Toeplitz Operators with Frequency Modulated Almost Periodic and Semi-Almost Periodic Symbols.

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Dedicated to ISRAEL GOHBERG on the occasion of his 80-th birthday.

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This talk is based on joint work with A. Böttcher, V.Dybin, E.Ramírez and I.Spitkovsky.

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2. A.Böttcher,S.Grudsky and I.Spitkovsky. "Toeplitz operators with frequency modulated semi-almost periodic symbols." The Journal of Fourier Analysis and Applications, Vol.7, 523-535, 2001.
3. A.Böttcher,S.Grudsky and I.Spitkovsky. "Block Toeplitz operators with frequency modulated semi-almost periodic symbols." International Journal of Mathematics and Mathematical Sciences, Vol.34, 2157-2176, 2003.
4. A.Böttcher,S.Grudsky and E.Ramíres de Arellano. "Algebras of Toeplitz operators with oscillating symbols." Revista Matemática Iberoamericana, Vol.20, N3, 647-671, 2004.

I.Gohberg and I.Feldman.

1. "Wiener-Hopf integro-difference equations." Doklad Akad.Nauk SSSR 183 (1968),25-28. English transl.in Soviet Math. Dokl. 9 (1968),1312-1316.
2. "Integro-difference Wiener-Hopf equations." Acta Sci.Math. 30 (1969),119-137.

$$(W\varphi)(t) = \sum_{-\infty}^{\infty} a_j \varphi(t-\delta_j) + \int_0^{\infty} K(t-s)\varphi(s)ds = f(t)(0 < t < \infty)$$

$$\delta_j \in R, \qquad \sum_{-\infty}^{\infty} |a_j| < \infty, \qquad \int_{-\infty}^{\infty} |K(t)|\delta t < \infty$$

$$W : L_p(0,\infty) \longrightarrow L_p(0,\infty)$$

Toeplitz operators with almost periodic symbols.

Wiener class of almost periodic functions:

$$a(x) \in AP_w(R) => a(x) = \sum_{-\infty}^{\infty} a_j e^{i\delta_j x}, \quad \delta_j \in R, \quad \sum_{-\infty}^{\infty} |a_j| < \infty$$

if all $\delta_j \geq 0 => a \in AP_w^+(R)$; if all $\delta_j \leq 0 => a \in AP_w^-(R)$

$(Sf)(x) = \frac{1}{\pi i} \int\limits_R \frac{f(\tau)}{\tau - x} d\tau, \quad x \in R, \quad S : L_p(R) \longrightarrow L_p(R)$

$P := \frac{1}{2}(1 + S)$ - analytic projector

$\mathrm{Im} P := L_p^+(R) = H_p^+(R)$

$T(a) = PaP : H_p^+(R) - H_p^+(R)$ - Toeplitz operator

$a =$ symbol of $T(a)$

Theorems of Gohberg and Feldman, 1968

Theorem (A)

Let $a \in AP_w(R)$ *and* $\inf\limits_{x \in R} |a(x)| > 0.$

Then the function a admits the following factorization of Wiener-Hopf type:

$$a(x) = a_+(x) \exp\{\sigma x\} a_-(x)$$

where the real number σ *is define as follows*

$$\sigma =: \sigma(a) = \lim_{\ell \to \infty} \frac{1}{2\ell} \arg a(x) \Big|_{-\ell}^{\ell}$$

and

$$a_+^{\pm 1}(x) \in AP_w^+(R), \qquad a_-^{\pm 1}(x) \in AP_w^-(R)$$

Theorem (B)

Let $a \in AP_w(R)$ *and* $\inf_{x \in \infty} |a(x)| > 0$.

Then if

1. $\sigma = 0$, *then the operator* $T(a)$ *is invertible* $H_p^+(R)$;
2. $\sigma > 0$, *then the operator* $T(a)$ *is left invertible* $H_p^+(R)$;
3. $\sigma < 0$, *then the operator* $T(a)$ *is right invertible* $H_p^+(R)$.

$$\arg a(x) = \sigma x + O(1)$$

Semi-almost periodic symbols.

$AP(R)$: the class of uniformly almost periodic functions.

