"Beltrami Equation Solver"
a numerical software for the Beltrami equation

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Contents

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Introduction

This program implements the algorithm presented in [1] for solving the Beltrami equation

$$\frac{\partial f(z)/\partial \overline{z}}{\partial f(z)/\partial z} = \mu(z) \tag{1}$$

in the complex plane numerically.

Here μ is a measurable function defined in the unit disk $\{|z| < 1\}$, or possibly in the left half-plane or the whole z-plane. The solution f is a function of the complex variable z, known as a quasiconformal or a μ-conformal mapping. It is normalized by fixing the three values 0, 1, and the point at infinity. For full details about quasiconformal mappings, see [2].

For installation, see Section 8. To open the program: click on the corresponding icon, or type `beltrami` in an appropriate terminal for your operating system.

If you find any interesting application for this program, or any bugs, I would be very happy to know about it.

1 Discretization of the Beltrami equation

The problem is discretized by triangulating the disk $\{|z| \leq 1\}$ as follows. Fix two integers M (=number of rings) and N (=number of sectors). The rings are bounded by $M + 1$ circles centered at the origin. (See Figure 1.)

The triangles subdividing these rings are defined through their vertices. Consider the $(M + 1)N$ points in the left half-plane $\{\operatorname{Re} Z \leq 0\}$ given by

$$Z_{jk} = \frac{\sqrt{3}\,\pi}{N} j + \begin{cases} \dfrac{2\pi k}{N} i, & j \text{ even}, \\ \dfrac{2\pi(k + 1/2)}{N} i, & j \text{ odd}, \end{cases} \tag{2}$$

where the integer indices j, k range in the intervals $-M \leq j \leq 0$ and $0 \leq k \leq N-1$. (Here i is the square root of -1.) The points $Z_{j,k}$ are the vertices of equilateral triangles obtained by joining each point to the six closest other points (or fewer than 6 near the boundary of this configuration).

The corresponding triangulation $\{z_{jk}\}$ of the unit disk has the vertices

$$z_{jk} = \exp(Z_{jk}). \tag{3}$$

When M and N are large, these triangles are approximately equilateral because they are small, and the exponential is conformal locally. In Figure 1, the triangles

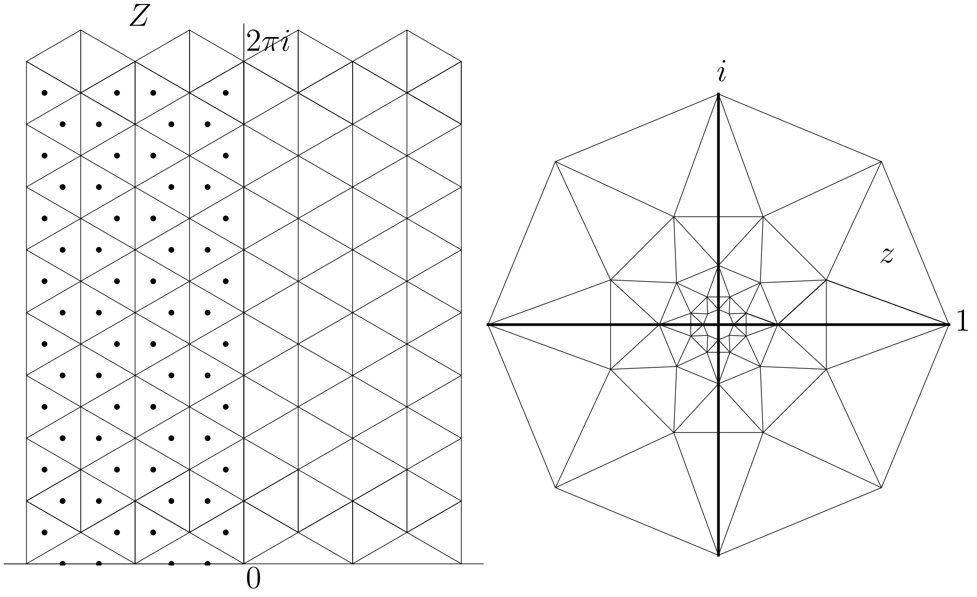

Figure 1: Triangulation in half-plane and disk models

in the right halfplane ($j > 0$) are also shown. They correspond to the exterior of the unit disk, and for this reason in this document their vertices are also referred to informally as “exterior” points.

