Double Barrier Options Under Lévy Processes and the Theory of Toeplitz Operators

Sergei Grudsky

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DOUBLE BARRIER OPTION

Let S_t be stock price at the instant of time t, and $\varphi : (0, \infty) \to [0, \infty)$ be a measurable function.

Definition 1 *An Up-Down-And-Out barrier option is an agreement between two persons (Writer and Holder) at time instant t according to which Writer is obliged to pay to Holder the amount $\varphi(S_T)$ at the future instant of time T (expiry date) if and only if during the option life (between t and T), S_t is always within the interval (S_1, S_2) (here $0 < S_1 < S_2$ are some levels, i.e., barriers, of the stock price).*

Note that if there exists some instant of time $t_{1,2} \leq T$ such that $S_{t_1} \geq S_2$ or $S_{t_2} \leq S_1$ then the option expires worthless.

Consider a market model which consists of a bond with a constant riskless rate of return $r > 0$
$A(t) = \exp\{-rt\}$, and of a stock with price $S_t = \exp\{X_t\}$ where X_t is a Lévy process.

LÉVY PROCESSES

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, where Ω is the space of elementary events and $\mathcal{F}$ is a σ-algebra of subsets of Ω.

Definition 2 . *An $\mathcal{F}$-adapted process X_t is called a Lévy process if the following conditions hold:*

1. *$X_0 = 0$ a.e.*
2. *X_t has stationary increment, that is, for arbitrary $t > s > 0$ the distribution of $(X_t - X_s)$ coincides with the distribution of X_{t-s}.*
3. *X_t has independent increments, that is, for arbitrary $0 \leq t_1 < t_2 < \ldots < t_n$, the random variables*

$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

   are independent.

4. *For each $w \in \Omega$ the function $X_t = X_t(w)$ is right-continuous on $(0, \infty)$ and there exists a left limit at all $t \geq 0$.*
5. *X_t is stochastically continuous, that is, for every $t \geq 0$ and $\epsilon > 0$*

$$\lim_{s \to t} \mathbf{P}[|X_s - X_t| > \epsilon] = 0$$

If X_t is a Lévy process, then according to the Lévy-Khintchine formula

$$E_{\mathbf{P}}[e^{i\xi X_t}] = e^{-t\psi^{\mathbf{P}}(\xi)}, \quad \xi \in \mathbb{R}, \tag{1}$$

where the function $\psi^{\mathbf{P}}(\xi)$ has the representation

$$\psi^{\mathbf{P}}(\xi) = \frac{1}{2}\sigma^2\xi^2 - i\mu\xi - \int_{-\infty}^{\infty}(e^{iu\xi} - 1 - i\xi u I_{(-1,1)}(u))\Pi(du) \tag{2}$$

with $\sigma \geq 0$, $\mu \in \mathbb{R}$, and Π is a measure on $\mathbb{R}$ satisfying the condition

$$\left| \int_{-\infty}^{\infty} \frac{u^2}{1+u^2}\Pi(du) \right| < \infty, \tag{3}$$

$$I_{(-1,1)}(u) = \begin{cases} 1, & |u| < 1; \\ 0, & |u| \geq 1. \end{cases}$$

The expectation of exponent $E_{\mathbf{P}}[e^{i\xi X_t}]$ is called the characteristic function, the function $\psi^{\mathbf{P}}(\xi)$ is called the characteristic exponent of X_t (under the probability measure $\mathbf{P}$), the triplet (a, γ, Π) is called the generating triplet of X_t.

EQUIVALENT MARTINGALE MEASURE (EMM)

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a probability space

$$S_t = e^{X_t}, \quad A(t) = \exp(-rt)$$

$S_t^* = e^{-rt}\, S_t$ - discounted price process.

Definition. *A new probability measure Q on same measure space $(\Omega, \mathcal{F})$ is called EMM if*

a) Q is absolutely continuously respect to $\mathbf{P}$ (historic measure);

b) $S_\tau^* = E^Q\,[S_t^* \mid \mathcal{F}_\tau], \quad \tau \le t.$

Typically there are infinitely many EMM **Eberlian and Jacod [1997]** give general description of all EMM for wide class of Lévy process.

