Cremona Transformations Based at Eight or Fewer Points

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Abstract

Cremona Transformations are birational automorphisms of $\mathbb{P}^2$. After presenting the basic concepts and properties about Blowups, we introduce the Quadratic plane Cremona Map and show it corresponds to a blowup of three points and a blowdown of three lines. Then we present three families of curves: the Cremona Lines, Minus One Curves and Zero Curves and show how a curve in any of these families behaves under the Quadratic Cremona Transformation. Using the properties found, we classify all the Cremona Lines and Minus One Curves through eight or fewer points. Finally, we discuss the relation between Cremona Lines and Minus One Curves and how each Cremona Line induces a Cremona Transformation.

0 Basic Definitions

Definition 0.1. *A **projective algebraic plane curve** is defined by a homogeneous polynomial in* $\mathbb{P}^2$

$$F(x, y, z) = 0.$$

*The **degree** of such a curve is the degree of the monomials, which is the same for all of them by homogeneity.*

Remark. *Given a projective algebraic curve of degree d, we can obtain the corresponding affine algebraic curve $f(x, y) = 0$ (polynomial in the variables x and y) of degree at most d (the maximum degree of of its terms) by:*

$$f(x, y) = F(x, y, 1)$$

Definition 0.2. *Given a point $p = [x_0 : y_0 : z_0]$ with $z_0 \neq 0$, without loss of generality we can assume $z_0 = 1$, so that its affine coordinates are $p = (x_0, y_0)$. We then say that p is a* ***singularity*** *of the curve C defined by $f(x, y) = 0$ if*

$$\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = f(p) = 0$$

If we translate p to the origin, this would be equivalent to $\tilde{f}$ (the translated curve) not having linear or constant terms.

Definition 0.3. *The previous concept can be extended in the following way: given a point p with affine coordinates (x_0, y_0), if*

$$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(p) = 0 \quad \text{for } i + j < m \quad \text{and}$$
$$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(p) \neq 0 \quad \text{for some i,j with } i + j = m$$

then we say that m is ***the multiplicity of the point p****.*

Notation. *We will use the notation:*

$$mult_p(C) = m.$$

Definition 0.4. *We say that the multiplicity of a point on a curve is* ***at least m****, $mult_p(C) \geq m$, if and only if*

$$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(p) = 0 \quad \text{for } i + j < m$$

Remark. *If we change the variables in $f(x, y)$ so that p is at the origin:*

$$\tilde{f}(u, v) = f(u + x_0, v + y_0)$$

the we have that by Taylor's formula (from [4]):

$$\tilde{f}(u, v) = f(x_0, y_0) + \sum_{i=1}^{d} \frac{1}{i!} \left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} \right)^i (x_0, y_0)$$

Then if $mult_p(C) = m$ we have that:

$$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(p) = 0 \quad \text{for } i + j < m$$

which implies that m is ***the minimum degree of a non zero term of*** *$\tilde{f}$.*

Remark. *Notice that by the definition of a singularity, a point is singular if its multiplicity is at least 2. However, if we want to study curves with a fixed maximum number of (singular) points, we will accept* 0 *or* 1 *as a possible value for the multiplicity of a point, even though in that case, the point is not a singularity.*

1 Blowups

Given a projective line $\mathbb{P}^1$ with homogeneous coordinates $[Y_1 : Y_2]$ we can think of it as a union of two affine lines in the following way:

$$\mathbb{P}^1 = \mathbb{A}^1_u \cup \mathbb{A}^1_v \quad \text{where} \quad u = \frac{Y_1}{Y_2} \text{ and } v = \frac{Y_2}{Y_1}$$

Definition 1.1. *Given the affine plane* $\mathbb{A}^2$ *in the variables* (x_1, x_2)*, we define the* ***Blowup Space*** *as the set:*

$$B = \{(x_1, x_2), [Y_1 : Y_2])|\, x_1 Y_2 = x_2 Y_1\} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

If we consider the projective line as a union of affine lines (as described before), we then get:

$$B \subset \{\mathbb{A}^2_{x_1,x_2} \times \mathbb{A}^1_u|\, x_1 = x_2 u\} \cup \{\mathbb{A}^2_{x_1,x_2} \times \mathbb{A}^1_v|\, x_1 v = x_2\}$$

We also have that:

$$\{\mathbb{A}^2_{x_1,x_2} \times \mathbb{A}^1_u|\, x_1 = x_2 u\} \cong \mathbb{A}^2_{x_2,u} = \mathbb{A}^2_{x_2,\frac{x_1}{x_2}}$$
$$\{\mathbb{A}^2_{x_1,x_2} \times \mathbb{A}^1_v|\, x_1 v = x_2\} \cong \mathbb{A}^2_{x_1,v} = \mathbb{A}^2_{x_1,\frac{x_2}{x_1}}.$$

Then:

$$B \subset \mathbb{A}^2_{x_2,u} \cup \mathbb{A}^2_{x_1,v} \quad \textit{where} \quad x_2 = x_1 v \quad \textit{and } u = \frac{1}{v}$$

Definition 1.2. *We define the* ***Exceptional Divisor*** *or* ***Exceptional Curve*** *as the subset:*

$$E = \{(0, u)\} \cup \{(0, v)\} \cong \{(0, 0)\} \times \mathbb{P}^1 \subset B$$

We can now consider the first projection map from B to $\mathbb{A}^2$ in terms of the coordinates x_1, x_2, u, v,:

$$\begin{aligned} \sigma :\ B &\longrightarrow \mathbb{A}^2_{x_1,x_2} \\ (x_2, u) &\mapsto (x_2 u, x_2) \\ (x_1, v) &\mapsto (x_1, x_1 v) \end{aligned}$$

Blowup of the origin: We have that given the origin $(0,0) \in \mathbb{A}^2_{x_1,x_2}$,

$$\begin{aligned}\sigma^{-1}(0,0) &= \{\{(0,0)\} \times \mathbb{P}^1\} \\ &= E\end{aligned}$$

We say that the point $(0,0) \in \mathbb{A}^2_{x_1,x_2}$ is **"blown up"** (to the Exceptional Curve), by σ^{-1}. That is why we call such a transformation a "blow up".

Generic points: Given any point $(x_1,x_2) \neq (0,0)$ we have:

$$\begin{aligned}\sigma^{-1}(x_1,x_2) &= \{(x_2,u)|\, x_2 \neq 0\} \cup \{(x_1,v)|\, x_1 \neq 0\} \\ &= ((x_1,x_2),[x_1 : x_2])\end{aligned}$$

which is a single point in B. Thus, we have that σ is an isomorphism from $B - E$ to $\mathbb{A}^2_{x_1,x_2} - \{(0,0)\}$.

Curve through the origin: Consider a curve $C \subset \mathbb{A}^2_{x_1,x_2}$ through the origin, defined by $f(x_1,x_2) = 0$ and suppose $mult_{(0,0)}(C) = m$. Then by Definition 0.4 we have that:

$$f(x_1,x_2) = f_m(x_1,x_2) + f_{m+1}(x_1,x_2) + \cdots$$

Since the curve contains the origin, then it's clear that $E \subset \sigma^{-1}(C)$, so $\sigma^{-1}(C)$ decomposes as a union of the Exceptional curve (with multiplicity m) and a new curve $\bar{C}$ in the following way:

$$\begin{aligned}\sigma^{-1}(C) &= \{(x_2,u)|\, f(x_2u,x_2) = 0\} \cup \{(x_1,v)|\, f(x_1,x_1v) = 0\} \\ &= \{(x_2,u)|\, f_m(x_2u,x_2) + f_{m+1}(x_2u,x_2) + \cdots = 0\} \\ &\quad \cup \{(x_1,v)|\,, f_m(x_1,x_1v) + f_{m+1}(x_1,x_1v) + \cdots = 0\}\end{aligned}$$

If we look at the term in the (x_2,u) chart (the (x_1,v) chart is similar), we have:

$$\begin{aligned}& f_m(x_2u,x_2) + f_{m+1}(x_2u,x_2) + \cdots = 0 \\ \Longrightarrow\ & x_2^m f_m(u,1) + x_2^{m+1} f_{m+1}(u,1) + \cdots = 0 \\ \Longrightarrow\ & x_2^m[f_m(u,1) + x_2 f_{m+1}(u,1) + \cdots] = 0\end{aligned}$$

so we have:

$$\begin{aligned}\sigma^{-1}(C) &= \{(x_2,u) \cup (x_1,v)|\, x_2^m[f_m(u,1) + x_2 f_{m+1}(u,1) + \cdots] = 0 \\ &\qquad \text{and } x_1^m[f_m(1,v) + x_1 f_{m+1}(1,v) + \cdots] = 0\} \\ &= \{(x_2,u) \cup (x_1,v)|\, (x_1x_2)^m[f_m(u,v) + x_1x_2 f_{m+1}(u,v) + \cdots] = 0\} \\ &= m\{\{(0,u)\} \cup \{(0,v)\}\} \cup \{[f_m(u,v) + x_1x_2 f_{m+1}(u,v) + \cdots] = 0\} \\ &= mE + \bar{C} \quad (\text{where } \bar{C} \text{ is defined by } [f_m(u,v) + x_1x_2 f_{m+1}(u,v) + \cdots] = 0)\end{aligned}$$

Remark. *If we start with curves or lines in the Blowup space, then we can always "blow them down" just by using the projection.*

A more in depth analysis of blowups can be found in [3].

