On the Regularity Property of Semi-Markov Processes with Borel spaces[1]

Óscar Vega-Amaya

Departamento de Matemáticas
Universidad de Sonora
México

October 27-28, 2011

[1] Coloquio de Sistemas Estocásticos 2011, 50 Aniversario del Departamento de Matemáticas, CINVESTAV-IPN, México, D.F.

Introduction

- A semi-Markov process (SMP) combines the probabilistic structure of Markov chain and a renewal process as follows:

  • it makes transitions according to Markov chain $X_n, n \in \mathbb{N}_0$;

  • the times spent between successive transitions are random variables $\delta_{n+1} \geq 0$;

  • the distribution of δ_{n+1} depends on the "present" state X_n;

  • $T_n := \delta_1 + \cdots + \delta_n,\ n \in \mathbb{N}$, the time for the nth transition

Introduction

- A semi-Markov process (SMP) combines the probabilistic structure of Markov chain and a renewal process as follows:

  • it makes transitions according to Markov chain $X_n, n \in \mathbb{N}_0$;

  • the times spent between successive transitions are random variables $\delta_{n+1} \geq 0$;

  • the distribution of δ_{n+1} depends on the "present" state X_n;

  • $T_n := \delta_1 + \cdots + \delta_n,\ n \in \mathbb{N}$, the time for the nth transition

- **Problem**: determine whether the SMP experiences finite or infinitely many transitions in bounded time periods:

$$T_n \to \infty \quad \text{or} \quad T_n \leq M \quad \forall n \in \mathbb{N} \ \text{ for some } \ M > 0?$$

- To guarantee the former property holds all what is need it is the transitions do not take place too quickly!
- Ross (1970), Cinlar (1975):

$$P[\delta_{n+1} \geq \varepsilon | X_n = x] \geq \theta \quad \forall x \in \mathbb{X}, n \in \mathbb{N}_0.$$

- To guarantee the former property holds all what is need it is the transitions do not take place too quickly!
- Ross (1970), Cinlar (1975):

$$P[\delta_{n+1} \geq \varepsilon | X_n = x] \geq \theta \quad \forall x \in \mathbb{X}, n \in \mathbb{N}_0.$$

- They also prove that

$$T_n \to \infty \ \ a.s.$$

for **countable** SMP's asumming the "embedded" Markov chain **reaches a recurrent state with probability one**.

Semi-Markov processes

Semi-Markov processes

- Let $Q(\cdot,\cdot|\cdot)$ be a stochastic kernel on $\mathbb{X}\times\mathbb{R}_+$ given $\mathbb{X}$.
- $\{(X_{n,}\delta_{n+1}) : n\in\mathbb{N}_0\}$ is the Markov chain defined on $(\Omega,\mathcal{F},\mathbb{P}_x)$:

$$\mathbb{P}_x[X_0 = x] = 1$$

$$\mathbb{P}_x[X_{n+1}\in B, \delta_{n+1}\leq t|X_n = y] = Q(B,[0,t]|y)$$

- $\{(X_{n,}\delta_{n+1}) : n\in\mathbb{N}_0\}$ is called *Markov renewal process* and it is usually thought of as a model of a stochastic system evolving as follows:

  • at time $t = 0$ the system is observed in some initial state $X_0 = x\in\mathbb{X}$;

  • it remains there for a nonnegative random time δ_1;

  • the conditional distribution function of δ_1 is given by

$$F(t|x) := Q(\mathbb{X},[0,t]|x) \quad \forall t\in\mathbb{R}_+, x\in\mathbb{X}$$

  and known as *holding time distribution*;

- • the *mean holding time function*

  $$\tau(x) := \int_0^\infty tF(dt|x), \ x \in \mathbb{X}$$

  • at time δ_1 the system jumps to a new state $X_1 = y \in \mathbb{X}$ according to the probability measure

  $$P(B|x) := Q(B, \mathbb{R}_+|x);$$

  • it remains in $X_1 = y$ up to the random time δ_2 and so on.
- The process $\{X_n, n \in \mathbb{N}_0\}$ is a Markov chain with transition probability $P(\cdot|\cdot)$.

- The system is tracked in continuous time by the process

$$Z_t = X_n \quad \text{if} \quad T_n \leq t < T_{n+1}, \quad n \in \mathbb{X}_0$$

- $\{Z_t, t \geq 0\}$ is called **semi-Markov process** with semi-Markov kernel $Q(\cdot, \cdot \mid \cdot)$

Definition (2.1)

The state $x \in \mathbb{X}$ is $regular$ if

$$\lim_{n \to \infty} T_n = \infty \quad \mathbb{P}_x - a.s.$$

The semi-Markov process is $regular$ if every state is regular.

