Delta$ is played by the operator $-P^\alpha$ (remember that $\Delta = -P^2$ over $\mathbb{R}$), and so the heat equation (8) becomes

$$\frac{\partial u}{\partial t}(x, t) = -(P^\alpha u)(x, t), \quad \text{i.e.:} \quad \frac{\partial u}{\partial t}(x, t) = -(\mathcal{F}^{-1} |.|^\alpha \mathcal{F} u)(x, t)$$

(11)

thus

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) = -|\xi|^\alpha \hat{u}(\xi, t)$$

(12)
Brownian motion and the heat equation over $K_{\mathbb{II}}$

giving

$$\hat{u}(\xi, t) = e^{-t|\xi|^\alpha}$$  \hspace{1cm} (13)

by a similar normalization as above. In analogy with the real case one now defines

$$p_t(x) = (\mathcal{F}^{-1}(e^{-t|\cdot|^\alpha}))(x) = \int_K e^{-t|\xi|^\alpha} \chi(x\xi) \, d\xi.$$  \hspace{1cm} (14)

The $(p_t)_{t>0}$ again form a semi-group under convolution (since clearly $(\hat{p}_t)_{t>0}$ form a semi-group under multiplication), and $\int_K p_t(x) \, dx = 1$ for all $t > 0$ (since $\hat{p}_t(0) = 1$ for all $t > 0$). Thus the only thing missing for the $(p_t)_{t>0}$ to generate a Wiener measure as above, is the positivity of the $(p_t)_{t>0}$. And this has been proved by several authors in various settings (see [Koc01, Ch. 4] and references therein, and [Var97]).
Brownian motion and the heat equation at the finite level I

For our finite model we pursue the above analogy and define

$$p_{t,n}(x) = (\mathcal{F}_n^{-1} e^{-t|\cdot|^\alpha})(x),$$  \hspace{1cm} (15)

in analogy with (14). Here we regard $e^{-t|\cdot|^\alpha}$ as a function on $X_n$. We still have

$$e^{-tP_n^\alpha} f = p_{t,n} * f$$  \hspace{1cm} (16)

since

$$(e^{-tP_n^\alpha} f)(x) = (e^{-t\mathcal{F}_n^{-1}Q_n^\alpha}\mathcal{F}_n f)(x) = (\mathcal{F}_n^{-1} e^{-tQ_n^\alpha} \mathcal{F}_n f)(x)$$
$$= (\mathcal{F}_n^{-1} (e^{-t|\cdot|^\alpha} \mathcal{F}_n f))(x) = (\mathcal{F}_n^{-1} (e^{-t|\cdot|^\alpha}) * f)(x)$$
$$= (p_{t,n} * f)(x),$$
Brownian motion and the heat equation
at the finite level II

where the convolution $\ast$ now is over $X_n$:

$$(f \ast g)(x) = \int_{X_n} f(y)g(x - y)d\mu_n(y) = q^{-n} \sum_{y \in X_n} f(y)g(x - y).$$

The one-parameter family $(p_{t,n})_{t>0}$ is a semi-group under convolution (since clearly $(\hat{p}_{t,n})_{t>0}$ is a multiplicative semi-group), and $\int_{X_n} p_{t,n}(x)dx = 1$ for all $n$ and for all $t > 0$ (since $\hat{p}_{t,n}(0) = 1$). It remains to show that the $p_{t,n}$ are positive:

**Lemma**

*We have $p_{t,n}(x) > 0$ for all $x \in X_n$, all $n$ and all $t > 0$, hence $(p_{t,n})_{t>0}$ defines a probability distribution over $X_n$.***
Probability measures on the space of Skorokhod functions

From now on we will work on a fixed time interval \([0, t]\). The space \(D[0, t]\) of Skorokhod functions are the functions defined on \([0, t]\) with values in \(K\) which satisfy the following two criteria:

1. For each \(s \in (0, t)\), \(f(s \pm 0)\) exist; \(f(0 + 0)\) and \(f(t - 0)\) exist.
2. \(f(s + 0) = f(s)\) for all \(s \in [0, t]\), and \(f(t) = f(t - 0)\).