2 Introduction to Quadratic Cremona Transformations

Definition 2.1. *Given curves of degree d in the plane and $p_1, \ldots p_n$ general points with multiplicities $m_1, \ldots, m_n$ respectively, if*

$$d^2 - \sum m_i^2 = 1$$
$$3d - \sum m_i = 3$$

Then the curves of such type are called ***Cremona Lines.***

Example 2.2. *The following are examples of Cremona Lines:*
A curve of degree $d = 5$ with 6 points of multiplicity 2.

A curve of degree $d = 13$ with 6 points of multiplicity 2 and 4 points of multiplicity 6.

A curve of degree $d = 16$ with 3 points of multiplicity 1 and 7 points of multiplicity 6.

Definition 2.3. *Given curves of degree d in the plane and $p_1, \ldots p_n$ general points with multiplicities $m_1, \ldots, m_n$ respectively, if*

$$d^2 - \sum m_i^2 = -1$$
$$3d - \sum m_i = 1$$

Then the curves of such type are called ***Minus One Curves.***

Definition 2.4. *Given curves of degree d in the plane and $p_1, \ldots p_n$ general points with multiplicities $m_1, \ldots, m_n$ respectively, if*

$$d^2 - \sum m_i^2 = 0$$
$$3d - \sum m_i = 2$$

*Then the curves of such type are called **Zero Curves.***

Remark. *In the previous definitions, the points we choose can have any multiplicity (including zero and one), so we are not necessarily looking at singularities of the curves. Also, we can choose any number of points on each curve so we are not always including all the singularities of the curve.*

Definition 2.5. *The **Self-intersection** of a curve of degree d passing through the points $p_1, \ldots, p_n$ with corresponding multiplicities $m_1, \ldots, m_n$, is defined by:*

$$C^2 = d^2 - \sum_{i=1}^{n} m_i^2.$$

Remark. *This implies that $d^2 \geq C^2$.*

Definition 2.6. *The **genus** of such a curve is defined by:*

$$2g - 2 = C^2 - (3d - \sum_{i=1}^{n} m_i)$$

Remark. *The notions of self-intersection and genus defined here depend on the points we choose and their corresponding multiplicities.*

Example 2.7.
For the Zero Curves, we have: $C^2 = 0 \quad g = 0$.
For the Minus One Curves, we have: $C^2 = -1, \quad g = 0$.
For the Cremona Lines, we have: $C^2 = 1, \quad g = 0$.

Definition 2.8. *We define the **Quadratic plane Cremona Map**:*

$$\begin{aligned} \phi : \mathbb{P}^2 &\longrightarrow \mathbb{P}^2 \\ [x : y : z] &\mapsto [yz : xz : xy] \end{aligned}$$

Remark.

- *ϕ is not defined at $[1 : 0 : 0], \quad [0 : 1 : 0], \quad [0 : 0 : 1]$.*
- *ϕ contracts the*

$$\begin{aligned} x = 0 &\quad \textit{line to } [1 : 0 : 0] \\ y = 0 &\quad \textit{line to } [0 : 1 : 0] \\ z = 0 &\quad \textit{line to } [0 : 0 : 1] \end{aligned}$$

Away from these lines ϕ is an isomorphism and we have the following property:

Claim 2.9. $\phi^{-1} = \phi$.

Proof.

$$\begin{aligned}\phi \circ \phi([x:y:z]) &= [xzxy : yzxy : yzxz] \\ &= [xyz(x) : xyz(y) : xyz(z)] \\ &= [x:y:z]\end{aligned}$$

□

Given the projective plane $\mathbb{P}^2$ with homogeneous coordinates $[x:y:z]$ we can think of it as a union of three affine planes in the following way:

$$\mathbb{P}^2 = \mathbb{A}^2_{u,\frac{1}{w}} \cup \mathbb{A}^2_{v,\frac{1}{u}} \cup \mathbb{A}^2_{w,\frac{1}{v}} \quad \text{where} \quad u = \frac{y}{x} \quad v = \frac{z}{y} \quad w = \frac{x}{z}$$

If we look at the three coordinate points in $\mathbb{P}^2$, we can see that:

$$\begin{aligned}&[1:0:0] \text{ is the origin for } \mathbb{A}^2_{u,\frac{1}{w}} \\ &[0:1:0] \text{ is the origin in } \mathbb{A}^2_{v,\frac{1}{u}} \\ &[0:0:1] \text{ is the origin in } \mathbb{A}^2_{w,\frac{1}{v}}\end{aligned}$$

If we blow up each of the origins the same way we did in Section 1, we then get that in the Blowup Space, each of the three affine planes will be replaced by two affine planes:

$$\mathbb{A}^2_{u,\frac{1}{w}} \text{ replaced by } \begin{cases} \mathbb{A}^2_{\frac{1}{w},\alpha} \\ \mathbb{A}^2_{u,\frac{1}{\alpha}} \end{cases} \qquad \text{where } u = \frac{\alpha}{w}$$

$$\mathbb{A}^2_{v,\frac{1}{u}} \text{ replaced by } \begin{cases} \mathbb{A}^2_{\frac{1}{u},\beta} \\ \mathbb{A}^2_{v,\frac{1}{\beta}} \end{cases} \qquad \text{where } v = \frac{\beta}{u}$$

$$\mathbb{A}^2_{w,\frac{1}{v}} \text{ replaced by } \begin{cases} \mathbb{A}^2_{\frac{1}{v},\gamma} \\ \mathbb{A}^2_{w,\frac{1}{\gamma}} \end{cases} \qquad \text{where } w = \frac{\gamma}{v}$$

If we reorganize these six affine planes into the following pairs:

$$\mathbb{A}^2_{v,\frac{1}{\beta}} \cup \mathbb{A}^2_{\frac{1}{v},\gamma}; \quad \mathbb{A}^2_{w,\frac{1}{\gamma}} \cup \mathbb{A}^2_{\frac{1}{w},\alpha}; \quad \mathbb{A}^2_{u,\frac{1}{\alpha}} \cup \mathbb{A}^2_{\frac{1}{u},\beta}$$

we can then blow each of these pairs down and we obtain the following affine planes:

$$\mathbb{A}^2_{\gamma,\frac{1}{\beta}} = \mathbb{A}^2_{wv,\frac{1}{uv}} = \mathbb{A}^2_{\frac{xz}{yz},\frac{xy}{yz}}$$
$$\mathbb{A}^2_{\alpha,\frac{1}{\gamma}} = \mathbb{A}^2_{uw,\frac{1}{wv}} = \mathbb{A}^2_{\frac{xy}{xz},\frac{yz}{xz}}$$
$$\mathbb{A}^2_{\beta,\frac{1}{\alpha}} = \mathbb{A}^2_{uv,\frac{1}{uw}} = \mathbb{A}^2_{\frac{yz}{xy},\frac{xz}{xy}}$$

These three affine planes correspond to a projective plane $\mathbb{P}^2$ with homogeneous coordinates $[yz : xz : xy]$.

