p-adic *N*-point open strings amplitudes

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Contents

String theory is the theory of tiny one-dimensional extended objects propagating in an underlying Riemannian or pseudo-Riemannian space-time manifold M. The string theory was motivated by need to understanding certain aspects of the strong interactions of elementary particles. This interactions are described by scattering amplitudes, which satisfy some general physical requirements.

In 1068, G. Veneziano proposed the following function for describing the interaction of four particles:

$$\widehat{A}_{\mathbb{R}}^{(4)}(s,t) = \int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}dx,$$

where $\alpha(s) = 1 + \frac{1}{2}s, s = -(\mathbf{k}_1 + \mathbf{k}_2)^2, t = -(\mathbf{k}_2 + \mathbf{k}_3)^2$

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where $\alpha(s) = 1 + \frac{1}{2}s, s = -(\mathbf{k}_1 + \mathbf{k}_2)^2, t = -(\mathbf{k}_2 + \mathbf{k}_3)^2$ This amplitudes can be generalized to the case of scattering N particles with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_N$ in the form

$$A_{\mathbb{R}}^{(N)}(\mathbf{K}) := \int_{\mathbb{R}^{N-3}} \prod_{i=2}^{N-2} |x_j|^{\mathbf{k}_1\mathbf{k}_j}|1-x_j|^{\mathbf{k}_{N-1}\mathbf{k}_j} \prod_{2\leq i<j\leq N-2} |x_i - x_j|^{\mathbf{k}_i\mathbf{k}_j} \prod_{i=2}^{N-2} dx_i.$$

The p- adic string theory started around 1987 with the work of Volovich, Freud and Witten, and Frampton and Okada. Volovich noted that the integral expresion for the Veneziano amplitude of the open bosonic string can be generalized to a $p-$ adic integral and to an adelic integral. The $p-$ adic string amplitudes for N particles have the form

$$A_p^{(N)}(\mathbf{K}) := \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_j|_p^{\mathbf{k}_1 \mathbf{k}_j} |1-x_j|_p^{\mathbf{k}_{N-1}\mathbf{k}_j} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_p^{\mathbf{k}_i \mathbf{k}_j} \prod_{i=2}^{N-2} dx_i$$

where $\mathbf{K} = (\mathbf{k}_1, \ldots \mathbf{k}_N)$, $\mathbf{k}_i = (k_{0,i}, \ldots, k_{l,i}) \in \mathbb{R}^{l+1}$, for $i = 1, \ldots, N$ $(N \geq 4)$

This talk is based on the paper "A. R. Fuquen-Tibatá, H. García-Compeán, W. A. Zúñiga-Galindo, Euclidean Quantum Field Formulation of p-Adic Open String Amplitudes. Nucl. Phys. B, 975 (2022)" we study in a rigorous mathematical way p-adic quantum field theories whose N-point amplitudes are the expectation of products of vertex operators. We show that this type of amplitudes admit a series expansion. The lowest term in this series is a regularized version of the p-adic open Koba-Nielsen string amplitude.

The field of p-adic numbers

Given x a nonzero rational number, we can represent x as $p^{\gamma}a/b$ where a and b are integers coprime with p. The p-adic norm in $\mathbb{Q}$ is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ p^{-\gamma} & \text{if} \quad x \neq 0. \end{cases}$$

The completion of the field $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ is called the field of p-adic numbers $\mathbb{Q}_p$.

The *p*-adic norm is extended to $\mathbb{Q}_p^N$ by taking

$$\|\mathbf{x}\|_p = \max_{1\leq i\leq N} |x_i|_p\,, \text{ for } \mathbf{x} = (x_1, \dots, x_N) \in \mathbb{Q}_p^N.$$

With this norm $\mathbb{Q}_p^N$ becomes an ultrametric space.

For $r \in \mathbb{Z}$, we denote by $B_r^N(\mathbf{a}) = \left\{\mathbf{x} \in \mathbb{Q}_p^N; ||\mathbf{x} - \mathbf{a}||_p \leq p^r\right\}$ *the ball of radius* p^r *with center at* $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N$.
We also denote by $S_r^N(\mathbf{a}) = \left\{\mathbf{x} \in \mathbb{Q}_p^N; ||\mathbf{x} - \mathbf{a}||_p = p^r\right\}$ *the sphere of radius* p^r *with center at* $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N$.
As a consequence of the ultrametricity, balls and spheres are both open and closed subsets in $\mathbb{Q}_p^N$ and the topological space $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is totally disconnected.