Definition: $AP(R)$ is the closure of the set of all almost periodic polynomials of the form

$P_n(x) = \sum_{j=1}^{n} c_j \exp\{i\delta_j x\}$ in the norm of $L_\infty(R)$.

$$AP_w(R) \subset AP(R) \quad (!)$$

$u_+ \in C(\bar{R}), u_+(+\infty) = 1, u_+(-\infty) = 0$ and $u_-(x) = 1 - u_+(x)$

Definition: $f(x) \in SAP <=> f = f_0 + u_+ f_+ + u_- f_-$

where $f\pm \in AP(R), \quad f_0 \in C(\bar{R}) \quad$ and $\quad f_0(\pm\infty) = 0.$

Sarason's Theorem.

$$f = (f_0 + u_+ f_+ + u_- f_-) \in SAP, \quad \inf_{x \in R} |f(x)| > 0.$$
$$\mu_\pm(f) = \sigma(f_\pm)$$

Definition: Let $\mu_\pm(f) = 0 => \log(f_\pm) \in AP(R)$, then

$$\lambda_\pm(f) := \exp\left\{ \lim_{\ell \to \infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} f_\pm(u) d \right\}$$

$$T(f) : H_2^+(R) \to H_2^+(R)$$

Theorem (D.Sarason, 1977)

Let $f \in SAP$ *and* $\inf_{x\in R} |f(x)| > 0.$

Then

1. *if* $\mu_+(f)\mu_-(f) < 0$*,then* $T(f)$ *is not semi-Fredholm;*
2. *if* $\mu_\pm(f) \geq 0$ *and* $\mu_+^2(f) + \mu_-^2(f) > 0$*, then* $T(f)$ *is left-invertible, but not right-invertible* $H_2^+(R)$*;*
3. *if* $\mu_\pm(f) \leq 0$ *and* $\mu_+^2(f) + \mu_-^2(f) > 0$*, then* $T(f)$ *is right-invertible, but not left-invertible;*
4. *if* $\mu_+(f) = \mu_-(f) = 0$*, then* $T(f)$ *is Fredholm,*
   if $(\lambda_+(f)/\lambda_-(f)) \notin (-\infty, 0)$ *and* $T(f)$ *is not Fredholm,*
   if $(\lambda_+(f)/\lambda_-(f)) \in (-\infty, 0)$

1. Scalar case $1 < p < \infty$:
   -R.Duduchava and A.Saginashvili, 1981.
2. Matrix case :
   -A.Böttcher, Yu.Karlovich and I.Spitkovsky.
   Series of Works 1983 - . . . ;

   -A.Böttcher, Yu.Karlovich and I.Spitkovsky
   "Convolution operators and factorization of almost periodic matrix functions." Operator Theory:Advances and Applications,131, Birkhäuser Verlag, Basel, XII+462pp, 2002.

Frequency Modulated Symbols.

Let $a \in AP(R)$ and $\inf_{x \in R} a(x)| > 0$. Then

$$\arg a(x) = \sigma x + a_0(x), \quad a_0 \in AP(R)$$

What happens if we change x by $\alpha(x)$, where $\alpha : R \to R$ is an orientation-preserving homeomorphism of R.

$$\arg a_\alpha(x) = \sigma\alpha(x) + O(1), \quad a_\alpha(x) = a(\alpha(x))$$

Problem: Let $a \in AP$ and let $T(a)$ be Fredholm, then:

is $T(a_\alpha)$ Fredholm?

Answer: In general No.

(A.Böttcher, S.Grudsky and I.Spitkovsky, 2001)

Sufficient conditions for $L\infty(R)$

Theorem (P.S.Muhly and J.Xia)

Let $\varphi : R \to \mathbb{T}, \; \varphi(x) = \frac{x-i}{x+i}, \; \psi : \mathbb{T} \to R, \; \psi(t) = i\frac{1+t}{1-t}$
Let $\alpha : R \to R$ *be a homeomorphism and* $\sigma \circ \alpha \circ \psi : \mathbb{T} \to \mathbb{T}$
be a bi-Lipshitz homeomorphism, such that $\sigma' \in VMO$.
Then $(T(b \circ \alpha) - T(b))$ *is compact for every* $b \in L_\infty(R)$

Consequence:

$$0 < \liminf_{x \to \infty} \frac{\alpha(x)}{x} \leq \limsup_{x \to \infty} \left| \frac{\alpha(x)}{x} \right| < \infty$$

Sufficient conditions

Definition. We call a real-valued function β defined for all sufficiently large $x > 0$ regular,if it is strictly monotonically increasing,if it is twice continuously differentiable and,if the following conditions are satisfied:

$$\liminf_{x \to +\infty} \frac{x\beta''(x)}{\beta'(x)} > -2,$$

$$\lim_{x \to +\infty} \frac{\beta''(x)}{(\beta'(x))^2} = 0,$$

$$\lim_{x \to +\infty} \frac{x^{1/2}\beta''(x)}{(\beta'(x))^{3/2}} = 0.$$

Theorem (S.Grudsky, 2001)

If the homeomorphism α is a regular function and if $\alpha(-x) = -\alpha(x)$ for all sufficiently large $x > 0$, then $e^{i\lambda\alpha} \in C + H^\infty$ for all $\lambda > 0$.