Proposition 3. *The Quadratic plane Cremona Map ϕ is a "blow up" of the three coordinate points and a "blow down" of the three coordinate lines.*

Proof. The result is obvious by the Definition 2.8 and the properties that follow it. □

Suppose $F(x, y, z) = 0$ is a curve C of degree d and the multiplicity of the point $[0 : 0 : 1]$ is greater or equal to m_1. That is:

$$mult_{[0:0:1]}(C) \geq m_1.$$

This means that there cannot be nonzero monomials in F of the form

$$x^i y^j z^k \quad \text{with} \quad i + j < m_1.$$

We get that

$$\begin{aligned} mult_{[0:0:1]}(C) \geq 1 &\iff \text{there is no pure } z^d \text{ term} \\ &\iff \text{there is no constant term in the affine equation of } C \\ &\quad\ \text{using the variables } (x, y) \\ &\iff C \text{ passes through the point } (0, 0) \text{ in the affine plane } z = 0. \end{aligned}$$

Also,

$$\begin{aligned} mult_{[0:0:1]}(C) \geq 2 &\iff \text{there cannot be terms of the form } z^d, \quad xz^{d-1}, \quad yz^{d-1}. \\ &\iff \text{there are no constant or linear terms in the affine equation of } C \\ &\quad\ \text{using the variables } (x, y). \end{aligned}$$

The same property holds for the points $[0:1:0]$ and $[1:0:0]$, so we have that if:

$$mult_{[0:0:1]}(C) \geq m_1$$
$$mult_{[0:1:0]}(C) \geq m_2$$
$$mult_{[1:0:0]}(C) \geq m_3$$

then all the nonzero terms in $F: \ x^i y^j z^k$ have:

$$i + j \geq m_1$$
$$i + k \geq m_2$$
$$j + k \geq m_3.$$

Since the curve has degree d then we have that $i + j + k = d$ for each monomial $x^i y^j z^k$. If we apply the Cremona Transformation, we have that:

$$x^i y^j z^k \mapsto (yz)^i (xz)^j (xy)^k = x^{j+k} y^{i+k} z^{i+j}.$$

The degree of each of these monomials is

$$j + k + i + k + i + j = 2(i + j + k) = 2d.$$

Let C^{cr} be the new curve obtained from the transformation, corresponing to F^{cr}. Since every term has $x^{m_3} y^{m_2} z^{m_1}$ as a factor, we can factor this last term out and we get new monomials of the form:

$$\frac{x^{j+k} y^{i+k} z^{i+j}}{x^{m_3} y^{m_2} z^{m_1}} = x^{j+k-m_3} y^{i+k-m_2} z^{i+j-m_1}$$

The degree of F^{cr} becomes then

$$2d - (m_1 + m_2 + m_3)$$

If we look at the sum of the power of x and the power of y we get that:

$$j + k - m_3 + i + k - m_2 = k + (d - m_3 - m_2) \geq d - m_3 - m_2$$
$$j + k - m_3 + i + j - m_1 = j + (d - m_3 - m_1) \geq d - m_3 - m_1$$
$$i + k - m_2 + i + j - m_1 = i + (d - m_2 - m_1) \geq d - m_2 - m_1$$

This implies that:

$$mult_{[0:0:1]}(C^{cr}) \geq d - m_3 - m_2.$$
$$mult_{[0:1:0]}(C^{cr}) \geq d - m_3 - m_1.$$
$$mult_{[1:0:0]}(C^{cr}) \geq d - m_2 - m_1.$$

If we now apply the Cremona transformation to any points p_i, p_j, p_k of C with multiplicities m_i, m_j, m_k, then we have that the new curve C^{cr} has degree:

$$2d - (m_i + m_j + m_k).$$

Let $s = m_i + m_j + m_k$ and $t = s - d$. Then the degree of the new curve is $2d - s = d - t$ and the multiplicities of the points of C^{cr} are now:

$$m_l^* = m_l - t \qquad for \quad l = i, j, k.$$
$$m_l^* = m_l \qquad for \quad l \neq i, j, k.$$

Remark. *Since we are working in $\mathbb{P}^2$ we can extend the properties of the Cremona Transformation to any 3-tuple of points in a curve (since we can bring them to the coordinate points). We can then think as the Cremona Transformation as a "blowup" of the three points and a "blowdown" of three lines that did not originally intercept the curve.*

4 Reduction of Degree Theorems

Remark. *Throughout this section, when refering to a curve C of degree d we will assume that it goes through n points.*

Lemma 4.1. *Given a curve C of degree d, let $s = m_i + m_j + m_k$, where m_i, m_j, m_k are the multiplicities of the points p_i, p_j, p_k and $t = s - d$. Then if we apply the Cremona Transformation:*

$$d^* \longleftarrow d - t$$
$$m_l^* \longleftarrow m_l - t \quad for \quad l = i, j, k.$$
$$m_l^* \longleftarrow m_l \quad for \quad l \neq i, j, k.$$

Then we have that:

$$d^{*^2} - \sum m_l^{*^2} = d^2 - \sum m_l^2$$
$$3d^* - \sum m_l^* = 3d - \sum m_l$$

Proof.

$$\begin{aligned}
d^{*^2} - \sum m_l^{*^2} &= (d-t)^2 - \sum_{l\neq i,j,k} m_l^2 - (m_i - t)^2 - (m_j - t)^2 - (m_k - t)^2 \\
&= d^2 - 2dt + t^2 - \sum_{l\neq i,j,k} m_l^2 - m_i^2 + 2m_i t - m_j^2 \\
&\qquad + 2m_j t - m_k^2 + 2m_k t - 3t^2 \\
&= d^2 - \sum m_l^2 - (2t^2 + 2dt - 2m_i t - 2m_j t - 2m_k t) \\
&= d^2 - \sum m_l^2 - 2t(t + d - m_i - m_j - m_k) \\
&= d^2 - \sum m_l^2 - 2t(s - d + d - s) \\
&= d^2 - \sum m_l^2
\end{aligned}$$

$$\begin{aligned}
3d^* - \sum m_l^* &= 3(d-t) - \sum m_l + 3t \\
&= 3d - \sum m_l
\end{aligned}$$

□

Corollary 4.2. *Cremona Lines, Zero Curves and Minus One Curves are preserved by such transformation.*

The following theorem is based on the the analysis found in [1].

Theorem 4.3 (General Degree Reduction for Genus Zero). *Given a curve C of degree $d \geq 1$ with self intersection C^2 and genus $g = 0$, if $m_3 \geq 1$, then*

$$\exists\, m_i, m_j, m_k \in \{m_1, \ldots, m_n\}$$
$$\text{such that } m_i + m_j + m_k > d.$$

Proof. Without loss of generality, we can suppose the elements $m_1, \ldots, m_n$ are in decreasing order.

Since

$$\sum m_i = 3d + 2g - 2 - C^2 \Longrightarrow \sum m_i m_3 = 3dm_3 - 2m_3 - C^2 m_3.$$

Also,

$$\sum m_i^2 = d^2 - C^2.$$

Subtracting this expression from the previous one, we get:

$$\sum m_i^2 - \sum m_i m_3 = d^2 - C^2 - 3dm_3 + 2m_3 + C^2 m_3$$
$$\Longrightarrow -\sum m_i^2 + \sum m_i m_3 + d^2 - C^2 - 3dm_3 + 2m_3 + C^2 m_3 = 0$$

LHS:

$$-m_1^2 - m_2^2 - \sum_{i\geq 4} m_i^2 + m_1 m_3 + m_2 m_3 + \sum_{i\geq 4} m_i m_3 + d^2$$
$$-C^2 - 3dm_3 + 2m_3 + C^2 m_3$$

Adding $(m_1 + m_2 + m_3 - d)(d + m_1 + m_2 - 2m_3)$

$$= m_1^2 + 2m_1 m_2 - m_1 m_3 + m_2^2 - m_2 m_3 - d^2 - 2m_3^2 + 3dm_3$$

on both sides of the equation, we get:

LHS:

$$\sum_{i\geq 4} m_i(m_3 - m_i^2) + 2m_1 m_2 - 2m_3^2 - C^2 + 2m_3 + C^2 m_3$$
$$= \sum_{i\geq 4} m_i(m_3 - m_i^2) + 2(m_1 m_2 - m_3^2) + 2m_3 + C^2(m_3 - 1)$$

RHS:

$$(m_1 + m_2 + m_3 - d)(d + m_1 + m_2 - 2m_3)$$

Since $m_1 \geq m_2 \geq \ldots \geq m_n$, we have that

$$2(m_1 m_2 - m_3^2) \geq 0$$
$$\text{and } m_1 + m_2 - 2m_3 \geq 0$$
$$(m_3 - m_i) \geq 0 \quad \text{for } i \geq 4$$
$$\Longrightarrow \sum_{i\geq 4} m_i(m_3 - m_i^2) \geq 0 \quad \text{for } i \geq 4$$

If $m_3 \geq 1$ then

$$C^2(m_3 - 1) \geq 0$$
$$2m_3 > 0.$$

Finally,

$$d \geq 1 \Longrightarrow (d + m_1 + m_2 - 2m_3) > 0$$

Then we have that

$$m_1 + m_2 + m_3 - d > 0,$$

which implies:

$$m_1 + m_2 + m_3 > d.$$

□

Corollary 4.4 (Noether's Inequality). *Given a Cremona Line of degree* $d \geq 2 \quad \exists\, m_i, m_j, m_k \in \{m_1, \ldots, m_n\}$ *such that* $m_i + m_j + m_k > d$.

Proof. For a Cremona Line the self intersection is 1. However, in order to have $m_3 \geq 1$ we need

$$d^2 \geq 1 + 3 \Longrightarrow d \geq 2$$

□

Corollary 4.5 (Minus One Curves). *Given a Minus One curve of degree* $d \geq 2\ \exists\, m_i, m_j, m_k \in \{m_1, \ldots, m_n\}$ *such that* $m_i + m_j + m_k > d$.