- The kernel $Q(\cdot, \cdot \mid \cdot)$ can be disintegrated as

$$Q(B, [0,t] \mid x) = \int_B G(t \mid x, y) P(dy \mid x)$$

where

$$G(t \mid x, y) = \mathbb{P}[\delta_{n+1} \leq t \mid X_n = x, X_{n+1} = y]$$

Proposition (2.2)

The random variables $\{\delta_n\}$ are conditionally independent given $\{X_n\}$:

$$\mathbb{P}[\delta_1 \leq t_1, \ldots, \delta_n \leq t_n \mid X_0, \ldots, X_n] = \prod_{k=0}^{n} G(t_k \mid X_{k-1}, X_k)$$

Recurrent Markov chains

Hernández-Lema and Lasserre (2003), Meyn and Tweedie (1993)

- A Markov chain $\{Y_n : n \in \mathbb{N}_0\}$ is said to be *irreducible* if there exists a nontrivial σ-finite measure $\nu(\cdot)$ on $(\mathbb{X}, \mathcal{B})$ such that

$$\mathbb{E}_x \sum_{n=0}^{\infty} \mathbb{I}_B(Y_n) > 0 \quad \forall x \in \mathbb{X}, \nu(B) > 0, \ B \in \mathcal{B};$$

  $\nu(\cdot)$ is called *irreducibility measure*.

- If the Markov chain $\{Y_n : n \in \mathbb{N}_0\}$ is irreducible, then there exists a *maximal irreducibility measure* $\psi(\cdot)$:

  •$\psi(\cdot)$ is an irreducibility measure;

  • if $\nu(\cdot)$ is an irreducibility measure, then $\nu(\cdot) \ll \psi(\cdot)$.

- An irreducible Markov chain $\{Y_n : n \in \mathbb{N}_0\}$ is said to be *recurrent* if

$$\mathbb{E}_x \sum_{n=0}^{\infty} \mathbb{I}_A(Y_n) = \infty \quad \forall x \in \mathbb{X}, A \in \mathcal{B}^+, \tag{1}$$

  where $\mathcal{B}^+ := \{B \in \mathcal{B} : \psi(B) > 0\}$.

- If instead of condition (1) we have

$$\sum_{n=0}^{\infty} \mathbb{I}_A(Y_n) = \infty \quad \mathbb{P}_x\text{-a.s. } \forall x \in \mathbb{X}, A \in \mathcal{B}^+,$$

  then the Markov chain is said to be *Harris recurrent*.

Theorem (3.1)

If the Markov chain $\{Y_n : n \in \mathbb{N}_0\}$ *is recurrent, then*

$$\mathbb{X} = H \cup N$$

where the measurable set H *is full and absorbing :*

- $\psi(N) = 0$;
- $P(H|x) = 1$ *for all* $x \in H$.

Moreover, the Markov chain restricted to H *is Harris recurrent, that is,*

$$\sum_{n=0}^{\infty} \mathbb{I}_A(X_n) = \infty \quad \mathbb{P}_x\text{-a.s. } \forall x \in H, A \subset H, A \in \mathcal{B}^+.$$

Theorem (3.2)

Suppose that $\{Y_n : n \in \mathbb{N}_0\}$ *has a* $unique\ invariant\ probability\ measure$ $\mu(\cdot)$*:*

$$\mu(B) = \int_B P(B \mid x) d\mu(x) \quad \forall B \in \mathcal{B}$$

(a) *Then, for each function* $v \in L_1(\mu)$ *there exists a set* $B_v \in \mathcal{B}$*, with* $\mu(B_v) = 1$*, such that*

$$\frac{1}{n} \sum_{k=0}^{n-1} v(Y_n) \to \mu(v) := \int_{\mathbb{X}} \quad \mathbb{P}_x\text{-a.s. } \forall x \in B_v. \qquad (2)$$

(b) *If in addition the Markov chain is Harris recurrent, then (2) holds for all* $x \in \mathbb{X}$*.*

Regularity, recurrence and invariant probability measures

Assumption (4.1)

The embedded Markov chain $X_n, n \in \mathbb{N}_0$, is Harris recurrent.