Theorem (A.Böttcher, S.Grudsky and I.Spitkovsky, 2001)

If there is a $\delta > 1$ such that $\alpha(x) - (\log x)^\delta$ is regular, then $u_+ e^{i\lambda\alpha} \in C + H^\infty$ for all $\lambda > 0$.

Examples of regular functions.

1. $\alpha(x) = cx^{\delta}, \qquad \delta > 0;$
2. $\alpha(x) = c(\log x)^{1+\delta}, \quad \delta > 0;$
3. $\alpha(x) = cx^{\nu}(\log x)^{\delta}, \quad \nu > 0, \delta > 0;$
4. $\alpha(x) = c_1 \exp(c_2 x^{\delta}), \quad \delta > 0.$

$x \geq M > 0, \qquad c, c_1, c_2$ are positive constants.

Let $a = (a_0 + u_+ a_+ + u_- a_-) \in SAP$ and $\inf_{x \in R} |a(x)| > 0$.

Let $\sigma_+ = \sigma(a_+)$, $\sigma_- = \sigma(a_-)$.

$a_\alpha(x) := a(\alpha(x))$, $T(a),\ T(a_\alpha) : H_p^+(R) \to H_p^+(R)$.

Theorem (A.Böttcher, S.Grudsky and I.Spitkovsky, 2001)

Assume that

$u_+ e^{\lambda\alpha} \in C + H^\infty$ *and* $u_- e^{i\lambda\alpha} \in C + H^\infty$, *for all* $\lambda > 0$.

Then

1. *if* $\sigma_- = \sigma_+ = 0$, *then for* $T(a_\alpha)$ *to be Fredholm it is necessary and sufficient that* $T(a)$ *be Fredholm. In this case* $indT(a_\alpha) = indT(a)$;
2. *if* $\sigma_\pm \geq 0$ *and* $\sigma_+^2 + \sigma_-^2 > 0$, *then* $T(a_\alpha)$ *is not Fredholm, but* $T(a_\alpha)$ *is left invertible;*
3. *if* $\sigma_\pm \leq 0$ *and* $\sigma_+^2 + \sigma_-^2 > 0$, *then* $T(a_\alpha)$ *is not Fredholm, but* $T(a_\alpha)$ *is right invertible;*

Theorem (A.Böttcher, S.Grudsky and I.Spitkovsky, 2001)

Assume that $\limsup\limits_{x\to+\infty} \frac{\alpha(x)}{\log x} = +\infty$ *and* $\liminf\limits_{x\to-\infty} \frac{\alpha(x)}{\log|x|} = +\infty.$

If $\sigma_+\sigma_- < 0$, *then* $T(a_\alpha)$ *is not normally solvable.*

Matrix case

$$H_2^{+(n)}(R), \quad AP^{(n\times n)}(R), \quad P^+ : L_2^{(n)}(R) \longrightarrow H_2^{+(n)}(R)$$
$$T(a) = P^+ a P^+ : H_2^{+(n)}(R) \longrightarrow H_2^{+(n)}(R)$$

Theorem (A.Böttcher, S.Grudsky and I.Spitkovsky, 2003)

Let $a \in AP^{(n\times n)}(R)$ and function α is regular. Then

1. *if $T(a)$ is invertible, then $T(a \circ \alpha)$ is a Φ - operator;*
2. *if $T(a)$ is left-invertible, then $T(a \circ \alpha)$ is a Φ_+ - operator;*
3. *if $T(a)$ is right-invertible, then $T(a \circ \alpha)$ is a Φ_- - operator.*

Theorem (A.Böttcher, S.Grudsky and I.Spitkovsky, 2003)

Let $a \in SAP_w^{(n\times m)}$ and function α be regular. Then if $T(a)$ is a Φ - operator, then $T(a \circ \alpha)$ is also Φ - operator.

Algebraic case

$P_\lambda(R)$ - is the class of all λ periodic functions.
$B_{P_\lambda(R)}$ - is the closure in $L(H_2^+(R))$ of all operators of the form

$$A = \sum_j \prod_k T(a_{j,k}), \quad a_{j,k} \in P_\lambda(R)$$

$B^\alpha_{P_\lambda(R)}$ - is the closure in $L_2(H_2^+(R))$ of all operators of the form

$$A = \sum_j \prod_k T(a_{j,k} \circ \lambda), \quad a_{j,k} \in P_\lambda(R)$$

$G_\alpha : B_{P_\lambda(R)} \longrightarrow B^\alpha_{P_\lambda(R)}$ is the natural map.

Theorem (A.Böttcher, S.Grudsky and E.Ramírez de Arellano, 2004)

Let α is the regular.
Then

1. *if A is invertible,then $G_\alpha(A)$ is a Φ - operator;*
2. *if A is left-invertible,then $G_\alpha(A)$ is a Φ_+ - operator;*
3. *if A is right-invertible,then $G_\alpha(A)$ is a Φ_- - operator;*