OPTION PRICE

$$U(x,t) = E^Q\left[e^{-(T-t)\tau} g(X_T)\,\mathbf{1}_{\eta>T}\,\Big|_{\mathcal{F}_t} X_t = x\right]$$

where $g(x) = \varphi(e^x)$

$\eta =: \inf\{\tau > t,\ X_\tau \in (-\infty, x_1) \cup (x_2, \infty)\}$ - is hitting time

$$S_1 = \exp\{x_1\}, \quad S_2 = \exp\{x_2\}$$

$$S_0\, U(x,t) - \text{is Option Price}$$

Let be

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(\xi)\, e^{-i\xi x}\, d\xi$$

Fourier Transform.

THEOREM 2.13 (Boyarchenko S.I., Levendorskii Sergei. Non-Gaussian Merton-Black-Scholes theory. 2002)

Let $g(x)(:= \varphi(e^x)) \in L_\infty(x_1, x_2)$ and let the process X_t satisfy the ACP-condition then the price (justified) of the

UP-DOWN-AND-OUT OPTION

the function $\mathcal{U}(x,t)$ is a *bounded* solution of the following problem.

$$\frac{\partial \mathcal{U}(x,t)}{\partial t} - (r - L_x^{\mathbf{Q}})\mathcal{U}(x,t) = 0, \ x \in (x_1, x_2), \ t < T, \quad (4)$$

$$\mathcal{U}(x,T) = g(x), \quad x \in (x_1, x_2), \quad (5)$$

$$\mathcal{U}(x,t) = 0, \quad x \in R \backslash (x_1, x_2), \quad t < T. \quad (6)$$

Here the pseudodifferential operator $L_x^{\mathbf{Q}}$ (acting on the variable x) is given by the formula

$$(L_x^{\mathbf{Q}} f)(x) = (\mathcal{F}^{-1}(-\psi^{\mathbf{Q}}(\cdot))\mathcal{F}f)(x), \quad (7)$$

where $\psi^{\mathbf{Q}}(\xi)$ is the characteristic exponent of the Lévy process X_t under the EMM $\mathbf{Q}$,

Definition 2.4 We will say that the Lévy process X_t satisfies the (ACP)-condition if the function

$$(U^r f)(x) := E_{\mathbf{Q}}[\int_0^\infty e^{-rt} f(X_t) dt | X_0 = x]$$

is continuous for every $f \in L_\infty(R)$.

Convolution equation and classes of symbols

Introduce the Laplace transform (LT) by variable $\tau = T - t$ and denote

$$v(x,w) := (\mathcal{L}u)(x,w) = \int_0^\infty e^{-w\tau} u(x,\tau) d\tau. \tag{8}$$

Thus we pass from problem (4)-(6) to the following problem

$$(-L_x^{\mathbf{Q}} + r + w)v(x,w) = g(x), \quad x \in (0,a), \tag{9}$$

$$v(x,w) = 0 \quad x \in R \backslash (0,a), \qquad a = x_2 - x_1 \tag{10}$$

we can rewrite the problem (9)-(10) as the following equation,

$$P_{(0,a)}(\mathcal{F}^{-1}(\psi^{\mathbf{Q}}(\xi) + r + w)\mathcal{F})v(x,w) = g(x), x \in (0,a), \tag{11}$$

$$\psi^Q(\xi) = \frac{1}{2}\sigma^2\xi^2 - i\mu\xi + \varphi(\xi)$$

where $\sigma \geq 0$, $\mu \in R$, and

$$\varphi(\xi) = -\int_{-\infty}^{\infty}(e^{iu\xi} - 1 - i\xi u I_{(-1,1)}(u))\Pi^{\mathbf{Q}}(du) \tag{12}$$

with the measure $\Pi^{\mathbf{Q}}$ satisfying the condition

$$\left| \int_{-\infty}^{\infty} \frac{u^2}{1+u^2} \Pi^{\mathbf{Q}}(du) \right| < \infty. \tag{13}$$

$$\sigma > 0, \quad \psi^{\mathbf{Q}}(\xi) \sim \frac{\sigma^2\xi^2}{2}, \tag{14}$$

$$\sigma = 0, \quad 1 \leq \nu < 2, \quad \psi^{\mathbf{Q}}(\xi) \sim |\xi|^\nu, \tag{15}$$

$$\sigma = 0, \quad \mu = 0, \quad 0 < \nu \leq 1, \quad \psi^{\mathbf{Q}}(\xi) \sim |\xi|^\nu, \tag{16}$$

$$\sigma = 0, \quad \mu \neq 0, \quad 0 < \nu < 1, \quad \psi^{\mathbf{Q}}(\xi) \sim \xi. \tag{17}$$