2 Solution procedure—overview

The user specifies the value of $\mu(z)$ corresponding to each of the triangles in the discretized problem. Beltrami Equation Solver converts these values, following the rules of the pullback of a Beltrami coefficient, to values $\nu(Z)$ corresponding to the triangles in the left half-plane. Then the program solves the Beltrami equation for a ν-conformal mapping $W = F(Z)$, and finally obtains the solution $w = f(z)$ by applying the exponential

$$w_{jk} = \exp(W_{jk}). \tag{4}$$

In normal use, the values of $\nu(Z)$ for $\operatorname{Re} Z < 0$ are reflected in the imaginary axis to define values for $\operatorname{Re} Z > 0$ in such a way that the solution in the whole W-plane will be symmetrical in the imaginary axis, and as a consequence the solution in the whole w-plane will be symmetrical in the unit circle $\{|z| = 1\}$.

If desired, the user may choose instead to have the Beltrami coefficient extended to be equal to zero in the right half-plane, so the resulting mapping will be conformal in the exterior $\{|z| > 1\}$.

A third possibility is for the user to define arbitrary values in the exterior domain as well (i.e., defining the Beltrami coefficient in the whole plane).

It is also possible for the user to define $\nu(Z)$ instead of $\mu(z)$, thus posing the Beltrami equation in the half-plane instead of the disk.

The normalization $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$ is achieved by normalizing $W = F(Z)$ to fix $-\infty + 0i$, 0, and $+\infty + 0i$, as well as imposing the periodicity condition $F(Z + 2\pi i) = F(Z) + 2\pi i$.

3 File structures for input and output

The user informs Beltrami Equation Solver of the data via ordinary text files, and then Beltrami Equation Solver writes text files with the results. The structure of these text files is as follows.

3.1 Beltrami coefficient data files (input)

Theoretically, it is most correct to integrate $\mu(z)$ over each triangle and then divide by its area in order to find the average value. However, for numerical purposes one usually prefers to do something requiring less calculation, for example,

- Average the values of $\mu(z)$ over the three vertices of the triangle; or
- Use the value of $\mu(z)$ at the center of the triangle.

Another possiblity, which we recommend, is

- Use the value of $\mu(\exp(Z))$ where Z is the center of the corresponding triangle in the left half-plane.

For a sufficiently fine mesh of triangles, the choice of strategy will make little difference in the result.

Although it is not required, it is recommended to use a file-naming system which describes the type of data contained. The default system is as follows.

description of mu`_d_int.txt,`   *description of mu*`_d_ext.txt`
description of nu`_h_int.txt,`   *description of nu*`_h_ext.txt`

Here “`d`,”, “`h`,” refer to the disk and the halfplane, while “`int`,”, “`ext`,” refer to $\{|z| < 1\}$ and $\{|z| > 1\}$ in the first case, and $\{\operatorname{Re} Z < 0\}$ and $\{\operatorname{Re} Z > 0\}$ in the second. All four file types have the same internal structure, but most users will only need the first one.

The internal structure of these four types of files, exemplified by the first case ... `_d_int.txt`, is as follows. Recall that there are N concentric rings, each divided into $2M$ triangles.

description of mu`_d_int.txt`

Row 1:

`DI` M N ("`DI`" means literally a "D" followed by an "I")

Rows $2, 3, ..., M+1$:

a_0 b_0 a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 ... ($4M$ floating point values)

The letters `DI` (or `DE`, `HI`, `HE`) indicate disk/halfplane and interior/exterior in a similar way to the filename. M and N are ordinary positive integers (customarily even), and $a_0, \ldots$ must be in floating point notation, where the format style .234e-5 is allowed. The numbers must not be separated by commas (only spaces or tabs).