Proof. For a Minus One Curve the self intersection is -1. However, in order to have $m_3 \geq 1$ we need

$$d^2 \geq -1 + 3 \Longrightarrow d \geq 2$$

□

Corollary 4.6 (Zero-Curves). *Given a Zero curve of degree* $d \geq 2$ $\exists\, m_i, m_j, m_k \in \{m_1, \ldots, m_n\}$ *such that* $m_i + m_j + m_k > d$.

Proof. In order to have $m_3 \geq 1$ we need

$$d^2 \geq 3 \Longrightarrow d \geq 2$$

□

Remark. *The fact that this process can be applied to any curve that follows the "numerical" conditions on d and the point multiplicities does not guarantee that we'll obtain a single curve at each step. We need to make sure that at each step* $m_1 + m_2 \leq d$ *(otherwise the line through the corresponding points would split). This is not necessarily obtained by the numerical properties.*

Example 4.7. *Consider the following curve of degree* 5 *through eight points with multiplicities:*

$$m_1 = m_2 = 3, \quad m_3 = m_4 = \ldots = m_8 = 1.$$

Then we have:

$$\begin{aligned} d^2 - \sum m_i^2 &= 25 - (9+9+6) \\ &= 1 \\ 3d - \sum m_i &= 15 - (3+3+6) \\ &= 3 \end{aligned}$$

but $m_1 + m_2 = 6 > d$*.*

Remark. *In spite of the previous observation, if we start with an actual irreducible curve* C*, the self inverse properties of the transformation guarantee that we get an irreducible curve at each step.*

Given a curve C of degree d such that $m_3 \geq 1$ Theorem 4.3 guarantees that we can find three points p_i, p_j, p_k with multiplicities m_i, m_j, m_k such that

$$s = m_i + m_j + m_k > d$$

and then we have that

$$t = s - d > 0.$$

By Lemma 4.1 we can then get a new curve of the same type of degree:

$$d^* = d - t < d \quad (\text{since } t > 0)$$

If we repeat this process enough times we can always transform the curve C down to a curve of the same type of degree 1.

- For the Cremona Lines, every such curve can be reduced to the curve with degree 1 and point multiplicities $m_1, \ldots, m_n = 0$ which is a line. For that reason Cremona Lines are sometimes called Pseudo-Lines.
- For the Minus One Curves, we can reduce any such curve to the curve with degree 1 and point multiplicities $m_1 = m_2 = 1 \quad m_3, \ldots, m_n = 0$.
- For the Zero Curves, we can reduce any curve of that type to one with degree 1 and point multiplicities $m_1 = 1 \quad m_2, \ldots, m_n = 0$.

Notation. *From now on, when referring to a curve of degree* d *and point multiplicities* $m_1, ..., m_n$*, we will use the notation:*

$$[d : m_1, m_1, \ldots, m_n]$$

This process can also be reversed. If we start with one of the curves of degree 1 we can always choose points p_i, p_j, p_k such that $s = m_i + m_j + m_k < d$. Then as we apply the transformation described in lemma 4.1 we have that $t = s - d < 0$ which implies the new curve has degree $d^* = d - t > d$.

Example 4.8. *If we start with the Cremona Line of degree* 1 *and fix the number of points to* 4*:*

$$[1 : 0, 0, 0, 0]$$

no matter which 3 *points we choose, we get* $s = 0$*. Without loss of generality, we will choose the first* 3*. Then the new curve obtained from the transformation defined in lemma 4.1 is:*

$$[2 : 1, 1, 1, 0].$$

If we look at different 3-tuples, we can then attempt to generate every possible curve in the same family, for any given fixed number of points. In the next section, we will study this problem for the families of Minus One Curves and Cremona Lines.

5 Classification of Curves

Lemma 5.1. *Given a curve* C *of degree* d *through* 9 *or more points with* $3d - \sum_{i=1}^{n} m_i \geq 1$*, then we have that there exists*

$$m_i, m_j, m_k \in \{m_1, \ldots, m_n\} \quad (n \geq 9)$$

$$\textit{such that } m_i + m_j + m_k < d.$$

Proof. Suppose $m_i + m_j + m_k \geq d$ for every i, j, $k \in \{m_1, \ldots, m_n\}$. Then we have that:

$$\sum_{i=1}^{n} m_i \geq 3d$$

$$\implies 3d - \sum_{i=1}^{n} m_i \leq 3d - 3d$$

$$\implies 1 \leq 3d - \sum_{i=1}^{n} m_i \leq 0$$

which is a contradiction! □

Corollary 5.2. *We have that the previous result holds for Minus One Curves and Cremona Lines.*

We then have that given a Minus One Curve (or a Cremona Line) through 9 or more points, we can always find 3 points that will transform the curve (through a Cremona Transformation) into a new curve of higher degree. This implies that there is an infinite number of such curves.

If we fix the number of points in the curves to be at most 8, we get a finite number of curves and we can explicitely find all of them by applying the Cremona Transformation to all the possible 3-tuples of points. This is shown in the next two theorems.

Theorem 5.3 (Classification of Minus One Curves). *There are* **240** *Minus One Curves through 8 points, divided into* **7** *different types.*

There are **56** *Minus One Curves through 7 points, divided into* **4** *different types.*

There are **27** *Minus One Curves through 6 points, divided into* **3** *different types.*

There are **16** *Minus One Curves through 5 points, divided into* **3** *different types.*

There are **10** *Minus One Curves through 4 points, divided into* **2** *different types.*

There are **6** *Minus One Curves through 3 points, divided into* **2** *different types.*

There are **3** *Minus One Curves through 2 points, divided into* **2** *different types.*

There is **1** *Minus One Curve through 1 point: the exceptional curve through that point.*

Proof. The following are the different types, the number of curves in each type for the different number of points and given any possible combination of 3-tuples of multiplicities, which type of curve it leads to when we apply the Cremona Transformation:

	Type **No. of Points**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
0	Exceptional Curve $[0 : -1, 0, 0, 0, 0, 0, 0, 0]$	8	7	6	5	4	3	2	1
	$-100 \to \mathbf{1}$								
1	$[1 : 1, 1, 0, 0, 0, 0, 0, 0]$	28	21	15	10	6	3	1	0
	$000 \to \mathbf{2}$ $100 \to \mathbf{1}$ $110 \to \mathbf{0}$								
2	$[2 : 1, 1, 1, 1, 1, 0, 0, 0]$	56	21	6	1	0	0	0	0
	$000 \to \mathbf{4}$ $100 \to \mathbf{3}$ $110 \to \mathbf{2}$ $111 \to \mathbf{1}$								
3	$[3 : 2, 1, 1, 1, 1, 1, 1, 0]$	56	7	0	0	0	0	0	0
	$110 \to \mathbf{4}$ $111 \to \mathbf{3}$ $210 \to \mathbf{3}$ $211 \to \mathbf{2}$								
4	$[4 : 2, 2, 2, 1, 1, 1, 1, 1]$	56	0	0	0	0	0	0	0
	$111 \to \mathbf{5}$ $211 \to \mathbf{4}$ $221 \to \mathbf{3}$ $222 \to \mathbf{2}$								
5	$[5 : 2, 2, 2, 2, 2, 2, 1, 1]$	28	0	0	0	0	0	0	0
	$211 \to \mathbf{6}$ $221 \to \mathbf{5}$ $222 \to \mathbf{4}$								
6	$[6 : 3, 2, 2, 2, 2, 2, 2, 2]$	8	0	0	0	0	0	0	0
	$222 \to \mathbf{6}$ $322 \to \mathbf{5}$								
Total: 7	**Total:**	240	56	27	16	10	6	3	1

□

Remark. *The previous results correspond to Proposition 5.7.5 from [2].*

Theorem 5.4 (Classification of Cremona Lines). *There are* **17280** *Cremona Lines through 8 points, divided into* **35** *different types.*

There are **560** *Cremona Lines through 7 points, divided into* **10** *different types.*

There are **72** *Cremona Lines through 6 points, divided into* **5** *different types.*
There are **16** *Cremona Lines through 5 points, divided into* **3** *different types.*
There are **5** *Cremona Lines through 4 points, divided into* **2** *different types.*
There are **2** *Cremona Lines through 3 points, divided into* **2** *different types.*
There is **1** *Cremona Line through 2 points:* $[1:0,0]$.
There is **1** *Cremona Line through 1 point:* $[1:0]$.
The different types are presented in the following table:

	Type		***Type***
1	$[1:0,0,0,0,0,0,0,0]$	**9c**	$[9:4,4,4,4,2,2,2,2]$
2	$[2:1,1,1,0,0,0,0,0]$	**10a**	$[10:5,5,3,3,3,3,3,2]$
3	$[3:2,1,1,1,1,0,0,0]$	**10b**	$[10:5,4,4,4,3,3,2,2]$
4a	$[4:2,2,2,1,1,1,0,0]$	**10c**	$[10:6,3,3,3,3,3,3,3]$
4b	$[4:3,1,1,1,1,1,1,0]$	**10d**	$[10:4,4,4,4,4,3,3,1]$
5a	$[5:2,2,2,2,2,2,0,0]$	**11a**	$[11:5,5,4,4,4,3,3,2]$
5b	$[5:3,2,2,2,1,1,1,0]$	**11b**	$[11:6,4,4,4,3,3,3,3]$
6a	$[6:3,3,3,2,1,1,1,1]$	**12a**	$[12:6,5,4,4,4,4,3,3]$
6b	$[6:3,3,2,2,2,2,1,0]$	**12b**	$[12:5,5,5,5,4,3,3,3]$
6c	$[6:4,2,2,2,2,1,1,1]$	**12c**	$[12:5,5,5,4,4,4,4,2]$
7a	$[7:4,3,3,2,2,2,1,1]$	**13a**	$[13:6,6,4,4,4,4,4,4]$
7b	$[7:3,3,3,3,2,2,2,0]$	**13b**	$[13:6,5,5,5,4,4,4,3]$
8a	$[8:5,3,3,2,2,2,2,2]$	**14a**	$[14:6,6,5,5,5,4,4,4]$
8b	$[8:4,4,3,3,2,2,2,1]$	**14b**	$[14:6,5,5,5,5,5,5,3]$
8c	$[8:4,3,3,3,3,3,1,1]$	**15**	$[15:6,6,6,5,5,5,5,4]$
8d	$[8:3,3,3,3,3,3,3,0]$	**16**	$[16:6,6,6,6,6,5,5,5]$
9a	$[9:4,4,4,3,3,3,2,1]$	**17**	$[17:6,6,6,6,6,6,6,6]$
9b	$[9:5,4,3,3,3,2,2,2]$		

Proof. The following are the different types, the number of curves in each type for the different number of points and given any possible combination of 3-tuples of multiplicities, which type of curve it leads to when we apply the Cremona Transformation:

	Type **No. of Points**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
1	[1 : 0, 0, 0, 0, 0, 0, 0, 0]	1	1	1	1	1	1	1	1
	000 → **2**								
2	[2 : 1, 1, 1, 0, 0, 0, 0, 0]	56	35	20	10	4	1	0	0
	000 → **4a** 100 → **3** 110 → **2** 111 → **1**								
3	[3 : 2, 1, 1, 1, 1, 0, 0, 0]	280	90	30	5	0	0	0	0
	000 → **6a** 100 → **5b** 110 → **4a** 111 → **3** 200 → **4b** 210 → **3** 211 → **2**								
4a	[4 : 2, 2, 2, 1, 1, 1, 0, 0]	560	140	20	0	0	0	0	0
	100 → **7a** 110 → **6b** 111 → **5a** 200 → **6c** 210 → **5b** 211 → **4a** 220 → **4a** 221 → **3** 222 → **2**								
4b	[4 : 3, 1, 1, 1, 1, 1, 1, 0]	56	7	0	0	0	0	0	0
	110 → **6a** 111 → **5b** 310 → **4b** 311 → **3**								
5a	[5 : 2, 2, 2, 2, 2, 2, 0, 0]	28	7	1	0	0	0	0	0
	200 → **8a** 220 → **6b** 222 → **4a**								

	Type No. of Points	8	7	6	5	4	3	2	1
5b	$[5:3,2,2,2,1,1,1,0]$	1120	140	0	0	0	0	0	0
	$110 \to$ **8b** $111 \to$ **7b** $210 \to$ **7a** $211 \to$ **6b** $220 \to$ **6a** $221 \to$ **5b** $222 \to$ **4b** $310 \to$ **6c** $311 \to$ **5b** $320 \to$ **5b** $321 \to$ **4a** $322 \to$ **3**								
6a	$[6:3,3,3,2,1,1,1,1]$	280	0	0	0	0	0	0	0
	$111 \to$ **9a** $211 \to$ **8c** $311 \to$ **7a** $321 \to$ **6a** $331 \to$ **5b** $332 \to$ **4b** $333 \to$ **3**								
6b	$[6:3,3,2,2,2,2,1,0]$	840	105	0	0	0	0	0	0
	$210 \to$ **9b** $220 \to$ **8b** $221 \to$ **7b** $222 \to$ **6b** $310 \to$ **8a** $320 \to$ **7a** $321 \to$ **6b** $322 \to$ **5b** $330 \to$ **6b** $331 \to$ **5a** $332 \to$ **4a**								
6c	$[6:4,2,2,2,2,1,1,1]$	280	0	0	0	0	0	0	0
	$111 \to$ **9c** $211 \to$ **8b** $221 \to$ **7a** $222 \to$ **6c** $411 \to$ **6c** $421 \to$ **5b** $422 \to$ **4a**								

	Type No. of Points	8	7	6	5	4	3	2	1
7a	$[7:4,3,3,2,2,2,1,1]$	1680	0	0	0	0	0	0	0
	$211 \rightarrow$ **10b** $221 \rightarrow$ **9a** $222 \rightarrow$ **8c** $311 \rightarrow$ **9b** $321 \rightarrow$ **8b** $322 \rightarrow$ **7a** $331 \rightarrow$ **7a** $332 \rightarrow$ **6c** $411 \rightarrow$ **8a** $421 \rightarrow$ **7a** $422 \rightarrow$ **6a** $431 \rightarrow$ **6b** $432 \rightarrow$ **5b** $433 \rightarrow$ **4a**								
7b	$[7:3,3,3,3,2,2,2,0]$	280	35	0	0	0	0	0	0
	$220 \rightarrow$ **10a** $222 \rightarrow$ **8d** $320 \rightarrow$ **9b** $322 \rightarrow$ **7b** $330 \rightarrow$ **8b** $332 \rightarrow$ **6b** $333 \rightarrow$ **5b**								
8a	$[8:5,3,3,2,2,2,2,2]$	168	0	0	0	0	0	0	0
	$222 \rightarrow$ **10b** $322 \rightarrow$ **9b** $332 \rightarrow$ **8a** $522 \rightarrow$ **7a** $532 \rightarrow$ **6b** $533 \rightarrow$ **5a**								
8b	$[8:4,4,3,3,2,2,2,1]$	1680	0	0	0	0	0	0	0
	$221 \rightarrow$ **11a** $222 \rightarrow$ **10d** $321 \rightarrow$ **10b** $322 \rightarrow$ **9a** $331 \rightarrow$ **9c** $332 \rightarrow$ **8b** $421 \rightarrow$ **9b** $422 \rightarrow$ **8b** $431 \rightarrow$ **8b** $432 \rightarrow$ **7a** $433 \rightarrow$ **6c** $441 \rightarrow$ **7b** $442 \rightarrow$ **6b** $443 \rightarrow$ **5b**								

	Type No. of Points	8	7	6	5	4	3	2	1
8c	$[8:4,3,3,3,3,3,1,1]$	168	0	0	0	0	0	0	0
	$311 \rightarrow$ **11b** $331 \rightarrow$ **9a** $333 \rightarrow$ **7a** $411 \rightarrow$ **10c** $431 \rightarrow$ **8c** $433 \rightarrow$ **6a**								
8d	$[8:3,3,3,3,3,3,3,0]$	8	1	0	0	0	0	0	0
	$330 \rightarrow$ **10a** $333 \rightarrow$ **7b**								
9a	$[9:4,4,4,3,3,3,2,1]$	1120	0	0	0	0	0	0	0
	$321 \rightarrow$ **12a** $331 \rightarrow$ **11a** $332 \rightarrow$ **10d** $333 \rightarrow$ **9a** $421 \rightarrow$ **11b** $431 \rightarrow$ **10b** $432 \rightarrow$ **9a** $433 \rightarrow$ **8b** $441 \rightarrow$ **9a** $442 \rightarrow$ **8c** $443 \rightarrow$ **7a** $444 \rightarrow$ **6a**								
9b	$[9:5,4,3,3,3,2,2,2]$	1120	0	0	0	0	0	0	0
	$222 \rightarrow$ **12b** $322 \rightarrow$ **11a** $332 \rightarrow$ **10b** $333 \rightarrow$ **9b** $422 \rightarrow$ **10a** $432 \rightarrow$ **9b** $433 \rightarrow$ **8a** $522 \rightarrow$ **9b** $532 \rightarrow$ **8b** $533 \rightarrow$ **7a** $542 \rightarrow$ **7b** $543 \rightarrow$ **6b**								
9c	$[9:4,4,4,4,2,2,2,2]$	70	0	0	0	0	0	0	0
	$222 \rightarrow$ **12c** $422 \rightarrow$ **10b** $442 \rightarrow$ **8b** $444 \rightarrow$ **6c**								