We will use $\Omega(t)$ to denote the characteristic function of the interval $[0, 1]$.
We also denote

$$\Delta_k(x) = \Omega\left(p^{-k}\|x\|_p\right), \; k \in \mathbb{Z}$$

and

$$\delta_k(x) = p^{Nk}\Omega\left(p^k\|x\|_p\right), \; k \in \mathbb{Z}.$$

The Bruhat-Schwartz space

A complex-valuated function $\phi : \mathbb{Q}_p^N \to \mathbb{C}$ is called *locally constant*, if for any $x \in \mathbb{Q}_p^N$ there exists $l(x) \in \mathbb{Z}$ such that:

$$\phi(y) = \phi(x); \quad y \in B^N_{l(x)}(x).$$

A *Bruhat-Schwartz function* or *test function* $\phi : \mathbb{Q}_p^N \to \mathbb{C}$ is a locally constant function with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^N) := \mathcal{D}$ ($\mathcal{D}_{\mathbb{R}}$ the $\mathbb{R}$-vector space).

Distributions

The $\mathbb{C}$-vector space $\mathcal{D}'\left(\mathbb{Q}_p^n\right) := \mathcal{D}'$ of all continuous linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$ is called the *Bruhat-Schwartz space of distributions*. We endow $\mathcal{D}'$ with the weak topology. The map

$$\begin{array}{ccc} \mathcal{D}' \times \mathcal{D} & \to & \mathbb{C} \\ (T, \varphi) & \to & T(\varphi) \end{array}$$

is a bilinear form which is continuous in T and φ separately.

L^ρ spaces

Given $\rho \in [1, \infty)$, we denote by $L^\rho := L^\rho\left(\mathbb{Q}_p^N\right) := L^\rho\left(\mathbb{Q}_p^N, d^N x\right)$, the $\mathbb{C}-$vector space of all the complex-valued and Borel measurable functions g satisfying

$$\int_{\mathbb{Q}_p^N} |g(x)|^\rho \, d^N x < \infty,$$

where $d^N x$ is the normalized Haar measure on $\left(\mathbb{Q}_p^N, +\right)$, the corresponding $\mathbb{R}$-vector spaces are denoted as $L^\rho_{\mathbb{R}}$. For $g \in L^\rho$ we define

$$\|g\|_\rho = \left\{\int_{\mathbb{Q}_p^N} |g(x)|^\rho \, d^N x\right\}^{\frac{1}{\rho}}.$$

The Fourier transform

The Fourier transform of $\varphi \in L^1(\mathbb{Q}_p^N)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(\xi \cdot x)\varphi(x) d^N x \quad \text{for } \xi \in \mathbb{Q}_p^N$$

where $d^N x$ is the normalized Haar measure on $\mathbb{Q}_p^N$.

The Vladimirov operator

The Vladimirov operator $\boldsymbol{D}:\mathcal{D}\left(\mathbb{Q}_p\right)\rightarrow L^2\left(\mathbb{Q}_p\right)$ is defined as

$$\boldsymbol{D}\theta\left(x\right)=\mathcal{F}^{-1}_{\xi\rightarrow x}\left[\left|\xi\right|_p\mathcal{F}_{x\rightarrow\xi}\theta\right]$$
$$=\frac{p^2}{p+1}\int_{\mathbb{Q}_p}\frac{\theta\left(x\right)-\theta\left(y\right)}{\left|x-y\right|_p^2}dy.$$

The Vladimirov operator can be rewritten as

$$\boldsymbol{D}\theta\left(x\right)=f_{-1}*\theta\left(x\right)=-\frac{p^{2}}{p+1}\left|x\right|_{p}^{-2}*\theta\left(x\right),\text{ for }\theta\in\mathcal{D}\left(\mathbb{Q}_{p}\right),$$

where

$$f_{-1}\left(x\right)=\frac{\left|x\right|_{p}^{-2}}{\Gamma_{p}\left(-1\right)}\text{ for }x\in\mathbb{Q}_{p},\tag{1}$$

is called the *Riesz Kernel* and $\Gamma_{p}\left(\alpha\right):=\dfrac{1-p^{\alpha-1}}{1-p^{-\alpha}}$.

Lizorkin spaces of second kind

Consider the space

$$\mathcal{L} := \mathcal{L}\left(\mathbb{Q}_p^N\right) = \left\{\varphi \in \mathcal{D}\left(\mathbb{Q}_p^N\right) ; \int_{\mathbb{Q}_p^N} \varphi(x)\, d^N x = 0\right\}$$

this space is called the p-adic Lizorkin space of second kind. $\left(\mathcal{L}_{\mathbb{R}} = \mathcal{L}\left(\mathbb{Q}_p^N\right) \cap \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)\right)$.