Example 1 (Kobol Family) For Lévy processes from this family the characteristic exponent $\psi(\xi)$ can have the following forms,

$$\begin{aligned}\psi(\xi) &= -i\mu\xi + c_+\Gamma(-\nu)[\lambda_-^\nu - (\lambda_- - i\xi)^\nu] \\ &\quad + c_-\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu] \end{aligned} \tag{18}$$

Example 2 (Normal Tempered Stable Lévy Processes) In this case the characteristic exponent is

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}] \tag{19}$$

where $\nu \in (0, 2)$, $\mu \in R$, $\delta > 0$, $\beta \in R$, $\alpha > |\beta|$.

Example 3 (Normal Inverse Gaussian Processes) If we put in (19) $\nu = 1$ we obtain the characteristic exponent of a normal inverse Gaussian Process

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}] \tag{20}$$

$$v_1(x,w) + \mathcal{F}^{-1}(\psi^{\mathbf{Q}}(\xi) + r + w)\mathcal{F}\, v(x,w) + v_2(x,w) = g(x) \quad (21)$$

where

$$g(x) \in H^{s_2}(0,a) \quad (22)$$
$$v_1(x,w) \in H^{s_2}(a,\infty) \quad (23)$$
$$v_2(x,w) \in H^{s_2}(-\infty,0) \quad (24)$$

Apply the Fourier transform to equation (21). Denote

$$\Phi_a^-(\xi,w) := (\mathcal{F}v)(\xi,w) \qquad (\in L_2(\mathbb{R},s_1); \quad (25)$$
$$e^{-ia\xi}\Phi^-(\xi,w) := (\mathcal{F}v_1)(\xi,w) \qquad (\in L_2(\mathbb{R},s_2); \quad (26)$$
$$\Phi^+(\xi,w) := (\mathcal{F}v_2)(\xi,w) \qquad (\in L_2(\mathbb{R},s_2)); \quad (27)$$
$$\hat{g}(\xi) = (\mathcal{F}g)(\xi) \qquad (\in L_2(\mathbb{R},s_2); \quad (28)$$

where $L_2(\mathbb{R},s)$ is Hilbert space with the norm

$$\|\Phi\|_{L_2^s} = \int_{-\infty}^{\infty} |\Phi(\xi)|^2(1+\xi^2)^s d\xi.$$

Thus we obtain the following boundary value problem

$$e^{-ia\xi}\Phi^-(\xi,w) + a(\xi,w)\Phi_a^-(\xi,w) + \Phi^+(\xi,w) = \hat{g}(\xi). \quad (29)$$

This problem is called the modified Wiener-Hopf equation and its solution is a triple $(\Phi^-, \Phi_a^-, \Phi^+)$ of unknown functions.

$$a(\xi,w) = (\psi^Q(\xi) + r + w)$$

$$P^+ := \mathcal{F}^{-1}\chi_{(0,\infty)}\mathcal{F} \quad \text{and} \quad P^- := \mathcal{F}^{-1}\chi_{(-\infty,0)}\mathcal{F}.$$

The projectors $P^\pm$ are bounded linear operators in the spaces $L_2(\mathbb{R}, s)$ for $s \in (-1/2, 1/2)$.

Denote

$$L_2^\pm(\mathbb{R}, s) = P^\pm(L_2(\mathbb{R}, s)).$$

The operator

$$T(a) := P^+ a P^+ : L_2^+(\mathbb{R}, s) \to L_2^+(\mathbb{R}, s)$$

is called a Toeplitz operator with symbol $a(x)$. If $a \in L_\infty(\mathbb{R})$ then $T(a)$ is a bounded operator on $L_2^+(\mathbb{R}, s)$ (for $s \in (-1/2, 1/2)$) and the conjugate operator $T^*(a) = T(\overline{a})$.

Definition 3 *The operator A acting on Hilbert space is called normally solvable if the subspace* $\mathrm{Im}A$ *is closed, i.e.,* $\mathrm{Im}A = \overline{\mathrm{Im}A}$.