	Type No. of Points	8	7	6	5	4	3	2	1
10a	[10 : 5, 5, 3, 3, 3, 3, 3, 2]	168	0	0	0	0	0	0	0
	332 → **12b** 333 → **11a** 532 → **10a** 533 → **9b** 552 → **8d** 553 → **7b**								
10b	[10 : 5, 4, 4, 4, 3, 3, 2, 2]	1680	0	0	0	0	0	0	0
	322 → **13b** 332 → **12c** 422 → **12a** 432 → **11a** 433 → **10b** 442 → **10b** 443 → **9b** 444 → **8a** 522 → **11b** 532 → **10b** 533 → **9c** 542 → **9a** 543 → **8b** 544 → **7a**								
10c	[10 : 6, 3, 3, 3, 3, 3, 3, 3]	8	0	0	0	0	0	0	0
	333 → **11b** 633 → **8c**								
10d	[10 : 4, 4, 4, 4, 4, 3, 3, 1]	168	0	0	0	0	0	0	0
	331 → **13a** 431 → **12a** 433 → **10d** 441 → **11a** 443 → **9a** 444 → **8b**								

	Type No. of Points	8	7	6	5	4	3	2	1
11a	[11 : 5, 5, 4, 4, 4, 3, 3, 2]	1680	0	0	0	0	0	0	0
	332 → **14a** 432 → **13b** 433 → **12c** 442 → **12b** 443 → **11a** 444 → **10a** 532 → **12a** 533 → **11a** 542 → **11a** 543 → **10b** 544 → **9b** 552 → **10d** 553 → **9a** 554 → **8b**								
11b	[11 : 6, 4, 4, 4, 3, 3, 3, 3]	280	0	0	0	0	0	0	0
	333 → **13b** 433 → **12a** 443 → **11b** 444 → **10c** 633 → **10b** 643 → **9a** 644 → **8c**								
12a	[12 : 6, 5, 4, 4, 4, 4, 3, 3]	840	0	0	0	0	0	0	0
	433 → **14a** 443 → **13b** 444 → **12a** 533 → **13a** 543 → **12a** 544 → **11b** 633 → **12a** 643 → **11a** 644 → **10b** 653 → **10d** 654 → **9a**								
12b	[12 : 5, 5, 5, 5, 4, 3, 3, 3]	280	0	0	0	0	0	0	0
	333 → **15** 433 → **14b** 533 → **13b** 543 → **12b** 553 → **11a** 554 → **10a** 555 → **9b**								

	Type	No. of Points	8	7	6	5	4	3	2	1
12c	$[12:5,5,5,4,4,4,4,2]$		280	0	0	0	0	0	0	0
	442 → **14a** 444 → **12c** 542 → **13b** 544 → **11a** 552 → **12c** 554 → **10b** 555 → **9c**									
13a	$[13:6,6,4,4,4,4,4,4]$		28	0	0	0	0	0	0	0
	444 → **14a** 644 → **12a** 664 → **10d**									
13b	$[13:6,5,5,5,4,4,4,3]$		1120	0	0	0	0	0	0	0
	443 → **15** 444 → **14b** 543 → **14a** 544 → **13b** 553 → **13b** 554 → **12a** 555 → **11b** 643 → **13b** 644 → **12b** 653 → **12c** 654 → **11a** 655 → **10b**									
14a	$[14:6,6,5,5,5,4,4,4]$		560	0	0	0	0	0	0	0
	444 → **16** 544 → **15** 554 → **14a** 555 → **13a** 644 → **14a** 654 → **13b** 655 → **12a** 664 → **12c** 665 → **11a**									
14b	$[14:6,5,5,5,5,5,5,3]$		56	0	0	0	0	0	0	0
	553 → **15** 555 → **13b** 653 → **14b** 655 → **12b**									

	Type No. of Points	8	7	6	5	4	3	2	1
15	[15 : 6, 6, 6, 5, 5, 5, 5, 4]	280	0	0	0	0	0	0	0
	554 → **16** 555 → **15** 654 → **15** 655 → **14a** 664 → **14b** 665 → **13b** 666 → **12b**								
16	[16 : 6, 6, 6, 6, 6, 5, 5, 5]	56	0	0	0	0	0	0	0
	555 → **17** 655 → **16** 665 → **15** 666 → **14a**								
17	[17 : 6, 6, 6, 6, 6, 6, 6, 6]	1	0	0	0	0	0	0	0
	666 → **16**								
Total: 35	**Total:**	17280	561	72	16	5	2	1	1

□

6 Minus One Curves that are disjoint from a Cremona Line

In the previous section, we obtained all the Cremona Lines and Minus One Curves through eight or fewer points by applying the Quadratic plane Cremona Map to different combinations of 3-tuples of points. In order to obtain any curve (Cremona Line or Minus one Curve) from the one of degree 1, we applied the Cremona transformation to a specific sequence of 3-tuples.

Example 6.1. *If we start with the Cremona Line of the class* **1***:*

$$[1 : 0, 0, 0, 0, 0, 0, 0, 0]$$

and we want to get the curve of the class **4b***:*

$$[4 : 3, 1, 1, 1, 1, 1, 1, 1],$$

we need to perform a Quadratic plane Cremona Map at 3-tuples of points with multiplicities:

000 *(from the class* **1** *producing class* **2***),* 100 *(from the class* **2** *producing class* **3***) and* 200 *(from the class* **3** *producing the class* **4b***).*

Remark. *If we want to start at any curve and "go back" to the curve of degree 1 we use a similar process, in reverse.*

If we look at the Cremona Lines of the class **2**: $[2:1,1,1]$ these correspond to quadratics through three non colinear points. The space of such curves is a 3-dimensional vector space and if the points are the coordinate points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, then the space has a basis: $\{F_0 = yz,\ F_1 = xz,\ F_2 = xy\}$. Consider the map:

$$\begin{aligned} \phi : \mathbb{P}^2 &\longrightarrow \mathbb{P}^2 \\ [x:y:z] &\mapsto [yz:xz:xy] \end{aligned}$$

This map takes the space of functions corresponding to $[1:0,0,0]$ (general line), to the space of functions corresponding to $[2:1,1,1]$. As we know, this is the Quadratic plane Cremona Map, which is a blowup of three points (the three points on the quadratic) and a blowdown of three lines that did not intercept the quadratic: $[1:1,0,1]$, $[1:0,1,1]$, $[1:1,1,0]$. These are all Minus One Curves.

Claim 6.2. *Given any Cremona Line class C of degree d through n general points: $[d:m_1,\dots,m_n]$, we have that the space of functions of that type is at least a 3-dimensional vector space.*

Proof. Clearly, we are considering a subspace of the vector space of polynomial functions, so we have a vector space. The dimension is at least:

$$\begin{aligned} \frac{d^2+3d+2}{2} - \sum_{i=1}^{n} \frac{m_i^2+m_i}{2} &= \frac{1}{2}(d^2 - \sum_{i=1}^{n} m_i^2) + \frac{1}{2}(3d - \sum_{i=1}^{n} m_i) + 1 \\ &= \frac{1}{2}(1) + \frac{1}{2}(3) + 1 \\ &= 3 \end{aligned}$$

The first term corresponds to the number of coefficients in a homogeneous polynomial of degree d. The second term corresponds to the conditions set by the multiplicities of the points. If those are independent conditions, then the dimension of the vector space is exactly 3. □

In fact, although we will not prove this result here, for curves through 8 or fewer points the conditions on the multiplicities are independent, so we always have a 3-dimensional vector space.

Given a basis for such a vector space: $\{F_0(x,y,z),\ F_1(x,y,z),\ F_2(x,y,z)\}$, we can consider the map:

$$\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$
$$[x:y:z] \mapsto [F_0(x,y,z) : F_1(x,y,z) : F_2(x,y,z)]$$

This map has the following properties:

- It's defined everywhere except at the n points.
- It corresponds to a blowup of the n points and a blowdown of n Minus One Curves that did not originally intersect the Cremona Line class of C.

This map takes the curves in the class $[1:0,0,0,0,0,0,0]$ *directly* to curve in the class of C. We then have that each Cremona Line induces a Cremona Transformation. The sequence of Quadratic Cremona maps that we found in the previous section corresponds then to a factorization of this new map.