We denote $\mathcal{L}' = \mathcal{L}'\left(\mathbb{Q}_p^N\right)$ the topological dual of the space $\mathcal{L}\left(\mathbb{Q}_p^N\right)$, this space is called the p-adic *Lizorkin space of distributions of the second kind*.

The operator **D** is invertible in $\mathcal{L}$ and its inverse is

$$\begin{array}{rccc} \boldsymbol{D}^{-1}: & \mathcal{L}\left(\mathbb{Q}_p\right) & \rightarrow & \mathcal{L}\left(\mathbb{Q}_p\right) \\ & \theta & \rightarrow & \boldsymbol{D}^{-1}\theta, \end{array}$$

where $\boldsymbol{D}^{-1}\theta\left(x\right) = \mathcal{F}^{-1}_{\xi\rightarrow x}\left[\left|\xi\right|_p^{-1}\mathcal{F}_{x\rightarrow\xi}\theta\right]$. Since $\left(\mathcal{F}_{x\rightarrow\xi}\theta\right)\left(0\right) = 0$, we have $\boldsymbol{D}^{-1}\theta\left(x\right) \in \mathcal{L}\left(\mathbb{Q}_p\right)$.

We set $\boldsymbol{K} := (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$, where $\boldsymbol{k}_j = (k_{0,j}, \ldots, k_{D-1,j}) \in \mathbb{R}^D$ is the momentum of a tachyon, $j = 1, \ldots, N$. The dimension $D \geq 1$ is fixed along this work. We also set

$$\boldsymbol{\varphi}(\cdot) = (\varphi_0(\cdot), \ldots, \varphi_{D-1}(\cdot)) \in (\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p))^D .$$

For $\boldsymbol{a} = (a_0, a_1, \ldots, a_{D-1})$, $\boldsymbol{b} = (b_0, \ldots, b_{D-1}) \in \mathbb{R}^D$, $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the standard scalar product in $\mathbb{R}^D$.

The naive Euclidean version of the p-adic N-point amplitudes is

$$\mathcal{A}^{(N)}(\boldsymbol{K}) = \frac{1}{Z_0^{phys}} \int D\varphi e^{-S(\varphi)} \int_{\mathbb{Q}_p^N} d^N x \; e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \varphi(x_j)}, \qquad (2)$$

where $d^N x = \prod_{j=1}^{N} dx_j$, $S(\varphi) = \frac{T_0}{2} \sum_{j=0}^{D-1} S_j(\varphi_j)$, with

$$S_j(\varphi_j) = \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \left\{ \frac{\varphi_j(x_j) - \varphi_j(y_j)}{|x_j - y_j|_p} \right\}^2 dx_j dy_j,$$

and

$$Z_0^{phys} = \int D\varphi e^{-S(\varphi)}.$$

For $\varphi_j \in \mathcal{D}(\mathbb{Q}_p)$,

$$S_j(\varphi_j) = 2\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{\varphi_j(x_j)(\varphi_j(x_j) - \varphi_j(y_j))}{|x_j - y_j|_p^2} dy_j dx_j$$
$$= 2\frac{(p+1)}{p^2}\int_{\mathbb{Q}_p} \varphi_j(x_j)\, \boldsymbol{D}\varphi_j(x_j)\, dx_j.$$

Then

$$S(\varphi) = \frac{T_0(p+1)}{p^2}\sum_{j=0}^{D-1}\int_{\mathbb{Q}_p} \varphi_j(x_j)\, \boldsymbol{D}\varphi_j(x_j)\, dx_j.$$

Gaussian processes and free quantum fields

We define the bilinear form $\mathbb{B}$

$$\begin{array}{rccc} \mathbb{B}: & \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)\times\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p) & \to & \mathbb{R} \\ & (\varphi,\theta) & \to & \langle\varphi,\boldsymbol{D}^{-1}\theta\rangle \end{array}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{Q}_p)$.

Lemma 1

$\mathbb{B}$ is a positive, continuous bilinear form from $\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)\times\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ into $\mathbb{R}$.

$\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ is a nuclear space that is dense and continuously embedded in $L^2_{\mathbb{R}}(\mathbb{Q}_p)$. Then we have the following Gel'fand triple:

$$\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p) \hookrightarrow L^2_{\mathbb{R}}(\mathbb{Q}_p) \hookrightarrow \mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p).$$

We denote by $\mathcal{B} := \mathcal{B}(\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p))$ the $\sigma-$algebra generated by the cylinder subsets of $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$.