Lemma 1*If the operator A is normally solvable, then the Hilbert space H may be represented as the following direct sum,*

$$\mathrm{Im}A \oplus \ker A^* = H.$$

Definition 4 *An operator A acting in Hilbert space H is called left-(right)- invertible if there exists an operator A_l^{-1} (A_r^{-1}) bounded on H such that $A_l^{-1}A = I$ $(AA_r^{-1} = I)$.*

It should be noted that a one-side invertible operator is normally solvable. Moreover if A is left-invertible, then $\ker A = \{0\}$; if A is right-invertible, then $\mathrm{Im}A = H$.

SECTORIALITY

Definition 5 *A linear bounded operator acting on a Hilbert space H is called a sectorial operator if*

$$\inf_{\|x\|_H=1} \operatorname{Re}(Ax,x) := \varepsilon > 0 \qquad (30)$$

where (Ax,y) denotes the scalar product in H.

If $a(x) \in L_\infty(\mathbb{R})$ then the Toeplitz operator with symbol operator $a(x)$ in the space $L_2(\mathbb{R}, s)$ is sectorial if and only if

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \operatorname{Re} a(x) = \varepsilon > 0. \qquad (31)$$

Definition 6 *We will call a function $a(x) \in L_\infty(\mathbb{R})$ sectorial if exists a number $\theta \in (-\pi, \pi)$ such that for the function $a_\theta(x) := e^{i\theta}a(x)$ the condition (31) holds.*

We formulate the famous result of Brown and Halmos.

Theorem 4 *Let A be a sectorial operator on a Hilbert space H. Then the operator A is invertible and,*

$$\|A^{-1}\|_H \leq 2\varepsilon^{-1}$$

where ε is the value from (30).

Let now G be a subspace of the Hilbert space $L_2^+(\mathbb{R}, s)$ and let $\mathcal{P}_G$ be the orthoprojector onto the space G. This means that an arbitrary function $f(x) \in L_2^+(\mathbb{R}, s)$ can be represented uniquely in the form

$$f(x) = g_1(x) + g_2(x) \tag{32}$$

where $g_1(x) \in G$, $g_2(x) \in G^{\perp}$, and $G^{\perp}$ denotes the orthogonal complement of the space G in $L_2^+(\mathbb{R}, s)$.

$$P^+ = \mathcal{P}_G^{\perp} + \mathcal{P}_G$$

Consider the operator

$$D = \mathcal{P}_G^{\perp} + P^+ a \mathcal{P}_G : L_2^+(\mathbb{R}, s) \to L_2^+(\mathbb{R}, s) \tag{33}$$

where the function a belongs to $L_\infty(\mathbb{R})$.

Theorem 5 *Let function $a(\in L_\infty(\mathbb{R}))$ be sectorial. Then the operator D (33) is invertible and for the solution x of the equation*

$$Dx = f, \quad f \in L_2^+(\mathbb{R}, s), \tag{34}$$

there holds the following estimate,

$$\|x_1\|_{L_2(\mathbb{R},s)} \leq 2\varepsilon^{-1} \|f_1\|_{L_2(\mathbb{R},s)} \tag{35}$$

where $x_1 = \mathcal{P}_G x$, $f_1 = \mathcal{P}_G f$, and ε is the value from (31).

$$e^{-ia\xi}\Phi^-(\xi,w) + a(\xi,w)\Phi_a^-(\xi,w) + \Phi^+(\xi,w) = \hat{g}(\xi) \tag{36}$$

$$\widetilde{\Phi}^\pm(\xi,w) = \Phi^\mp(-\xi,w), \quad \widetilde{\Phi}_a^+(\xi,w) = \Phi_a^-(-\xi,w).$$

Then we can rewrite (36) in the form

$$e^{ia\xi}\widetilde{\Phi}^+(\xi,w) + (1+\xi^2)^{\nu/2}\widetilde{c}(\xi,w)\widetilde{\Phi}_a^+(\xi,w) + \widetilde{\Phi}^-(\xi,w) = \hat{g}(-\xi), \tag{37}$$

$$\widetilde{\Phi}^\pm(\xi,w) \in L_2^\pm(\mathbb{R}, -\nu/2+s); \tag{38}$$

$$\widetilde{\Phi}_a^+(\xi,w) \in L_2^+(\mathbb{R}, \nu/2+s). \tag{39}$$

Consider the so-called Wiener-Hopf factorization of the function $\gamma(\xi) := (1+\xi^2)^{\nu/2}$,

$$\gamma(\xi) = (1+i\xi)^{\nu/2}(1-i\xi)^{\nu/2} := \gamma_-(\xi)\gamma_+(\xi).$$