- Define

$$\Delta(x) := \int_{\mathbb{R}_+} \exp(-t) F(dt \mid x), \qquad x \in \mathbb{X}.$$

$\bullet\ 0 < \Delta(\cdot) \leq 1 \ \text{ and } \ \tau(\cdot) \geq 0$

$\bullet\ \Delta(x) = 1 \Leftrightarrow F(0 \mid x) = 1 \Leftrightarrow \tau(x) = 0.$

Assumption (4.2)

The embedded Markov chain is irreducible and for some $\alpha < 1$

$$B := \{x \in \mathbb{X} : \Delta(x) \leq \alpha\} \in \mathcal{B}^+$$

Remark (4.3)

The conditional independence of $\{\delta_n : n \in \mathbb{N}_0\}$ *implies that*

$$\mathbb{E}_x[\exp(-T_n) \mid X_0, \ldots, X_n] = \Delta(X_0) \cdots \Delta(X_n) \;\; \mathbb{P}_x - a.s.$$

for every $n \in \mathbb{N}_0$. *Thus,*

$$T_n \to \infty \;\; \mathbb{P}_x\text{-}a.s. \Leftrightarrow \Delta(X_0) \cdots \Delta(X_n) \to 0 \;\; \mathbb{P}_x\text{-}a.s.$$

Theorem (4.4)

If Assumptions 4.1 and 4.2 hold, then the SMP is regular:

$$T_n \to \infty \quad \mathbb{P}_x\text{-}a.s.$$

Theorem (4.4)

If Assumptions 4.1 and 4.2 hold, then the SMP is regular:

$$T_n \to \infty \ \ \mathbb{P}_x\text{-}a.s.$$

Proof: $\sigma_1 := \inf\{k > 0 : X_k \in B\}, \quad \sigma_{n+1} := \inf\{k > \sigma(n) : X_k \in B\}$

Theorem (4.4)

If Assumptions 4.1 and 4.2 hold, then the SMP is regular:

$$T_n \to \infty \;\; \mathbb{P}_x\text{-}a.s.$$

Proof: $\sigma_1 := \inf\{k > 0 : X_k \in B\}$, $\sigma_{n+1} := \inf\{k > \sigma(n) : X_k \in B\}$

$$S_n := \sum_{k=1}^{n} \mathbb{I}_B(X_k)$$

Theorem (4.4)

If Assumptions 4.1 and 4.2 hold, then the SMP is regular:

$$T_n \to \infty \ \ \mathbb{P}_x\text{-}a.s.$$

Proof: $\sigma_1 := \inf\{k > 0 : X_k \in B\}, \quad \sigma_{n+1} := \inf\{k > \sigma(n) : X_k \in B\}$

$$S_n := \sum_{k=1}^{n} \mathbb{I}_B(X_k)$$

Now observe

$$\Delta(X_0)\cdots\Delta(X_n) \leq \Delta(X_{\sigma(1)})\Delta(X_{\sigma(2)})\cdots\Delta(X_{\sigma(S_n)}) \leq \alpha^{S_n}$$

on the set $[S_n \neq 0]$.

Theorem (4.4)

If Assumptions 4.1 and 4.2 hold, then the SMP is regular:

$$T_n \to \infty \ \ \mathbb{P}_x\text{-}a.s.$$

Proof: $\sigma_1 := \inf\{k > 0 : X_k \in B\}$, $\sigma_{n+1} := \inf\{k > \sigma(n) : X_k \in B\}$

$$S_n := \sum_{k=1}^{n} \mathbb{I}_B(X_k)$$

Now observe

$$\Delta(X_0)\cdots\Delta(X_n) \leq \Delta(X_{\sigma(1)})\Delta(X_{\sigma(2)})\cdots\Delta(X_{\sigma(S_n)}) \leq \alpha^{S_n}$$

on the set $[S_n \neq 0]$.

Thus, since $B \in \mathcal{B}^+$, Assumption 1 implies that $S_n \to \infty$ $\mathbb{P}_x$-a.s. for all $x \in \mathbb{X}$. Hence

Theorem (4.4)

If Assumptions 4.1 and 4.2 hold, then the SMP is regular:

$$T_n \to \infty \quad \mathbb{P}_x\text{-}a.s.$$

Proof: $\sigma_1 := \inf\{k > 0 : X_k \in B\}$, $\sigma_{n+1} := \inf\{k > \sigma(n) : X_k \in B\}$

$$S_n := \sum_{k=1}^{n} \mathbb{I}_B(X_k)$$

Now observe

$$\Delta(X_0) \cdots \Delta(X_n) \leq \Delta(X_{\sigma(1)})\Delta(X_{\sigma(2)}) \cdots \Delta(X_{\sigma(S_n)}) \leq \alpha^{S_n}$$

on the set $[S_n \neq 0]$.

Thus, since $B \in \mathcal{B}^+$, Assumption 1 implies that $S_n \to \infty$ $\mathbb{P}_x$-a.s. for all $x \in \mathbb{X}$. Hence

$$\Delta(X_0) \cdots \Delta(X_n) \to 0 \quad \mathbb{P}_x\text{-a.s. for all } x \in \mathbb{X},$$

which proves that the process is regular.■

Theorem (4.5)

Suppose Assumption 2 holds. If the embedded Markov chain is recurrent and

$$\sigma := \inf\{k > 0 : X_k \in H\} < \infty \quad \mathbb{P}_x\text{-}a.s. \quad \forall x \in \mathbb{X}$$

where H is the subset as in Theorem 3.1 , then the SMP is regular.