Consider the mapping

$$\begin{array}{rccc} \mathcal{C}: & \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) & \rightarrow & \mathbb{R} \\ & f & \rightarrow & e^{\frac{-1}{2}\mathbb{B}(f,f)}. \end{array}$$

$\mathcal{C}$ defines a characteristic functional in $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$. By Bochner-Minlos theorem, there exists a unique probability measure $\mathbb{P}$ on $\left(\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right), \mathcal{B}\right)$ given by its characteristic functional as

$$\int_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sqrt{-1}(W,f)} d\mathbb{P}\left(W\right) = e^{-\frac{1}{2}\mathbb{B}(f,f)}, \quad f \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right), \tag{3}$$

We denote by

$$\bigotimes_{j=0}^{D-1} \mathbb{P}\left(\varphi_j\right) = \mathbb{P}_D\left(\boldsymbol{\varphi}\right),$$

the product probability measure on the product $\sigma-$algebra $\mathcal{B}^D$. We set

$$\mathcal{L}_{\mathbb{R}}^D\left(\mathbb{Q}_p\right) = \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) \times \cdots \times \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right), \text{ } D\text{-times.}$$

Intuitively, the N-point amplitudes are the expectation values of the products of the vertex operators with respect to the measure $\mathbb{P}_D$.

$$\left\langle \prod_{j=1}^{N} \int_{\mathbb{Q}_p} dx_j e^{\boldsymbol{k}_j \cdot \varphi(x_j)} \right\rangle_{\mathbb{P}_D} = \frac{1}{Z_0} \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} \int_{\mathbb{Q}_p^N} d^N x e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \varphi(x_j)} d\mathbb{P}_D(\varphi). \tag{4}$$

We set

$$\mathcal{A}_R^{(N)}(\boldsymbol{K}) = \frac{1}{Z_0^{phys}} \int D\varphi e^{-S(\varphi)} \int_{B_R^N} d^N x e^{\sum_{j=1}^N \boldsymbol{k}_j \cdot \varphi(x_j)},$$

where R is a positive integer and $B_R^N = \left\{ x \in \mathbb{Q}_p^N ; \|x\|_p \leq p^R \right\}$.

Definition 2

For a positive integer R, we define the p-adic N-point amplitudes as $\mathcal{A}^{(N)}(\boldsymbol{K}) = \lim_{R\to+\infty} \mathcal{A}_R^{(N)}(\boldsymbol{K})$, where

$$\mathcal{A}_R^{(N)}(\boldsymbol{K}) := \frac{1}{Z_0} \int_{B_R^N} \left\{ \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \varphi(x_j)} d\mathbb{P}_D(\varphi) \right\} \prod_{j=1}^{N} dx_j .$$

Corollary 3

For R fixed, $\mathcal{A}_R^{(N)}(\boldsymbol{K}) < \infty$ *for any* $\boldsymbol{K}$. *Furthermore,*

$$\mathcal{A}_R^{(N)}(\boldsymbol{K}) = \frac{1}{Z_0}\int\limits_{B_R^N}\left\{\int\limits_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=1}^N \boldsymbol{k}_j\cdot\varphi(x_j)}\,d\mathbb{P}_D(\varphi)\right\}\prod_{j=1}^N dx_j$$

$$= \frac{1}{Z_0}\int\limits_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)}\left\{\int\limits_{B_R^N} e^{\sum_{j=1}^N \boldsymbol{k}_j\cdot\varphi(x_j)}\prod_{j=1}^N dx_j\right\}d\mathbb{P}_D(\varphi).$$

Proposition 1

The amplitude $\mathcal{A}_R^{(N)}(\boldsymbol{K})$ *satisfies*

$$\mathcal{A}_R^{(N)}(\boldsymbol{K}) =$$

$$\frac{1}{Z_0}\int\limits_{B_R^N}\prod_{j<i}^{N}|x_j - x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N}\boldsymbol{k}_j\cdot\widetilde{\varphi}(x_j)}\,d\widetilde{\mathbb{P}}_D(\widetilde{\varphi})\prod_{j=0}^{N}dx_j.$$

We now introduce the 'convention' that the insertion points x_1, $x_2, \ldots, x_{N-1}$, x_N, with $N \geq 4$, belong to the p-adic projective line, and then by using the Möbius group, we may take the normalization

$$x_1 = 0, x_{N-1} = 1, x_N = \infty.$$

In our framework, the convention $x_N = \infty$ means that the N-point amplitudes do not depend on x_N.