We will use the densities \(p_{s,n}\) to construct, for each \(n\) and for each \(a \in X_n\), a probability measure \(P^n_a\) on the space \(D[0, t]\), and subsequently show that these measures converge weakly to the measure \(P_a\) on \(D[0, t]\) which is constructed from \(p_s\). The measure \(P^n_a\) will give full measure to the paths which take values in the grid \(X_n\).
Construction of measures on the space of Skorokhod functions I

Pick a point \( a \in X_n \), fix \( N \) time points \( t_1 < t_2 < \cdots < t_N \), and for each \( i = 1, \ldots, N \), pick a Borel subset \( J_i \) of \( K \). We define a measure \( \mathbf{P}^n_a \) on the cylinder sets \( \{ \omega : [0, t] \to K : \omega(t_i) \in J_i \} \) by

\[
\mathbf{P}^n_a(\omega(t_i) \in J_i) = \sum_{b_i \in J_i \cap X_n, 1 \leq i \leq N} p_{t_1,n}(b_1 - a) \cdots p_{t_{N-1},n}(b_N - b_{N-1})q^{-nN}.
\]

By Kolmogorov’s Extension Theorem [Øks98, Thm. 2.1.5], \( \mathbf{P}^n_a \) has a unique extension to a probability measure on \( \Omega[0, t] \), the space of all functions \( \omega : [0, t] \to K \), equipped with the \( \sigma \)-algebra generated by all cylinder sets. To get a probability measure on \( D[0, t] \), equipped with the Borel sets coming from the Skorokhod topology, we need to check the Čentsov
Construction of measures on the space of Skorokhod functions II

criterion, which says: If there are constants $\alpha, \beta, \gamma, C > 0$ such that

$$E_{P^a_n}(\left| Y_{t_1} - Y_{t_2} \right|^\alpha \left| Y_{t_2} - Y_{t_3} \right|^\beta) \leq C |t_1 - t_3|^{1+\gamma}$$

(19)

for all $0 < t_1 < t_2 < t_3$, then there is a unique measure on $D[0, t]$ which satisfies the condition (17). Here $E_{P^a_n}$ denotes the expectation w.r.t. the measure $P^a_n$, and $Y_s$ denotes the random variable $Y_s(\omega) = \omega(s)$, $\omega \in \Omega[0, t]$, $s \in [0, t]$. The random variables $Y_s$ define a process with independent increments with respect to each of the measures $P^a_n$. 
Weak Convergence of Unconditioned Measures

Theorem
Let \( a_n \in X_n, \ a \in K \) be such that \( a_n \to a \) as \( n \to \infty \). Then

\[
P^n_{a_n} \Rightarrow P_a \text{ as } n \to \infty
\]

where \( \Rightarrow \) denotes weak convergence of measures.
Conditioned measures

Let $a, b \in X_n$. We define the conditioned measure $P_{a,b,t}^n$ of a Borel set $A \subset D[0, t]$ by

$$P_{a,b,t}^n(A) = \frac{P_a^n(A \cap (\omega(t) = b))}{P_a^n(\omega(t) = b)}. \quad (20)$$

Let $a_n$ and $b_n$ be sequences of grid points converging to $a$ and $b$ respectively. Then for time points $t_1, \ldots, t_N$ in $[0, t]$ and balls $J_i$ in $K$

$$P_{a_n,b_n,t}^n(\omega(t_i) \in J_i) = \frac{P_{a_n}^n((\omega(t_i) \in J_i) \cap (\omega(t) = b_n))}{P_{a_n}^n(\omega(t) = b_n)}. \quad (21)$$

\footnote{Here and in the following we use the probabilist’s notation for sets: $(\omega(t) = b)$ is a shortcut notation for the set $\{\omega : \omega(t) = b\}$.}
Weak Convergence of Conditioned Measures

Theorem
If \( a_n \in X_n \rightarrow a \in K \) and \( b_n \in X_n \rightarrow b \in K \), then \( P_{a_n,b_n,t}^n \Rightarrow P_{a,b,t} \).
The convergence is uniform when \((a, b)\) varies in compact subsets of \( K \times K \).

Theorem
For each \( a \in X_n \) the measure \( P_{a}^n \) gives full measure to the paths on the grid, that is,

\[
P_{a}^n(\omega : \omega(s) \in X_n, \forall s \in [0, t]) = 1 \tag{22}
\]
Feynman-Kac formulas I

For the Hamiltonian $H$ over $K$:

$$(e^{-tH}f)(x) = \int_K K_t(x,y)f(y)\,dy, \quad f \in L^2(K), \quad (23)$$

where

$$K_t(x,y) = \int_{D[0,t]} e^{-\int_0^t v(\omega(s))\,ds} dP_{x,y,t}(\omega) \cdot p_t(y - x). \quad (24)$$
Feynman-Kac formulas II