The rows (one for each ring) must be given according to decreasing radii of the rings (i.e. starting next to the unit circle and approaching inward towards the origin as we advance in the file), with the complex numbers

$$\mu_0 = a_0 + b_0 i, \quad \mu_1 = a_1 + b_1 i, \ldots$$

denoting the values of the Beltrami coefficient as the triangles move counterclockwise around the ring, starting with the one which has its center on the real axis. Note that these triangles alternate pointing outwardly and inwardly.

(For the ... `_h_int.txt` files, the data points correspond exactly via the exponential to the ... `_d_int.txt` files. Thus successive rows move leftwards from the imaginary axis, and in each row of the data file the triangles move upwards from the real axis.)

In rows $2, 4, 6, \ldots$, the first, third, fifth,... points are centers of triangles which are inward pointing, and alternate with the outward pointing triangles. In rows $3, 5, 7, \ldots$, it is the other way around.

Files containing rows with too many or too few numbers will cause an error message. Files with too few rows will cause also cause an error. However, extra rows will be ignored. This facilitates modifying the number N for experimentation purposes, leaving M fixed, without otherwise modifying the file. (It would usually make no sense to modify M by truncating a row.) Extra white space within lines will be ignored. Commas, semicolons, etc., are not allowed. Blank lines are not allowed before the last data line.

A line whose first character is `#` is ignored when the file is read (in such a case the descriptive row numbering mentioned above would be altered accordingly).

For the exterior files `..._d_ext.txt`, `..._h_ext.txt`, the same principle applies, in that the successive rows are farther from the unit circle or the imaginary axis, corresponding to for increasing $|z|$ or $\operatorname{Re} Z$. Thus each data value in an `_ext.txt` file corresponds to the reflection of the point at the same position in the `_int.txt` file.

3.2 Triangulation coefficient data files (output)

After the Beltrami equation is solved, the result may be downloaded to a text file. (However, a possibly more effective method is described below, using subsets.)

The format of the file produced by Beltrami Equation Solver is similar to the input, but the data describes a triangulation instead of a Beltrami coefficient. Therefore now there are $N + 2$ rows of numerical data because there are $N + 1$ circles bordering the rings.

trianglevertices`_d_int.txt`

Row 1:
DI N M

Row 2:
# *Comment containing date-time stamp*

Rows 3,...,N+3:
$\operatorname{Re} w_{-j,0}$ $\operatorname{Im} w_{-j,0}$ $\operatorname{Re} w_{-j,1}$ $\operatorname{Im} w_{-j,1}$ ... ($2M$ floating point values)

Here j is 3 less than the number of the row, i.e., $0 \le j \le M + 1$.

3.3 Subset files (input and output)

A *subset* is simply a list of polygonal curves of arbitrary length. (A polygonal curve is a list of complex numbers.) After solving the Beltrami equation, you may load a file containing the coordinates of the points in a subset, and then Beltrami Equation Solver will find the images of these points (by interpolation within the corresponding triangles.). You can save the image values in another subset file.

A subset file has no header. Apart from optional comments, simply list the real and imaginary parts of the points of each curve one after another, with one line for each curve:

subset`_sub.txt`
$\operatorname{Re} z_1$ $\operatorname{Im} z_1$ $\operatorname{Re} z_2$ $\operatorname{Im} z_2$ ...
...

There is no distinction between subset files for the disk and halfplane models (so be careful).

Note that in the Beltrami coefficient files, the z-value corresponds to the *position* of the data in the file. In contrast, the *data* contained in the subset files refer to the z-value, regardless of the position.

4 All the features of Beltrami Equation Solver

Any user who understands what the Beltrami equation is could probably simply open the program and guess how to use all of the functions. Thus the principal usefulness of this section would be to help decide whether one wants to install and use the program. (See also Section 7.)

The program contains four "screens" or work areas, each accesible by a "tab" figure near the top.