Example 6.3. *We have already verified this results in section 2 for the curve [2,1,1,1] and the corresponding Cremona Transformation, which is the Quadratic cremona transformation 2.8.*

Example 6.4. *Consider the curves of the class* $[3:2,1,1,1,1]$*. These correspond to cubics through five points, one of them of multiplicity* 2*. Given a cubic* $F(x,y,z)=0$ *through the points* $P_2=[1:0:0]$, $P_3=[0:1:0]$, $P_4=[0:0:1]$, $P_5=[1:1:1]$ *(we can assume any cubic goes through these points that are in general position), suppose the point* $P_1=[1:-1:3]$ *is on the curve and has multiplicity 2. We have four conditions set on the curves: one given by the fact that it passes through the first four points and three more given by the partial derivatives being zero at the fifth point (since it has multiplicity 2). If we solve the system of restrictions we get a basis for the vector space of such curves given by:*

$$F_0(x,y,z) = 7xy^2 + xz^2 - 2yx^2 - \frac{10}{3}zx^2 - \frac{8}{3}zy^2$$
$$F_1(x,y,z) = -8xy^2 + yx^2 + yz^2 + \frac{5}{3}zx^2 + \frac{13}{3}zy^2$$
$$F_2(x,y,z) = -xy^2 - yx^2 + \frac{1}{3}zx^2 + \frac{2}{3}zy^2 + xyz$$

We can then define the Cremona Transformation:

$$\begin{aligned} \phi : \mathbb{P}^2 &\longrightarrow \mathbb{P}^2 \\ [x : y : z] &\mapsto [F_0(x,y,z) : F_1(x,y,z) : F_2(x,y,z)] \end{aligned}$$

Consider the following curves:

C_1 *defined by* $3y + z = 0$ *which is a Minus One Curve of the type:* $[1 : 1, 1, 0, 0, 0]$
C_2 *defined by* $3x - z = 0$ *which is a Minus One Curve of the type:* $[1 : 1, 0, 1, 0, 0]$
C_3 *defined by* $x + y = 0$ *which is a Minus One Curve of the type:* $[1 : 1, 0, 0, 1, 0]$
C_4 *defined by* $2x - y - z = 0$ *which is a Minus One Curve of the type:* $[1 : 1, 0, 0, 0, 1]$
C_5 *defined by* $-3xy + 2yz + xz = 0$ *which is a Minus One Curve of the type:* $[2 : 1, 1, 1, 1, 1]$

Given a point on C_1*:* $[x : y : -3y]$ *we have that:*

$$\begin{aligned} F_0(x,y,-3y) &= -\frac{8}{27}z^3 + \frac{16}{9}xz^2 - \frac{8}{3}zx^2 \\ F_1(x,y,-3y) &= \frac{4}{27}z^3 - \frac{8}{9}xz^2 + \frac{4}{3}zx^2 \\ F_2(x,y,-3y) &= \frac{2}{27}z^3 - \frac{4}{9}xz^2 + \frac{2}{3}zx^2 \end{aligned}$$

so we have:

$$\begin{aligned} \phi([x : y : -3y]) &= [F_0(x,y,-3y) : F_1(x,y,-3y) : F_2(x,y,-3y)] \\ &= \left[-\frac{8}{27}z(3x^2 - z)^2 : \frac{4}{27}z(3x^2 - z)^2 : \frac{2}{27}z(3x^2 - z)^2\right] \\ &= \left[-\frac{8}{27} : \frac{4}{27} : \frac{2}{27}\right] \\ &= [-4 : 2 : 1] \end{aligned}$$

The curve C_1 *gets blown down to the point* $[-4 : 2 : 1]$ *by the transformation.*
If we perform the same calculations for $C_2, \ldots, C_5$ *we will find that:*

- C_2 *is blown down to the point* $\left[-\frac{1}{27} : \frac{5}{27} : \frac{1}{27}\right] = [-1 : 5 : 1]$.

- C_3 *is blown down to the point* $[-1 : 1 : 0]$.

- C_4 *is blown down to the point* $\left[\frac{8}{3} : -\frac{10}{3} : -\frac{2}{3}\right] = [4 : -5 : -1]$.

- C_5 *is blown down to the point* $\left[-\frac{2}{3} : \frac{1}{3} : 0\right] = [-2 : 1 : 0]$.

We have that ϕ *blows up the five given points and blows down the five curves* $C_1, \ldots, C_5$.

If we now consider the different types of Cremona Lines through 8 points, we can find the 8 corresponding Minus One Curves (these are all disjoint from each type of curve) that will be blown down by the Cremona Transformation ϕ induced by the curve type. The following table shows for each of the types of Cremona Lines through 8 points found in Theorem 5.4 the 8 corresponding Minus One Curves.

Note: The same table can be used to identify the correspondences for curves through fewer points.

	Type
1	[1 : 0, 0, 0, 0, 0, 0, 0, 0]
	[0 : −1, 0, 0, 0, 0, 0, 0, 0]
	[0 : 0, −1, 0, 0, 0, 0, 0, 0]
	[0 : 0, 0, −1, 0, 0, 0, 0, 0]
	[0 : 0, 0, 0, −1, 0, 0, 0, 0]
	[0 : 0, 0, 0, 0, −1, 0, 0, 0]
	[0 : 0, 0, 0, 0, 0, −1, 0, 0]
	[0 : 0, 0, 0, 0, 0, 0, −1, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
2	[2 : 1, 1, 1, 0, 0, 0, 0, 0]
	[0 : 0, 0, 0, −1, 0, 0, 0, 0]
	[0 : 0, 0, 0, 0, −1, 0, 0, 0]
	[0 : 0, 0, 0, 0, 0, −1, 0, 0]
	[0 : 0, 0, 0, 0, 0, 0, −1, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[1 : 0, 1, 1, 0, 0, 0, 0, 0]
3	[3 : 2, 1, 1, 1, 1, 0, 0, 0]
	[0 : 0, 0, 0, 0, 0, −1, 0, 0]
	[0 : 0, 0, 0, 0, 0, 0, −1, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[1 : 1, 0, 0, 1, 0, 0, 0, 0]
	[1 : 1, 0, 0, 0, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
4a	[4 : 2, 2, 2, 1, 1, 1, 0, 0]
	[0 : 0, 0, 0, 0, 0, 0, −1, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[1 : 0, 1, 1, 0, 0, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 0, 1, 1, 0, 0]
4b	[4 : 3, 1, 1, 1, 1, 1, 1, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[1 : 1, 0, 0, 1, 0, 0, 0, 0]
	[1 : 1, 0, 0, 0, 1, 0, 0, 0]
	[1 : 1, 0, 0, 0, 0, 1, 0, 0]
	[1 : 1, 0, 0, 0, 0, 0, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]

	Type
5a	[5 : 2, 2, 2, 2, 2, 2, 0, 0]
	[0 : 0, 0, 0, 0, 0, 0, −1, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 0, 1, 1, 0, 0]
	[2 : 1, 1, 0, 1, 1, 1, 0, 0]
	[2 : 1, 0, 1, 1, 1, 1, 0, 0]
	[2 : 0, 1, 1, 1, 1, 1, 0, 0]
5b	[5 : 3, 2, 2, 2, 1, 1, 1, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[1 : 1, 0, 0, 1, 0, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 1, 0, 0, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
6a	[6 : 3, 3, 3, 2, 1, 1, 1, 1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 0, 1, 1, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 1, 0, 0, 1, 0]
	[2 : 1, 1, 1, 1, 0, 0, 0, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
6b	[6 : 3, 3, 2, 2, 2, 2, 1, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 0, 1, 1, 0, 0]
	[2 : 1, 1, 0, 1, 1, 1, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
6c	[6 : 4, 2, 2, 2, 2, 1, 1, 1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[1 : 1, 0, 0, 1, 0, 0, 0, 0]
	[1 : 1, 0, 0, 0, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[3 : 2, 1, 1, 1, 1, 0, 1, 1]

	Type
7a	[7 : 4, 3, 3, 2, 2, 2, 1, 1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 0, 1, 1, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
7b	[7 : 3, 3, 3, 3, 2, 2, 2, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 1, 0, 0, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
	[3 : 1, 1, 2, 1, 1, 1, 1, 0]
	[3 : 1, 1, 1, 2, 1, 1, 1, 0]
8a	[8 : 5, 3, 3, 2, 2, 2, 2, 2]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[1 : 1, 0, 1, 0, 0, 0, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[3 : 2, 1, 1, 1, 1, 0, 1, 1]
	[3 : 2, 1, 1, 1, 0, 1, 1, 1]
	[3 : 2, 1, 1, 0, 1, 1, 1, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
8b	[8 : 4, 4, 3, 3, 2, 2, 2, 1]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 1, 0, 0, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
8c	[8 : 4, 3, 3, 3, 3, 3, 1, 1]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 0, 1, 1, 0, 0]
	[2 : 1, 1, 0, 1, 1, 1, 0, 0]
	[2 : 1, 0, 1, 1, 1, 1, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]