At the finite level we can similarly prove:

**Theorem (Feynman-Kac at the finite level)**

\[
(e^{-tH_n}f)(x) = \int_{X_n} K^n_t(x, y)f(y) \, d\mu_n(y)
\]

\[
= q^{-n} \sum_{y \in X_n} K^n_t(x, y)f(y), \quad f \in L^2(X_n)
\]

where

\[
K^n_t(x, y) = \int_{D[0,t]} e^{-\int_0^t v_n(\omega(s)) \, ds} \, d\mathbf{P}^n_{x,y,t}(\omega) \cdot p_{t,n}(y-x), \quad x, y \in X_n.
\]
Convergence of propagators

Lemma

$K^n_t$ converges continuously to $K_t$, i.e., if $x_n \in X_n \to x \in K$ and $y_n \in X_n \to y \in K$ as $n \to \infty$, then

$$K^n_t(x_n, y_n) \to K_t(x, y).$$

In particular, $K^n_t$ converges uniformly to $K_t$ on compact sets.

The proof uses the Feynman-Kac formula (21).
Stochastic proof of main theorem

Theorem
For any $t > 0$,

$$\text{Tr}(e^{-tH_n}) \to \text{Tr}(e^{-tH})$$  \hspace{1cm} (27)

$$\|e^{-tH_n} - e^{-tH}\|_1 \to 0$$  \hspace{1cm} (28)

as $n \to \infty$.

Convergence in trace norm implies convergence in operator norm which gives convergence of eigenvalues and eigenfunctions (see pp. 289-290 in [RS80]). Thus we have reproved by stochastic methods the main convergence theorem (16).

However, the stochastic method gives an even stronger convergence result, namely uniform convergence on compacta of the eigenfunctions:
Uniform Convergence on Compacta of Eigenfunctions

Theorem

Let $f_{n,j}$ and $f_j$ be eigenfunctions of $H_n$ and $H$ corresponding to the eigenvalues $\lambda_{n,j}$ and $\lambda_j$ respectively. Assume that $\lambda_{n,j}$ converges to $\lambda_j$ and that $f_{n,j}$ converges to $f_j$ in $L^2(K)$. Then

$$f_{n,j} \rightarrow f_j$$

(29)

uniformly on compacta.


Outline

1. Finite approximations over the reals
   - Discrete spectrum
   - Mixed spectrum: Atomic potential (from the thesis of Erik M. Bakken)
     - Hamiltonian inside a finite box
     - Convergence theorems

2. Finite approximations over local fields (discrete spectrum)
   - Preliminaries
   - Finite models
   - Standard methods
   - Stochastic methods
     - Stochastics at the finite level

3. References

4. Appendix: Numerical results
   - Numerical results over \( \mathbb{R} \)
   - Numerical results over \( K \)
The Schwinger model

In order to illustrate the remarkable accuracy of the Schwinger model, we present on the following pages some numerical results for the harmonic oscillator with \( N = 5, 21, \) and 81 points in the grid. The finite eigenfunctions and the Hermite functions are rendered in the same diagram, and, as can be seen, the finite eigenfunctions lie smack on the Hermite functions, already at \( N = 5 \). Also, the eigenvalues show good agreement already at \( N = 5 \), and at \( N = 81 \) the first 11 eigenvalues are exact up to machine accuracy.

We also present some tables where we compare the Schwinger model with the finite difference model, showing clear superiority of the former over the latter.
Schwinger eigens with $N = 5$

Eigenvalues for $H_5$
- $0.4969786369997017$
- $1.538153655416401$
- $2.273277799898967$
- $3.512928870280916$
- $4.745031651763187$

Eigenfunction no. 0 for $H_5$ (dotted) and $H$ (smooth).

Eigenfunction no. 1 for $H_5$ (dotted) and $H$ (smooth).

Eigenfunction no. 2 for $H_5$ (dotted) and $H$ (smooth).
Schwinger eigens with $N = 21$

Eigenvalues for $H_{21}$

0.4999999999999396
1.50000000000396
2.499999999873056
3.50000002436515
4.4999963136251
5.50000389175935
6.49996311530578
7.50024950572093
8.49832644686019
9.500769902078664
10.49608851334482

Eigenfunction no. 0
for $H_{21}$ (dotted) and $H$ (smooth).

Eigenfunction no. 1
for $H_{21}$ (dotted) and $H$ (smooth).