4.1 Beltrami Equation Solver work area

The Beltrami Equation Solver work area (accesible by the leftmost tab) is shown upon startup. First choose the appropiate options at the right:

- Disk: to work with the disk model;
- Halfplane: to work with the halfplane model.

- Symmetric: for loading a Beltrami coefficient for the interior, and having it reflected automatically;
- Conformal: for loading a Beltrami coefficient for the interior, and having it extended automatically as zero;
- General: for loading a Beltrami coefficient for the interior, and then a separate file for the exterior.

Then click on the "Load Beltrami coefficient" button at the left. You will be asked to choose a file, and then the Beltrami equation will be solved automatically and graphed without requiring further action on your part.

If desired, then click on the "Load subset" button. You will be asked to choose a file and the subset will be loaded, and graphed as well as its image. In the case

General, the second (exterior) file must have the same values of M,N as the first (interior) file.

The names of the currently loaded files will be displayed in the upper-right area of the window.

At this point you may change the Disk/Halfplane setting at any time to see immediately the change in the drawing areas on the application of the exponential or logarithm.

4.2 Graphics work area

The Graphics work area allows you to change the view of the z-plane and w-plane drawings. Each side has three sliders,

- x-offset: moves the picture to the left and right
- y-offset: moves the picture up and down
- Scale: makes the picture bigger or smaller (the base-10 logarithm of the scale factor is shown)

Each set of graphics controls is accompanied by a button labeled "0" which resets the sliders to the zero position.

There are three control buttons,

- Triangulation: toggles whether the triangulation is visible or not
- Axes: toggles whether the axes are visible or not
- Subset: toggles whether the subset (if one is loaded) is visible or not

4.3 Technical work area

The Technical work area contains a button labeled "Documentation" which you may press to read a compressed description of the file structures and other essential data.

Above the w-plane drawing area there appears a summary of numerical data relevant to the currently solved Beltrami equation: The values of M and N, the largest value of $|\mu|$ or $|\nu|$, the largest and smallest values of z and w.

When the value of M is small compared to the value of N, i.e., when there are too few rings for a given number of sectors, the triangulation will have a visible "hole" around the origin. Then the mesh will not be fine enough to give precise results within this hole. On the other hand, when the value of M is very large, there will be a very large number of rings of small radius, formed of very small triangles. Possibly the calculation involved in calculating the images of these small triangles will not be useful. An "optimal" situation, when the central "hole" is not too big compared to the size of the triangles of the smallest triangles, is given by

$$M = 4\lceil N \log N/(4\sqrt{3}\pi)\rceil \tag{5}$$

where $\lceil x \rceil$ is the least integer no larger than x. This value is also displayed. However it should be noted that a combination of M,N which makes the "hole" reasonable in the z-plane may not work at all in the w-plane.

In any case, Beltrami Equation Solver uses the average μ-value of the smallest ring as the constant μ-value for the entire "hole" in the z-plane.

4.4 File work area

The File work area contains six buttons for saving the numerical data.

- Save z-triangulation
- Save z-subset
- Save z-graphics
- Save w-triangulation
- Save w-subset
- Save w-graphics

The top four buttons write the data to text files in the format described above. The two bottom buttons write to BMP format graphics files. The 'Save z-subset" button is included only for completeness, since it writes the same data that has been loaded for the current subset.

The "Quit" button, located below the "About" button, closes the program. One may also close the program by closing the window containing it or by pressing the Esc key. However, the "Quit" button will also save the current configuration for the following session, as described in the following paragraph.

4.5 Configuration file.

Beltrami Equation Solver looks for a text file named `.config.beltrami` upon startup, and if it finds it, reads the information. (This is a hidden file on Unix-type systems.) Most of the information it contains concerns the most recent folder and the most recent filename used in the various operations of loading or writing files. Some lines of `.config.beltrami` also control the graphics colors. Upon startup Beltrami Equation Solver also makes an approximate test of CPU speed and can estimate how long a proposed calculation will take. The number of seconds above which a warning message will be displayed can also be controlled by a line in `.config.beltrami`. This configuration file may be edited by hand; some guidelines are given in the file itself. If any problem occurs, simply erase the file or change its name; then a new one with default settings will be created when the "Quit" button is next activated.