Divide all terms of (37) by $\gamma_-(\xi)$ and write

$$\Psi_a^+(\xi,w) := \gamma_+(\xi)\widetilde{\Phi}_a^+(\xi,w); \tag{40}$$

$$\Psi^\pm(\xi,w) := \frac{\widetilde{\Phi}^\pm(\xi,w)}{\gamma_\pm(\xi)}. \tag{41}$$

Then we obtain

$$e^{ia\xi}u(\xi)\Psi^{+}(\xi,w)+\widetilde{c}(\xi,w)\Psi^{+}_{a}(\xi,w)+\Psi^{-}(\xi,w)=\frac{\hat{g}(-\xi)}{\gamma_{-}(\xi)} \quad (42)$$

where

$$u(\xi):=\frac{\gamma_{+}(\xi)}{\gamma_{-}(\xi)}=\left(\frac{1-i\xi}{1+i\xi}\right)^{\nu/2}. \quad (43)$$

$$\widetilde{c}(\xi,w)=\frac{a(\xi,w)}{\gamma_{-}(\xi)}$$

It is easy to see that

$$\Psi^{+}_{a}(\xi) \in L^{+}_{2}(\mathbb{R},s); \quad (44)$$

$$\Psi^{\pm}(\xi) \in L^{\pm}_{2}(\mathbb{R},s). \quad (45)$$

Apply the projector P^+ to all terms of equation (42). Then we have

$$(T_{u_a}\Psi^+)(\xi, w) + P^+(\widetilde{c}(\xi, w)\Psi_a^+(\xi, w) = f^+(\xi) \qquad (46)$$

where T_{u_a} is the Toeplitz operator with symbol

$$u_a(\xi) := e^{ia\xi}u(\xi), \qquad (47)$$

and

$$f^+(\xi) = P^+(\hat{g}(-\xi)/\gamma_-(\xi)), \qquad (48)$$

$$\Psi_a^+(\xi, w) \in \ker T_{\overline{u}_a}\Big|_{L_2^+(\mathbb{R},s)}, \qquad (49)$$

$$\Psi^+(\xi, w) \in L_2^+(\mathbb{R}, s). \qquad (50)$$

$$\mathrm{im}T_{u_a} \oplus \mathrm{ker}T_{\overline{u}_a} = L_2^+(\mathbb{R}, s) \quad \text{since} \quad T_{u_a}^* = T_{\overline{u}_a}.$$

$$\mathcal{P}_{u_a}(L_2^+(\mathbb{R}, s)) = \mathrm{ker}T_{\overline{u}_a},$$

$$\mathcal{P}_{u_a}^{\perp}(L_2^+(\mathbb{R}, s)) = \mathrm{im}T_{u_a}$$

$$D_{u_a} := \mathcal{P}_{u_a}^{\perp} + P^{+}\widetilde{c}(\xi, w)\mathcal{P}_{u_a} : L_2^{+}(\mathbb{R}, s) \to L_2^{+}(\mathbb{R}, s). \tag{51}$$

$$\psi^{\mathbf{Q}}(\xi) = \frac{1}{2}\sigma^2\xi^2 - i\mu\xi + \varphi(\xi). \tag{52}$$

$$\sigma = 0, \tag{53}$$

$$c(\xi) := \frac{\varphi(\xi)}{(1+\xi^2)^{\nu/2}} \in L_{\infty}(\mathbb{R}), \tag{54}$$

for some $M > 0$ satisfies

$$\inf_{|\xi| \geq M} \operatorname{Re} \widetilde{c}(\xi) = \varepsilon_1 > 0, \tag{55}$$

$$\begin{cases} \text{if} \quad \mu \neq 0 \text{ then } \quad 1 < \nu < 2, \\ \text{if} \quad \mu = 0 \text{ then } \quad 0 < \nu < 2. \end{cases} \tag{56}$$

$$r + \operatorname{Re} w \geq \varepsilon_2 > 0. \tag{57}$$

Lemma 2 *Let the conditions (52)-(57) hold. Then the function $\widetilde{c}(\cdot, \xi)$ is sectorial, and if the value ε_2 in (57) is independent of w then there exists a number ε indepen nt of w such that*

$$\inf_{\xi \in \mathbb{R}} \operatorname{Re} c(\xi, w) \geq \varepsilon > 0. \tag{58}$$