Eigenfunction no. 10
for $H_{21}$ (dotted) and $H$ (smooth).
Schwinger eigens with $N = 81$

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Eigenfunction no. 0
for $H_{81}$ (dotted) and $H$ (smooth).

Eigenfunction no. 1
for $H_{81}$ (dotted) and $H$ (smooth).

Eigenfunction no. 40
for $H_{81}$ (dotted) and $H$ (smooth).
Comparison of Schwinger model and finite difference model, $N = 81$

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Comparison of Schwinger model and finite difference model, $N = 241$

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Coulomb potential

Table: Numerical result with $n = 350$ and $\varepsilon = (2\pi/n)^{1/2}$

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</tbody>
</table>
Outline

1 Finite approximations over the reals
   Discrete spectrum
   Mixed spectrum: Atomic potential (from the thesis of Erik M. Bakken)
   Hamiltonian inside a finite box
   Convergence theorems

2 Finite approximations over local fields (discrete spectrum)
   Preliminaries
   Finite models
   Standard methods
   Stochastic methods
   Stochastics at the finite level

3 References

4 Appendix: Numerical results
   Numerical results over $\mathbb{R}$
   Numerical results over $K$
Numerical results for the Schrödinger operator over $\mathbb{Q}_3[\sqrt{3}]$

Let as before $H = P^\alpha + V$ denote the Schrödinger operator over a local field $K$. The eigenfunctions of $H$ can be divided into two main types, corresponding to two complementary subspaces of $L^2(K)$: those which are supported on a single spherical shell (which we shall call shell functions), and those which are radial\(^7\). Of these, only the shell functions are completely understood: They belong to eigenvalues which can be determined from Diophantine equations, and there are explicit formulae for them. For radial eigenfunctions no such explicit formulae seem to be known.

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\(^7\)With terminology as in [VVZ94], the set of shell functions comprises all the type I functions plus the shell functions of type II; the radial functions are all of type II.
In this numerical study we specialize to the case of the Schrödinger operator \( H = \frac{1}{2}(P^2 + Q^2) \) of the harmonic oscillator over the local field \( \mathbb{Q}_3[\sqrt{3}] \), which is a quadratic and totally ramified extension of \( \mathbb{Q}_3 \). We were interested in the following questions:

- Do eigenfunctions of both types (shell functions and radial functions) show up already at the finite level?
- Is there good agreement between the theoretical and numerical eigenvalues?
- Is there good agreement between the theoretical and numerical eigenfunctions?
- Are multiplicities correct?

The answer to all these questions was ‘yes’. To illustrate this, we sum up some of the results in Table 2.
The extension $\mathbb{Q}_3[\sqrt{3}]/\mathbb{Q}_3$ is totally ramified, so with standard notation we have $e = 2$, and hence $f = 1$ since $ef = [\mathbb{Q}_3[\sqrt{3}] : \mathbb{Q}_3] = 2$. Further, from $q = p^f$ follows $q = p = 3$, and as uniformizer we can take $\beta = \sqrt{3}$, hence $|\beta| = 1/q = 1/3$. For the exponent of the different we have $d = 1$, so the canonical character $\chi$ associated with these data becomes

$$\chi(x) = \exp \left( 2\pi i \left\{ \text{Tr}_{\mathbb{Q}_3[\sqrt{3}]/\mathbb{Q}_3} (\sqrt{3}^{-1} x) \right\} \right), \ x \in \mathbb{Q}_3[\sqrt{3}].$$
For the finite model we did experiments with $n = 1, 2, 3, 4$, so we were working with finite grids of sizes $|X_1| = 9$, $|X_2| = 9^2 = 81$, $|X_3| = 9^3 = 729$, and $|X_4| = 9^4 = 6561$, respectively. Of particular interest to us was how the eigenfunctions came out: Would they clearly exhibit characteristics as shell functions or radial functions? They did. To illustrate this we give in the last table an excerpt from the value tables of three eigenfunctions: one is radial, one is a linear combination of two shell functions, and one is a pure shell function. We also wanted to compare our numerically computed eigenfunctions to the theoretical ones (evaluated on the grid). To do this, we measured the distance from each of the former to the linear span of the latter. Up to machine accuracy ($10^{-16}$), the distance came out as zero. We find this quite remarkable.
The following two tables should be self-explanatory. The data are taken from a computer run with $n = 2$ (i.e., 81 points in the finite grid). Each of the functions in the last table is represented with 28 values, with values coming from each of the 5 shells which occur for $n = 2$.

\[\text{[VVZ94, p. 190]}\]

\[\text{In the estimate for the lowest eigenvalue in Table 2 (first entry in column 1) we are assuming that the estimate given in [VVZ94, p. 190] is valid also in our setting.}\]
Table: Numerical approximations to the spectral data of $H = \frac{1}{2}(P^2 + Q^2)$ over $Q_3[\sqrt{3}]$.