5 Caveats

Use of very large values of M,N may cause Beltrami Equation Solver to freeze or shut down. Larger values may even cause your operating system to freeze.

A meaningful Beltrami equation requires $|\mu(z)| < 1$. Even when $\mu(z) \leq \mu_0 < 1$, when the values of $\mu(z_{jk})$ are too large for a given set of values of M and N, the solution may not produce a valid triangulation.

When μ is discontinuous (in practice, this means μ varies greatly between adjacent triangles), the resulting vertices may also not give a valid triangulation. In fact, there is no theoretical justification that Beltrami Equation Solver should give good results when μ is not differentiable, but it seems to work quite well with "nice" discontinuities.

The computing time is basically determined by M,N, and does not depend on $\sup|\mu|$.

Beltrami Equation Solver has not been tested much for M or N odd.

Under very high magnification, the graphics routines may behave badly, producing spurious line segments. However, one can often get a useful view of what is happening near the origin in spite of this software defect.

6 Code for generating coefficient files

As we explained above, Beltrami Equation Solver does not create the files containing the data of Beltrami coefficients. You can calculate them with a slide rule and type the values into a text file if you like. Here we show how to produce such files in MATLAB and *Mathematica*. Programmers in C++, Python and a variety of other languages will have no trouble in adapting these routines.

6.1 MATLAB–Octave

```
## This is the file trianglecenter.m:
function [x,y] = trianglecenter(numrings,numsecs,j,k)
  x = (sqrt(3.)*pi/numsecs)*(j - sign(j)*(2/3-mod(j,2)/3.0) );
  y = k * pi / numsecs;
endfunction


## This is the file writebcfdisk.m:
function [] = writebcfdisk(filename,mu,numrings,numsecs,comment)
  fil = fopen(filename,'w');
  fprintf(fil, "DI %d %d \n", numrings,numsecs);
  fprintf(fil, "# %s\n",comment);
  ## Double loop over j,k
  for j=-1:-1:-numrings ## For exterior region, use j,1,numrings
    for k=0:2*numsecs-1
      ## For the half-plane, just remove 'exp':
      value = mu( exp( trianglecenter(numrings,numsecs,j,k) ) );
      fprintf(fil, "%f %f ",real(value), imag(value) );
    end;
    fprintf(fil, "\n"); # End of output line
  end;
  fclose(fil);
endfunction
```

6.2 *Mathematica*

```
trianglecenter[ numsecs_, {j_, k_} ] :=
  Module[ {x, y},
    x = (Sqrt[3.]Pi/numsecs) (
      j - Sign[j]If[ EvenQ[j+k], 2/3, 1/3 ] );
    y = k Pi/numsecs;
    x + I y
  ]
writediscretebeltramidifferentialdisk[ filename_,
      mu_, {numrings_, numsecs_}, comment_ ] :=
Module[ {fstr,j,k,value},
  fstr = OpenWrite[filename];
  WriteString[ fstr,"DI ",numrings," ",numsecs ];
  (* for left half-plane use "HI " *)
  WriteLine[fstr, " "]; (* End of output line *)
  WriteLine[fstr, "# " <> comment ];
  Do[ (* Double loop over j,k *)
    Do[ value = (* For the half-plane, just remove 'Exp' *)
      mu[ Exp[ trianglecenter[numrings,numsecs,{j,k}] ] ];
      WriteString[ fstr,
          CForm[Re[value]], " ", CForm[Im[value]], "  " ];
      , {k, 0, 2*numsecs-1}
    ];
    WriteLine[ fstr, " " ] (* End of output line *)
    , {j, -1, -numrings, -1}
      (* for exterior region, use {j,1,numrings} *)
  ];
  Close[ fstr ];
]
```

7 Examples

Example files are distributed with Beltrami Equation Solver to illustrate its use. This section may be used as a quick tutorial.