Theorem (4.5)

Suppose Assumption 2 holds. If the embedded Markov chain is recurrent and

$$\sigma := \inf\{k > 0 : X_k \in H\} < \infty \;\; \mathbb{P}_x\text{-}a.s. \;\; \forall x \in \mathbb{X}$$

where H is the subset as in Theorem 3.1 , then the SMP is regular.

Proof: Observe $B' := B \cap H \in \mathcal{B}^+$. Now, proceed as in the proof of Theorem 4.4.

Assumption (4.6)

(a) *The embedded Markov chain has a unique invariant probability measure;*

Assumption (4.6)

(a) *The embedded Markov chain has a unique invariant probability measure;*

(b) $\mu(\Delta) = \int_{\mathbb{X}} \Delta(x) d\mu(x) < 1.$

Assumption (4.6)

(a) *The embedded Markov chain has a unique* $invariant\ probability\ measure$*;*

(b) $\mu(\Delta) = \int_{\mathbb{X}} \Delta(x) d\mu(x) < 1.$

Theorem (4.7)

(a) *If Assumption holds, then the SMP is regular for* μ*-almost all* $x \in \mathbb{X}$*.*

(b) *If additionally, the embedded Markov chain is Harris recurrent, then the SMP is regular.*

Proof: (a) Note that

$$[\Delta(X_0)\cdots\Delta(X_n)]^{1/(n+1)} \leq \frac{1}{n+1}\sum_{k=0}^{n}\Delta(X_k)$$

Proof: (a) Note that

$$[\Delta(X_0)\cdots\Delta(X_n)]^{1/(n+1)} \leq \frac{1}{n+1}\sum_{k=0}^{n}\Delta(X_k)$$

From Theorem 3.2(a) there exists a subset $B_\Delta \in \mathcal{B}$ with $\mu(B_\Delta) = 1$ such that

$$\frac{1}{n+1}\sum_{k=0}^{n}\Delta(X_k) \to \mu(\Delta) < 1 \quad \mathbb{P}_x\text{-a.s.} \quad \forall x \in B_\Delta$$

Proof: (a) Note that

$$[\Delta(X_0)\cdots\Delta(X_n)]^{1/(n+1)} \leq \frac{1}{n+1}\sum_{k=0}^{n}\Delta(X_k)$$

From Theorem 3.2(a) there exists a subset $B_\Delta \in \mathcal{B}$ with $\mu(B_\Delta) = 1$ such that

$$\frac{1}{n+1}\sum_{k=0}^{n}\Delta(X_k) \to \mu(\Delta) < 1 \;\; \mathbb{P}_x\text{-a.s.} \;\; \forall x \in B_\Delta$$

Therefore,

$$\Delta(X_0)\cdots\Delta(X_n) \to 0 \;\; \mathbb{P}_x\text{-a.s.} \;\; \forall x \in B_\Delta.$$

Proof: (a) Note that

$$[\Delta(X_0)\cdots\Delta(X_n)]^{1/(n+1)} \leq \frac{1}{n+1}\sum_{k=0}^{n}\Delta(X_k)$$

From Theorem 3.2(a) there exists a subset $B_\Delta \in \mathcal{B}$ with $\mu(B_\Delta) = 1$ such that

$$\frac{1}{n+1}\sum_{k=0}^{n}\Delta(X_k) \to \mu(\Delta) < 1 \;\; \mathbb{P}_x\text{-a.s.} \;\; \forall x \in B_\Delta$$

Therefore,

$$\Delta(X_0)\cdots\Delta(X_n) \to 0 \;\; \mathbb{P}_x\text{-a.s.} \;\; \forall x \in B_\Delta.$$

(b) The second statement follows from Theorem 3.2(b). ■

References

R.N. Bhattacharya, M. Majumdar (1989), Controlled semi-Markov models–The discounted case, *J. Statist. Plann. Inf.* **21**, 365-381.

E. Çinlar (1975), *Introduction to Stochastic Processes* (Prentice-Hall, Englewood Cliffs, New Jersey.

O. Hernández-Lerma, J.B. Lasserre (2003), *Markov Chains and Invariant Probabilities*, Birkhäuser.

S.P. Meyn, R.L. Tweedie (1993), *Markov Chains and Stochastic Stability* , Springer-Verlag, London.

S.M. Ross (1970), *Applied Probability Models with Optimization Applications* , Holden-Day, San Francisco.

THANKS!