<table>
<thead>
<tr>
<th>Theoretical eigenvalue</th>
<th>Numerical eigenvalue</th>
<th>Theoretical multiplicity</th>
<th>Numerical multiplicity</th>
<th>Type of eigenfunction</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; \lambda_0 &lt; 9/13 \approx 0.6923$</td>
<td>0.6684</td>
<td>1</td>
<td>1</td>
<td>radial</td>
<td></td>
</tr>
<tr>
<td>?</td>
<td>4.6922</td>
<td>?</td>
<td>1</td>
<td>radial</td>
<td></td>
</tr>
<tr>
<td>?</td>
<td>4.7158</td>
<td>?</td>
<td>1</td>
<td>radial</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5.0000</td>
<td>2</td>
<td>2</td>
<td>shell function</td>
<td>$2 = 1 + 1$: Coming from two different shells.</td>
</tr>
<tr>
<td>9</td>
<td>9.0000</td>
<td>4</td>
<td>4</td>
<td>shell function</td>
<td>All supported on the same shell.</td>
</tr>
<tr>
<td>?</td>
<td>40.5213</td>
<td>?</td>
<td>2</td>
<td>radial</td>
<td></td>
</tr>
<tr>
<td>$40 + \frac{5}{9} = 40.5555 \ldots$</td>
<td>40.5555</td>
<td>2</td>
<td>2</td>
<td>shell function</td>
<td>$2 = 1 + 1$: Coming from two different shells.</td>
</tr>
<tr>
<td>41</td>
<td>41.0000</td>
<td>8</td>
<td>8</td>
<td>shell function</td>
<td>$8 = 4 + 4$: Coming from two different shells.</td>
</tr>
<tr>
<td>45</td>
<td>45.0000</td>
<td>24</td>
<td>24</td>
<td>shell function</td>
<td>$24 = 12 + 12$: Coming from two different shells.</td>
</tr>
</tbody>
</table>
The following page is extracted from [BD15]. The function values 0 in columns 2 and 3 are rounded values of numbers of the order $10^{-16}$ or smaller (machine accuracy).
Table 2. Eigenfunctions for three different eigenvalues, 28 values for each function, coming from all the 5 shells. Both kinds of eigenfunctions occur (shell functions and radial functions). – Shell no. \( k (k = 2, 1, 0, -1, -\infty) \) is the shell \(|x| = 3^k \) (so shell no. \(-\infty\) is the shell \(|x| = 3^{-\infty} = 0\).

<table>
<thead>
<tr>
<th>Eigenfunction for shell no. ( k ) ( (\lambda \approx 0.6684) )</th>
<th>Eigenfunction for ( \lambda = 5 ). Eigenfunctions here are linear combinations of shell functions from two different shells (shells 1 and 0). As should be expected, the function below exhibits non-radial behavior, being non-constant on each shell where it doesn’t vanish (shells 1 and 0).</th>
<th>Eigenfunction for ( \lambda = 9 ). It exhibits a perfect shell function behavior, with support on shell no. 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shell no.</td>
<td>Shell no.</td>
<td>Shell no.</td>
</tr>
<tr>
<td>3.5818432 ( \cdot 10^{-1} )</td>
<td>( -\infty )</td>
<td>( -\infty )</td>
</tr>
<tr>
<td>5.5430722 ( \cdot 10^{-3} )</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5.5430722 ( \cdot 10^{-5} )</td>
<td>2</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>5.5430722 ( \cdot 10^{-5} )</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1.2747433 ( \cdot 10^{-2} )</td>
<td>1</td>
<td>-2.3459638 ( \cdot 10^{-1} )</td>
</tr>
<tr>
<td>1.2747433 ( \cdot 10^{-2} )</td>
<td>1</td>
<td>-2.3459638 ( \cdot 10^{-1} )</td>
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<td>2.3459638 ( \cdot 10^{-1} )</td>
</tr>
<tr>
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<td>3.9500330 ( \cdot 10^{-2} )</td>
</tr>
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<td>-3.9500330 ( \cdot 10^{-2} )</td>
</tr>
<tr>
<td>3.5768544 ( \cdot 10^{-1} )</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>