The subset files

example_bubbles_sub.txt
example_concentric_sub.txt
example_gridconcentric_sub.txt
example_gridconcentricradial_sub.txt
example_gridhorizontal_sub.txt
example_gridradial_sub.txt
example_gridradiusneg_sub.txt
example_gridradiuspos_sub.txt
example_gridrectangular_sub.txt
example_gridvertical_sub.txt
example_radius_sub.txt
example_radiusneg_sub.txt
example_radiuspos_sub.txt
example_shiftedcircle_sub.txt
example_spiral_sub.txt
example_spirals_sub.txt

are fairly self-explanatory. These subsets, especially those containing a "grid," can be helpful in visualizing the quasiconformal mapping after loading any of the following Beltrami coefficient files. Note that after having solved a Beltrami equation and graphed the resulting triangulation, Beltrami Equation Solver permits you to load any number of subset files consecutively (but showing only one at a time). The following explanations have as somewhat technical mathematical content.

example1a_d_int.txt: μ is identically zero. Load this with the options Disk +Symmetric, and note that the w-triangulation is identical to the z-triangulation because the normalized conformal mapping of the plane is the identity mapping. Then load example_spiral_sub.txt. When you change the mode to Halfplane, you will see that the lifting of the spiral extends far beyond the basic triangulation. (Use the Graphics controls to view the extent of the lifting.) Beltrami Equation Solver does the best it can to handle such situations, but occasionally it can be befuddled.

example1a_h_int.txt: ν is identically zero. Load this with the options Halfplane+Symmetric, and note that again the w-triangulation is identical to the z-triangulation (all triangles are equilateral). If you load example_spiral_sub.txt now, and then switch to Disk mode, you will obtain a different subset from the previous example.

example1b_d_int.txt: μ is constant. Load this with Disk+Symmetric.

example1b_h_int.txt: μ is constant. Load this with Halfplane+Symmetric, and note that all of the image triangles are similar (although not equilateral). Although the constant is the same as in the previous example, the triangulation is different because the pullback of a constant (in either direction) is not constant.

example2a_d_int.txt: Here μ is discontinuous, taking the value 0 in the lower half-plane, and a constant nonzero value in the upper half-plane. Load this with

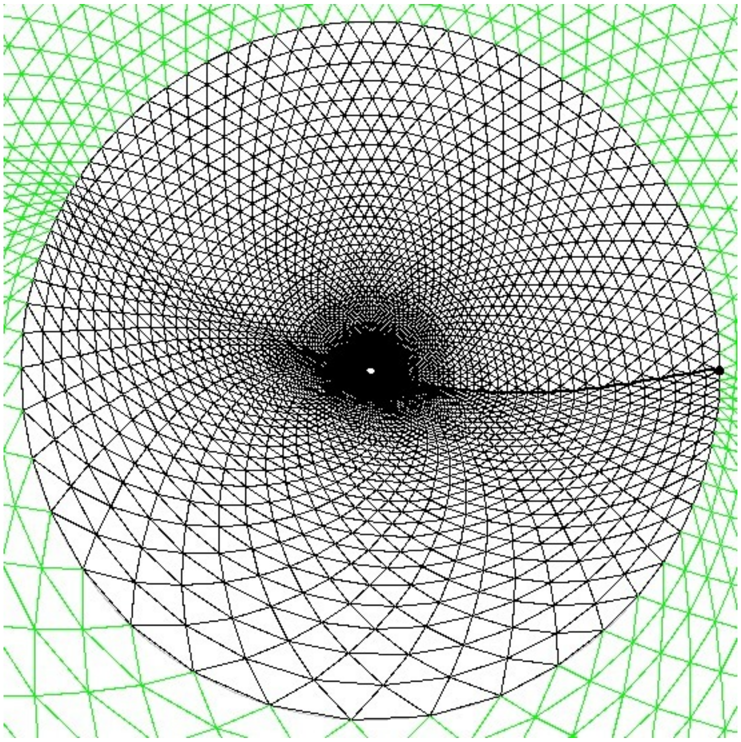

Figure 2: Discontinuous μ of Example 2d. The subset file for the positive real radius was also loaded. Note the equilateral triangles in the image of the upper half-plane.

