Ultrametric space in Teichmüller theory

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This in collaboration with Prof. Alberto Verjovsky.
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$$Z_g := \lambda^g \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} dy_i d\bar{y}_i |F_g(y)|^2 \det(1 - z(y)\bar{z}(y))^{-13}$$
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where the moduli space $\mathcal{M}_g$ is taken as a fundamental domain in the Teichmüller space $T_g$ respect to the mapping class group.
Introduction

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The $g$-loop Polyakov action of the closed bosonic string is an integral over the moduli space $\mathcal{M}_g$ of genus $g$ compact Riemann surfaces:

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where the moduli space $\mathcal{M}_g$ is taken as a fundamental domain in the Teichmüller space $T_g$ respect to the mapping class group.
An action for the theory is obtained by summing all the contributions:

$$Z := \sum_{g \in \mathbb{N}_0} Z_g$$
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$$Z = \int_{\mathcal{M}} \prod_{i=1}^{\infty} dc_i d\bar{c}_i |\tilde{F}(c)|^2 \det(1 - Z(c)^\dagger Z(c))^{-\frac{13}{2}}$$

where the space $\mathcal{M}$ is a fundamental domain of the universal Teichmüller space $T(1)$. respect to the mapping class group.
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where the space $\mathcal{M}$ is a fundamental domain of the universal Teichmüller space $T(1)$ respect to the mapping class group. Unfortunately, the last expression cannot be formalized. One of the problems is that $T(1)$ is non separable; i.e. It is too big. Another direction is to work on the closure of the inductive limit of finite Teichmüller space:

$$T_\infty := \bigcup_{g \in \mathbb{N}_0} T_g$$

This is a separable space.
We show that the space $T_\infty$ can be seen as a space of finite dimensional valued fields over an ultrametric space.
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In particular, heuristically, we can write the resulting string theory on $T_{\infty}$ as a Quantum Field Theory of these fields:

$$Z = \int D\varphi D\bar{\varphi} \, e^{S(\varphi, \bar{\varphi})}$$
Consider a Riemann surface $\Sigma$ ...
Consider a Riemann surface $\Sigma$ . . .
How we deform its complex structure?
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How we deform its complex structure?

Consider the Poincaré-Koebe uniformization:

$$\Delta \to \Sigma$$

and the representation of $G := \pi_1(\Sigma)$ as a Fuchsian group:

$$\alpha : G \to Isom^+(\Delta)$$
Introduction

Deformation

Universal Hyperbolic Lamination

Renormalized Weil-Petersson metric

Main results
Definition

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- \( \mu \) is a \( G \)-periodic differential if it is a differential and:

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\alpha(g)^* \mu = \mu \quad \forall \ g \in G
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Remark

The pullback of a differential in $\Sigma$ by the uniformization map is $G$-periodic differential in the Poincaré disk.
How a Beltrami differential (deformation parameter) actually realizes a deformation?
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A Beltrami differential can be seen as an $\infty$-measurable field of ellipses:

$$K = \frac{1 + |\mu|}{1 - |\mu|}$$

$$\arg(\mu)/2$$
Consider the Ahlfors-Bers equation:

\[ \partial \bar{z} f = \mu \partial_z f \]
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Is there a solution to this equation on the disk $\Delta$? ... Equivalently, Is there a map $f$ on the disk straightening all the infinitesimal ellipses into infinitesimal circles?
Theorem

There are quasiconformal homeomorphisms solutions to the Ahlfors-Bers equation. Moreover, these solutions uniquely extends to a homeomorphism on the boundary and there is a unique solution $f^\mu$ fixing 1, $i$ and $-1$. 

Remark

$f^\mu$ is $G$-equivariant if and only if $\mu$ is $G$-invariant.

If $\mu = 0$, then $\partial \bar{z} f^\mu = 0$ and by the Weil Lemma, $f^\mu$ is holomorphic.

There are at most $84(g-1)G$-equivariant biholomorphisms of the disk; i.e. $|\text{Aut}(\Sigma_g)| \leq 84(g-1)$.
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- There are at most $84(g - 1)$ $G$-equivariant biholomorphisms of the disk; i.e. $|\text{Aut}(\Sigma_g)| \leq 84(g - 1)$. 
In particular, abusing of notation, for every $\mu \in L_\infty(\Sigma)_1$ we have a quasiconformal deformation $f^\mu : \Sigma \to \Sigma$. 
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$$\mathcal{A} = \{(U, \varphi_U)\} \rightsquigarrow \mathcal{A}_\mu = \{(U, f \circ \varphi)\}$$
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$$A = \{(U, \varphi_U)\} \leadsto A_\mu = \{(U, f \circ \varphi)\}$$

We define $\Sigma_\mu$ as the surface $\Sigma$ with the deformed atlas:

$$\Sigma_\mu := (\Sigma, A_\mu)$$
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$$J^\mu := df^\mu \circ J \circ d(f^\mu)^{-1}$$
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We say that $\Sigma_\mu \sim \Sigma_{\mu'}$ if there is a homeomorphism $h$ such that the following diagram commutes:

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\begin{array}{ccc}
\Sigma & \xrightarrow{f_\mu} & \Sigma \\
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This relation gives the coarse moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g$. 
To produce a fine moduli we strength the relation: We say that $\Sigma_\mu \overset{T}{\sim} \Sigma_\mu'$ if there is a homeomorphism $h$ isotopic to the identity such that the following diagram commutes:

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![Diagram](image)

**Proposition**

$\Sigma_\mu \overset{T}{\sim} \Sigma_{\mu'}$ if and only if $f^\mu|_{\partial \Delta} = f^{\mu'}|_{\partial \Delta}$. 
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**Theorem**

$T(\Sigma_g)$ is a complex domain of complex dimension $3g - 3$. 
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We have the following model for the \textit{Teichmüller space}:

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\begin{theorem}
\(T(\Sigma_g)\) is a complex domain of complex dimension \(3g - 3\).
\end{theorem}

\[
\mathcal{M}(\Sigma) = T(\Sigma)/\text{MCG}(\Sigma)
\]

\[
\text{MCG}(\Sigma) := \text{Homeo}(\Sigma)/\text{Homeo}_0(\Sigma)
\]
We define the *universal Teichmüller space* as follows:

\[ T(1) := L_\infty(\Delta)/\sim \]
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$$T(1) := L_\infty(\Delta) / \sim$$

It is Universal in the sense that it contains all the finite dimensional Teichmüller spaces:

$$T(\Sigma) \subset T(1)$$
Universal Hyperbolic Lamination

Consider the inverse system of finite index subgroups of $G = \pi_1(\Sigma)$ and inclusions.
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For every finite index subgroup $G'$, consider the finite disk pile $(G' \setminus G) \times \Delta$ and its diagonal action:

$$g \cdot (f, x) := (f \cdot g, \alpha(g)(x))$$
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The quotient by this action is the Riemann surface $\Sigma_{G'}$:

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\]

The diagonal action is equivariant respect to the Fuchsian representation \( \alpha \) hence we have a finite holomorphic covering:

\[
G' \backslash G \longrightarrow \Sigma_{G'}
\]

\[
\downarrow
\]

\[
\Sigma
\]
Because the construction is functorial, we actually have an inverse system of finite holomorphic coverings of $\Sigma$: 

$$
\Sigma = \ldots
$$
Consider the profinite completion group $G_\infty$ of $G$:

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$$G'/G \quad [G':G]<\infty$$
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Because the Fuchsian group $G$ is residually finite, the completion is a group extension and we have a dense immersion:

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The collection of finite index subgroups of $G$ is a neighborhood system of the identity and by translation it defines a topology on $G$ whose completion is the group $G_\infty$ just defined. As a topological space, the group $G_\infty$ is a compact totally disconnected Hausdorff space; i.e. a Cantor set.
The group $G_{\infty}$ is an ultrametric space:
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Because the group $G$ is finitely generated, there must be a finite amount of subgroups of a given index hence the normal subgroups $A_n$ are of finite index as well. Define the following valuation $\text{val} : G \to \mathbb{N} \cup \{\infty\}$ such that:

$$\text{val}(g) := \max\{n \in \mathbb{N} \mid g \in A_n\}$$

if $g$ is not the neutral element $e$ and $\text{val}(e) := \infty$. Define the translation invariant metric $d$ on the group $G$ such that:

$$d(g, h) := e^{-\text{val}(g^{-1}h)}$$
The group $G_\infty$ is an ultrametric space:
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Geometrically, the disoriented Cayley graph of $G = \pi_1(\Sigma)$ is the barycentric subdivision of a tessellation. In particular, $G_\infty$ is the Cantor in the ideal boundary.
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For example, consider the cusped torus. Its fundamental group is the free product $\mathbb{Z} \ast \mathbb{Z}$ and its Cayley graph is the following:
The inverse limit of the covering tower \((\Sigma_{G'})\) defined before is the \textit{Universal Hyperbolic Lamination}: \[
\Sigma_\infty := \lim_{\overset{\rightarrow}{G'} < G} \Sigma_{G'}
\]

By functoriality of the construction, we have:

\[
\Sigma_\infty := \Sigma_{G_\infty} \times \Delta / G
\]

This is a lamination whose leaves are densely immersed disks \(\Delta\). The leaf space is \(\Sigma_{G_\infty} / G\).
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The following is a picture of the lamination:
Define the baseleaf \( \iota : \Delta \to \Sigma_\infty \) as the composite map:

\[
\Delta \hookrightarrow G_\infty \times \Delta \to \Sigma_\infty
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such that the first map is \( x \mapsto (e, x) \) where \( e \) is the neutral element of \( G_\infty \).
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Its basic open sets are:

\[
V(U, G') := \bigcup_{g \in G'} \alpha(g)(U)
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where \( U \) is an open set of the disk and \( G' \) is a finite index subgroup of \( G \).
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where $U$ is an open set of the disk and $G'$ is a finite index subgroup of $G$.