	Type
8d	[8 : 3, 3, 3, 3, 3, 3, 3, 0]
	[0 : 0, 0, 0, 0, 0, 0, 0, −1]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
	[3 : 1, 1, 2, 1, 1, 1, 1, 0]
	[3 : 1, 1, 1, 2, 1, 1, 1, 0]
	[3 : 1, 1, 1, 1, 2, 1, 1, 0]
	[3 : 1, 1, 1, 1, 1, 2, 1, 0]
	[3 : 1, 1, 1, 1, 1, 1, 1, 0]
9a	[9 : 4, 4, 4, 3, 3, 3, 2, 1]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 0, 1, 1, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
	[3 : 1, 1, 2, 1, 1, 1, 1, 0]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]
9b	[9 : 5, 4, 3, 3, 3, 2, 2, 2]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[3 : 2, 1, 1, 1, 1, 0, 1, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 2, 1, 1, 2, 1, 1, 1]
9c	[9 : 4, 4, 4, 4, 2, 2, 2, 2]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[2 : 1, 1, 1, 1, 0, 0, 1, 0]
	[2 : 1, 1, 1, 1, 0, 0, 0, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 1, 2, 2, 1, 1, 1, 1]
	[4 : 1, 2, 2, 2, 1, 1, 1, 1]
10a	[10 : 5, 5, 3, 3, 3, 3, 3, 2]
	[1 : 1, 1, 0, 0, 0, 0, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 2, 1, 1, 2, 1, 1, 1]
	[4 : 2, 2, 1, 1, 1, 2, 1, 1]
	[4 : 2, 2, 1, 1, 1, 1, 2, 1]

	Type
10b	[10 : 5, 4, 4, 4, 3, 3, 2, 2]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[2 : 1, 1, 1, 1, 0, 1, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 1, 2, 2, 1, 1, 1, 1]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]
10c	[10 : 6, 3, 3, 3, 3, 3, 3, 3]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[3 : 2, 1, 1, 1, 1, 0, 1, 1]
	[3 : 2, 1, 1, 1, 0, 1, 1, 1]
	[3 : 2, 1, 1, 0, 1, 1, 1, 1]
	[3 : 2, 1, 0, 1, 1, 1, 1, 1]
	[3 : 2, 0, 1, 1, 1, 1, 1, 1]
	[6 : 3, 2, 2, 2, 2, 2, 2, 2]
10d	[10 : 4, 4, 4, 4, 4, 3, 3, 1]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
	[3 : 1, 1, 2, 1, 1, 1, 1, 0]
	[3 : 1, 1, 1, 2, 1, 1, 1, 0]
	[3 : 1, 1, 1, 1, 2, 1, 1, 0]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]
	[5 : 2, 2, 2, 2, 2, 1, 2, 1]
11a	[11 : 5, 5, 4, 4, 4, 3, 3, 2]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 2, 1, 1, 2, 1, 1, 1]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]
	[5 : 2, 2, 2, 2, 2, 1, 2, 1]
11b	[11 : 6, 4, 4, 4, 3, 3, 3, 3]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[3 : 2, 1, 1, 1, 1, 0, 1, 1]
	[3 : 2, 1, 1, 1, 0, 1, 1, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 1, 2, 2, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[6 : 3, 2, 2, 2, 2, 2, 2, 2]

	Type
12a	[12 : 6, 5, 4, 4, 4, 4, 3, 3]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 2, 1, 1, 1, 1, 1, 0, 1]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 2, 1, 1, 2, 1, 1, 1]
	[4 : 2, 2, 1, 1, 1, 2, 1, 1]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]
	[6 : 3, 2, 2, 2, 2, 2, 2, 2]
12b	[12 : 5, 5, 5, 5, 4, 3, 3, 3]
	[2 : 1, 1, 1, 1, 1, 0, 0, 0]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 1, 2, 2, 1, 1, 1, 1]
	[4 : 1, 2, 2, 2, 1, 1, 1, 1]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]
	[5 : 2, 2, 2, 2, 2, 1, 2, 1]
	[5 : 2, 2, 2, 2, 2, 1, 1, 2]
12c	[12 : 5, 5, 5, 4, 4, 4, 4, 2]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[3 : 1, 2, 1, 1, 1, 1, 1, 0]
	[3 : 1, 1, 2, 1, 1, 1, 1, 0]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]
	[5 : 2, 2, 2, 2, 2, 1, 2, 1]
	[5 : 2, 2, 2, 2, 1, 2, 2, 1]
	[5 : 2, 2, 2, 1, 2, 2, 2, 1]
13a	[13 : 6, 6, 4, 4, 4, 4, 4, 4]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[4 : 2, 2, 1, 1, 2, 1, 1, 1]
	[4 : 2, 2, 1, 1, 1, 2, 1, 1]
	[4 : 2, 2, 1, 1, 1, 1, 2, 1]
	[4 : 2, 2, 1, 1, 1, 1, 1, 2]
	[6 : 3, 2, 2, 2, 2, 2, 2, 2]
	[6 : 2, 3, 2, 2, 2, 2, 2, 2]
13b	[13 : 6, 5, 5, 5, 4, 4, 4, 3]
	[3 : 2, 1, 1, 1, 1, 1, 1, 0]
	[4 : 2, 2, 2, 1, 1, 1, 1, 1]
	[4 : 2, 1, 2, 2, 1, 1, 1, 1]
	[4 : 2, 2, 1, 2, 1, 1, 1, 1]
	[5 : 2, 2, 2, 2, 2, 2, 1, 1]
	[5 : 2, 2, 2, 2, 2, 1, 2, 1]
	[5 : 2, 2, 2, 2, 1, 2, 2, 1]
	[6 : 3, 2, 2, 2, 2, 2, 2, 2]

	Type
14a	$[14 : 6, 6, 5, 5, 5, 4, 4, 4]$
	$[\ 4 : 2, 2, 2, 1, 1, 1, 1, 1]$
	$[\ 4 : 2, 2, 1, 2, 1, 1, 1, 1]$
	$[\ 4 : 2, 2, 1, 1, 2, 1, 1, 1]$
	$[\ 5 : 2, 2, 2, 2, 2, 2, 1, 1]$
	$[\ 5 : 2, 2, 2, 2, 2, 1, 2, 1]$
	$[\ 5 : 2, 2, 2, 2, 2, 1, 1, 2]$
	$[\ 6 : 3, 2, 2, 2, 2, 2, 2, 2]$
	$[\ 6 : 2, 3, 2, 2, 2, 2, 2, 2]$
14b	$[14 : 6, 5, 5, 5, 5, 5, 5, 3]$
	$[\ 3 : 2, 1, 1, 1, 1, 1, 1, 0]$
	$[\ 5 : 2, 2, 2, 2, 2, 2, 1, 1]$
	$[\ 5 : 2, 2, 2, 2, 2, 1, 2, 1]$
	$[\ 5 : 2, 2, 2, 2, 1, 2, 2, 1]$
	$[\ 5 : 2, 2, 2, 1, 2, 2, 2, 1]$
	$[\ 5 : 2, 2, 1, 2, 2, 2, 2, 1]$
	$[\ 5 : 2, 1, 2, 2, 2, 2, 2, 1]$
	$[\ 6 : 3, 2, 2, 2, 2, 2, 2, 2]$
15	$[15 : 6, 6, 6, 5, 5, 5, 5, 4]$
	$[\ 4 : 2, 2, 2, 1, 1, 1, 1, 1]$
	$[\ 5 : 2, 2, 2, 2, 2, 2, 1, 1]$
	$[\ 5 : 2, 2, 2, 2, 2, 1, 2, 1]$
	$[\ 5 : 2, 2, 2, 2, 1, 2, 2, 1]$
	$[\ 5 : 2, 2, 2, 1, 2, 2, 2, 1]$
	$[\ 6 : 3, 2, 2, 2, 2, 2, 2, 2]$
	$[\ 6 : 2, 3, 2, 2, 2, 2, 2, 2]$
	$[\ 6 : 2, 2, 3, 2, 2, 2, 2, 2]$
16	$[16 : 6, 6, 6, 6, 6, 5, 5, 5]$
	$[\ 5 : 2, 2, 2, 2, 2, 2, 1, 1]$
	$[\ 5 : 2, 2, 2, 2, 2, 1, 2, 1]$
	$[\ 5 : 2, 2, 2, 2, 2, 1, 1, 2]$
	$[\ 6 : 3, 2, 2, 2, 2, 2, 2, 2]$
	$[\ 6 : 2, 3, 2, 2, 2, 2, 2, 2]$
	$[\ 6 : 2, 2, 3, 2, 2, 2, 2, 2]$
	$[\ 6 : 2, 2, 2, 3, 2, 2, 2, 2]$
	$[\ 6 : 2, 2, 2, 2, 3, 2, 2, 2]$
17	$[17 : 6, 6, 6, 6, 6, 6, 6, 6]$
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	$[\ 6 : 2, 3, 2, 2, 2, 2, 2, 2]$
	$[\ 6 : 2, 2, 3, 2, 2, 2, 2, 2]$
	$[\ 6 : 2, 2, 2, 3, 2, 2, 2, 2]$
	$[\ 6 : 2, 2, 2, 2, 3, 2, 2, 2]$
	$[\ 6 : 2, 2, 2, 2, 2, 3, 2, 2]$
	$[\ 6 : 2, 2, 2, 2, 2, 2, 3, 2]$
	$[\ 6 : 2, 2, 2, 2, 2, 2, 2, 3]$

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