Halfplane+Symmetric, and note (Figure 2) how half of the image triangles are approximately equilateral. Files `example2b_d_int.txt`, `example2c_d_int.txt` use the same μ with larger values of M,N. You may observe these parameters in the File Chooser preview area before choosing the files, and also in the Technical data area.

`example3a_d_int.txt`: Here μ is given by

$$\mu(z) = \frac{\alpha - 1}{\alpha + 1}\frac{z}{\overline{z}} = \frac{\alpha - 1}{\alpha + 1}e^{2i\theta}$$

where $z = re^{i\theta}$. This is the Beltrami derivative of the mapping

$$f(z) = r^{\alpha}e^{i\theta}.$$

In this file $\alpha = 0.25$. In `example2b_d_int.txt` the Beltrami coefficient is provided for $\alpha = 2.0$, and in `example2c_d_int.txt` for $\alpha = 3.0$. By means of the explicit formula for $f(z)$, one can verify that the error in the solution found by Beltrami E quation Solver in these three examples is approximately 0.02, 0.0002, and 0.0001 respectively.

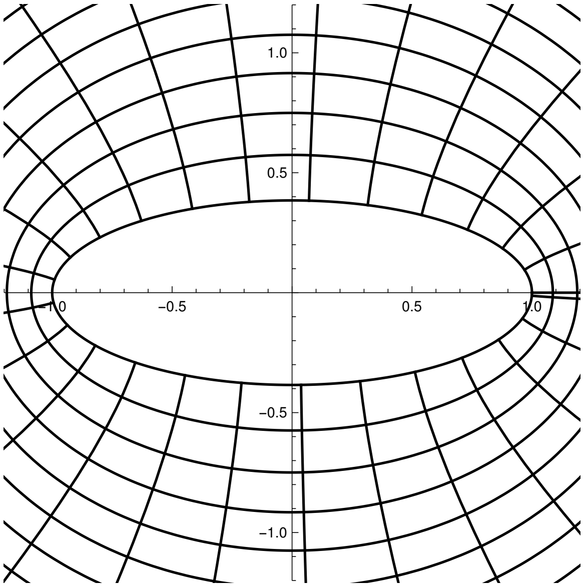

Figure 3: Exterior conformal mapping for $a = 1.5$

`example4a_d_int.txt`: Here

$$\mu(z) = \frac{-\frac{i\beta}{2}\frac{z^2}{|z|}}{1 - \frac{i\beta}{2}|z|}$$

and

$$f(z) = ze^{i\beta(1-r)} = re^{i(\theta+\beta(1-r))}$$

with $\beta = 1.5$. In `example4a_d_int.txt` the value $\beta = 2.0$ is used. The errors in both cases are around 0.0002.

In all of the above examples the Symmetric mode is applied. In such cases the unit circle is left invariant. For the next example, consider the conformal mapping from the exterior region $\{|z| > 1\}$ to the exterior of an ellipse,

$$f(z) = \frac{a^2 z + 1/z}{1 + a^2} \quad (|z| > 1)$$

where $a > 1$ (Figure 3).

The boundary values for $|z| = 1$, namely

$$f(e^{i\theta}) = \frac{1}{1 + a^2}(a^2 e^{i\theta} + e^{-i\theta})$$

can be extended to $|z| < 1$ quasiconformally in many ways. In this example, simple radial interpolation is used to define the following quasiconformal mapping from the disk to the interior of the ellipse:

$$f(z) = \frac{r}{1 + a^2}(a^2 e^{i\theta} + e^{-i\theta}) = \frac{a^2}{1 + a^2} z + \frac{1}{1 + a^2}\overline{z} \quad (|z| < 1).$$