The map $\iota : \Delta_{Emb} \hookrightarrow \Sigma_\infty$ is an embedding.
Because $\Delta \xrightarrow{id} \Delta_{Emb}$ is continuous, we have:

$$C(\Delta_{Emb}, \mathbb{C}) \xrightarrow{\mathbb{C}} C(\Delta, \mathbb{C})$$
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$$limit - periodic := C(\Delta_{Emb}, \mathbb{C}) \hookrightarrow C(\Delta, \mathbb{C})$$
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$$\text{limit – periodic} := C(\Delta_{Emb}, \mathbb{C}) \hookrightarrow C(\Delta, \mathbb{C})$$

**Proposition**

Consider a function $f : \Delta \rightarrow \mathbb{C}$. Then,

- $f$ is limit-periodic iff it is the uniform limit of periodic functions.
- $f$ is limit-periodic iff there is a continuous function $g : \Sigma_{\infty} \rightarrow \mathbb{C}$ such that $\iota^*g = f$.  

We have the following chain of proper inclusions:

\[ T(\Sigma) \subset T(\Sigma_{G'}) \subset \ldots \bigcup_{[G' : G] < \infty} T(\Sigma_{G'}) \subset T(1) \]
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The following definition is due to D.Sullivan:

**Definition**

\[ T(\Sigma_{\infty}) := \bigcup_{G' < G \atop [G':G] < \infty} T(\Sigma_{G'}) \]
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Continuous Beltrami differentials respect to $\Delta_{Emb}$, by $L_\infty(\Delta_{Emb})_1$. 
Denote the space of limitperiodic Beltrami differentials; i.e. Continuous Beltrami differentials respect to $\Delta_{Emb}$, by $L_\infty(\Delta_{Emb})_1$. The following is a model for the Sullivan’s Teichmüller space:

**Proposition**

$$T(\Sigma_\infty) = L_\infty(\Delta_{Emb})/ \sim \subset T(1)$$
What metric should we put in $T(\Sigma_\infty)$?...
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Problems:
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**Problems:**

1. The universal Weil-Petersson metric $g_{WP}$ doesn’t work. It is only defined for differentials $\nu$ such that:

   $$d_0 f^\mu |_{\partial \Delta(\nu)} \in C^{3/2+\epsilon}$$

2. If we consider nets of periodic differentials converging uniformly to the limit-periodic differentials respectively then:

   $$\lim_{\leftarrow} G' \langle G' : G \rangle < \infty$$

   $$WP(\mu_G', \nu_G') = 0 \text{ or } \infty$$

   where $WP$ is the usual Weil-Petersson metric.
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We define the *renormalized Weil-Petersson metric*:

**Definition**

*Consider nets of periodic differentials converging uniformly to the limitperiodic differentials respectively then:*

\[
(\mu, \nu)_{WP} = \lim_{\substack{G' < G \\ [G' : G] < \infty}} \frac{1}{[G' : G]} WP(\mu_{G'}, \nu_{G'})
\]

*where* \( WP \) *is the usual Weil-Petersson metric.*
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*Consider nets of periodic differentials converging uniformly to the limit-periodic differentials respectively then:*

\[
(\mu, \nu)_{WP} = \lim_{\substack{G' < G \\ [G' : G] < \infty}} \frac{1}{[G' : G]} \, WP(\mu_{G'}, \nu_{G'})
\]

*where* \( WP \) *is the usual Weil-Petersson metric.*

**Proposition**

*The renormalized Weil-Petersson metric is well defined; i.e. It converges in the space of limit-periodic differentials and is independent of the choice of the nets.*
Remark

The renormalized Weil-Petersson metric is an extension of the usual one for $G$-periodic differentials.
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Actually, a physicist would think on this result as follows: The inverse limit of coverings is the renormalization group of the theory and the number of sheets of the covering is the renormalization factor of the respective energy level. Then, to get the measured observables on the respective energy level we have to quotient by the renormalization factor; i.e. by the index $[G', G]$. The limit gives the observable at fundamental scale.
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The complex analytic Kähler coadjoint orbit:

\[
\text{Diff}^+(S^1)/\text{Möb} \hookrightarrow T(1)
\]

is transversal to the Teichmüller space of the lamination in the universal one:

\[
\text{Diff}^+(S^1)/\text{Möb} \cap T(\Sigma_\infty)
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This can be seen as Kähler coordinates of the Teichmüller space of the lamination, labelled by an ultrametric space. The previous result is functorial; i.e. The following diagram commutes:

\[
\begin{align*}
C \left( G_\infty, T(\Sigma) \right) \xrightarrow{\hat{f}} \sim & \quad T(\Sigma_\infty) \\
T(\Sigma)^n \xrightarrow{\sim} & \quad C \left( G' \setminus G, T(\Sigma) \right) \xrightarrow{\sim} \quad T(\Sigma_{G'})
\end{align*}
\]
The following is Theorem B:

**Theorem**

The \((g - 1)\)-times alternating product of the moduli space of genus two compact Riemann surfaces is a discrete fiber complex analytic Kähler covering of the moduli space of genus \(g\) compact Riemann surfaces:

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\text{Alt}^{g-1}(M_2) \twoheadrightarrow M_g
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For example, we have a covering:

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Thank you very much!!!