Theorem 6 *Let the function $\widetilde{c}(\xi, w)$ satisfy conditions (52)-(56) and w satisfies (57). Then the following statements are true:*

i) The operator D_{u_a} (51) is invertible

ii)

The problem (37), has the unique solution

$$\widetilde{\Phi}^+_a(\xi, w) = \frac{1}{\gamma^+(\xi)}\left(\mathcal{P}_{u_a} D^{-1}_{u_a} P^+ \frac{\hat{g}(-\xi)}{\gamma_-(\xi)}\right); \qquad (59)$$

$$\widetilde{\Phi}^+(\xi, w) = \gamma^+(\xi)\left(T^{-1}_{u_a}\mathcal{P}^{\perp}_{u_a} D^{-1}_{u_a} P^+ \frac{\hat{g}(-\xi)}{\gamma_-(\xi)}\right); \qquad (60)$$

$$\widetilde{\Phi}^-(\xi, w) = \gamma_-(\xi)\Psi^-(\xi, w); \qquad (61)$$

$$\mathcal{U}(x,t) = \int_{L_{\theta_0}}\int_{-\infty}^{\infty} \widetilde{\varphi}^+_a(\xi, w) e^{i\xi x + (T-t)w} d\xi dw \qquad (62)$$

$$\mathcal{U}(x, T-t) \in C^0([0,\infty), H^{\frac{\nu}{2}+s}(0,a)), \quad |s| < 1/2.$$

This means that for each fixed $\tau \leq 0$ $\;u(\cdot, \tau) \in H^{\frac{\nu}{2}+s}(0, a)$, and the function $F(\tau) := ||u(\cdot,\tau)||_{H^{\frac{\nu}{2}+s}}$ is continuous on $[0, \infty)$ with $\lim_{\tau\to\infty} F(\tau) = 0$.

Theorem 7 *Let $\nu \in (0,2)$, let the function $g(x) \in H^{-\frac{\nu}{2}+s}(0,a)$, for some $s \in (-1/2, 1/2)$ and let the be characteristic exponent under a $EMM \quad Q$, the function $\psi^Q(\xi)$, such that the symbol $\widetilde{c}(\xi, w)$ satisfies the conditions (52)-(56).*

Then the problem (4)-(6) has a unique solution in the space $C^0([0,\infty), H^{\frac{\nu}{2}+s)}(0,a))$ and this solution has the form (62).

$$\frac{\nu}{2} + s > \frac{1}{2} \tag{63}$$

It is well known that in this case

$$H^{\frac{\nu}{2}+s}(0,a) \subset C[0,a] \tag{64}$$

Theorem 8 *Let all conditions of Theorem 7 are hold and inequality (63) hold. Then the solution of the problem (4)-(6) is bounded.*

Finally suppose that $g(x)$ is a piecewise smooth function on the segment $[0,a]$. It is easy to see that in this case $g(x) \in H^{\mu}(0,a)$ for any $\mu < \frac{1}{2}$. For arbitrary $\frac{\nu}{2} \in (0,1)$ we always can choose $s \in [0, 1/2)$ such that condition (6.9) holds and moreover we have

$$\mu = -\frac{\nu}{2} + s < \frac{1}{2}.$$

Thus in this case according to Theorem 8 the problems (4)-(6) have bounded solutions. Since the Theorem 7 implies that this solution is unique, it has the form (62).

Conclusion

In this report we treat some power cases of characteristic functions. These cases involve wide classes of Lévy processes which are used in option theory. However, there exist many other cases which could be considered with the help of the methods worked out in this article.

1. The case $\sigma > 0$ is important because it corresponds to the processes with non trivial Gaussian components. This case can be realized as the case $\nu < 2$ considered in these notes.

2. The case $\sigma = 0$, $\mu \neq 0$ and $0 < \nu < 1$.

3. Logarithmic cases,

4. Power logarithmic case,

5. Rational case. In this case not only the solvability theory can worked out but one can obtain the solution in explicit form.

6. Periodic case. The Poisson process generates a periodic characteristic function. It is interesting to get explicit formulae and to analyze them in this case. It is very interesting also because here X_t is sum of a Gaussian process and a discrete-jumping process. In this area the theory of matrix Toeplitz operators with periodic and almost periodic symbols (worked out by Karlovich-Spitkovsky-Böttcher) could be applied.