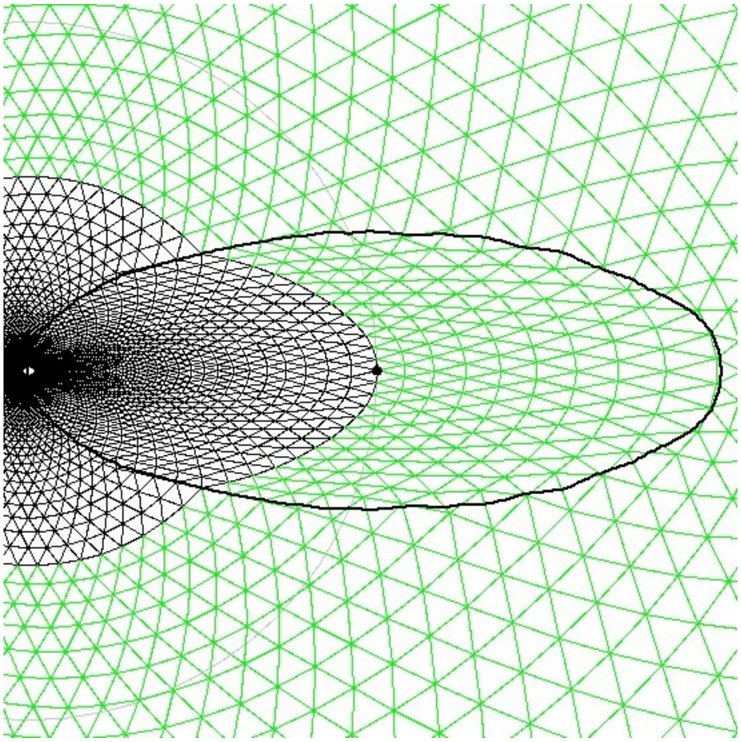

Figure 4: Mapping to shifted ellipse by distortion in interior, $a = 1.5$

Its Beltrami derivative is the constant value

$$\mu(z) = a^{-2}.$$

The file `example5_d_int.txt` was created with the parameter $a = 1.5$. Load it with the option Disk+Conformal, and observe that $|z| = 1$ is sent to the ellipse corresponding to this value of a.

In the next example, the files `example6_d_int.txt` and `example6_d_ext.txt` contain the values of the previous example shifted via the transformation $z \mapsto z + 1$. Thus the Beltrami differential is not symmetric in the unit circle, and not identically zero either in the interior or the exterior. Load these files with the option Disk+General. After the Beltrami equation is solved, then load the subset file `example_shiftedcircle_sub.txt`, which describes the circle $\{|z - 1| = 1\}$. Since in this example μ is constant inside this circle and vanishes outside, the image is again an ellipse. Note that there is less accuracy in the exterior region $|z| > 1$ because the mesh triangles are larger there (Figure 4).

Finally, `example7_d_int.txt` shows how garbage is produced when an attempt is made to solve the Beltrami equation with values in the range $|\mu| > 1$.

8 Installation

For Linux users. Download the executable file, and copy it to wherever you want. No other installation is required, except that it may be necessary to execute

```
chmod +x ./beltrami
```

before the first use. To run the program, either (1) locate the file `beltrami` in a file manager and double-click on the image, or (2) open a terminal, navigate via cd to the directory containing the file and type

```
./beltrami
```

For Windows mac/OS users. Consult the associated document, "Beltrami Equation Solver, Instrucions for Windows/macOS Users" by H. Shimauchi.

9 Acknowledgements

Beltrami Equation Solver was written in C++. The following software is incorporated into Beltrami Equation Solver:

- Eigen: `eigen.tuxfamily.org`
- FLTK (Fast Light Toolkit): `www.fltk.org`
- QDBMP (Quick n' Dirty BMP library): `qdbmp.sourceforge.net`

10 Disclaimer

Beltrami Equation Solver is provided at no cost and on an "as is" basis, without warranty of any kind. The programmer will be in no event liable for any kind of damage or loss caused by the program, which you accept using at your own risk.

References

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