$$\mathcal{A}_R^{(N)}(\boldsymbol{K}) = \frac{C_0}{Z_0} \int_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$

$$\times \prod_{2\leq i,j\leq N-2} |x_j - x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=2}^{N-2}\boldsymbol{k}_j\cdot\widetilde{\varphi}(x_j)} d\widetilde{\mathbb{P}}_D(\widetilde{\varphi}) \prod_{j=2}^{N-2} dx_j.$$

Theorem 4

$$\mathcal{A}_R^{(N)}(\boldsymbol{K}) = \frac{CC_0}{Z_0} \int\limits_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} \, |1-x_i|_p^{2\frac{(p-1)}{\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$

$$\times \prod_{2\leq i,j\leq N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \prod_{j=2}^{N-2} dx_j$$

$$+ \frac{C_0}{Z_0}\sum_{r=1}^{\infty} \int\limits_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_j} \, |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$

$$\times \prod_{2\leq i,j\leq N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} G_r(\boldsymbol{k},\boldsymbol{x}) \prod_{j=2}^{N-2} dx_j.$$

we introduce the notation:

$$A_R^{(N)}(\boldsymbol{K}) = \frac{CC_0}{Z_0} \int_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$

$$\times \prod_{2\leq i,j\leq N-2} |x_j - x_i|_p^{2\frac{(p-1)}{\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \prod_{j=2}^{N-2} dx_j,$$

$$Z_{G_r,R}^{(N)}(\boldsymbol{K}) = \frac{C_0}{Z_0} \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$

$$\times \prod_{2\leq i,j\leq N-2} |x_j - x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} 1_{B_R^{N-3}}(\boldsymbol{x}) G_r(\boldsymbol{k},\boldsymbol{x}) \prod_{j=2}^{N-2} dx_j.$$

$$\frac{Z_0}{CC_0}\lim_{R\to\infty} A_R^{(N)}(\boldsymbol{K}) = \lim_{R\to\infty}\left(Z_R^{(N)}(\boldsymbol{s})\mid_{s_{ij}=2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\right) = Z^{(N)}(\boldsymbol{s})\mid_{s_{ij}=2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}.$$

By using the fact that $Z^{(N)}(\boldsymbol{s})$ is a holomorphic function in a certain domain of $\mathbb{C}^{D_0}$, we conclude that $\lim_{R\to\infty} A_R^{(N)}(\boldsymbol{K})$ exists for $\boldsymbol{K}$ belonging a non-empty subset of $\mathbb{C}^{D_0}$.

We may assume that $1_{B_R^{N-3}}(\boldsymbol{x})G_r(\boldsymbol{K},\boldsymbol{x}) = \phi$ is a test function in $\boldsymbol{x}$, and then $Z^{(N)}_{G_r,R}(\boldsymbol{K}) = \frac{C_0}{Z_0} Z^{(N)}_{\phi}(\boldsymbol{s}) \mid_{s_{ij}=2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}$ is a multivariate local zeta function.

Furthermore, $\left|Z^{(N)}_{G_r,R}(\boldsymbol{K})\right| \leq \frac{C_0 \ C_r(\boldsymbol{K},R)}{Z_0} Z^{(N)}(\boldsymbol{k})$, where

$$C_r(\boldsymbol{K},R) = \sup_{\boldsymbol{x}\in B_R^{N-3}} |G_r(\boldsymbol{K},\boldsymbol{x})|.$$

Since $Z^{(N)}(\boldsymbol{K})$ converges in a non-empty open set, we conclude that all the $Z^{(N)}_{G_r,R}(\boldsymbol{K})$s converges in the open set where $Z^{(N)}(\boldsymbol{K})$ converges.

Theorem 5

The amplitudes $\mathcal{A}_R^{(N)}(\boldsymbol{K})$ satisfy the following. For R fixed,

$$\mathcal{A}_R^{(N)}(\boldsymbol{K}) = A_R^{(N)}(\boldsymbol{K}) + \sum_{r=1}^{\infty} Z_{G_r,R}^{(N)}(\boldsymbol{K}),$$

where $A_R^{(N)}(\boldsymbol{K})$, and all the $Z_{G_r,R}^{(N)}(\boldsymbol{K})$'s are multivariate Igusa's local zeta functions, all of them converging in a common non-empty open set. Furthermore,

$$\lim_{R\to\infty} A_R^{(N)}(\boldsymbol{K}) = \frac{CC_0}{Z_0} Z^{(N)}(\boldsymbol{K}),$$

which is the p-adic Koba-Nielsen open string amplitude.

Thank You.

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