Can spectral value sets of Toeplitz band matrices jump?

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Received 8 January 2001; accepted 24 July 2001
Submitted by D. Hinrichsen

Abstract

The spectral value set $sp_{\varepsilon}^{B,C} A$ of a bounded linear operator $A$ on $l^2$ is the union of the spectra of all operators of the form $A + BKC$, where $\|K\| < \varepsilon$ and $B$, $C$ are fixed bounded linear operators. It turns out that small changes of $\varepsilon$ may cause drastic changes of the set $sp_{\varepsilon}^{B,C} A$. We conjecture that this can never happen if $B$ or $C$ is compact and $A$ is given by an infinite Toeplitz band matrix. In the present paper, this conjecture is proved for certain interesting operators $B$ and $C$ and for several classes of Toeplitz band matrices, including Hessenberg matrices and matrices of small bandwidth. Our approach is based on working with the monodromy group of the Riemann surface associated with the generating function of the matrix. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: Primary: 47B35; Secondary: 15A18; 30F10; 93B03; 93D09

Keywords: Spectral value set; Pseudospectrum; Toeplitz matrix; Perturbation; Monodromy group

1. Introduction

Let $l^2 = l^2(Z)$ be the usual complex Hilbert space of sequences over an at most countable set $Z$. The spectrum of a bounded linear operator $A$ on $l^2$ is denoted...
by \( \text{sp} A \). Given bounded linear operators \( A, B, C \) on \( l^2 \) and a number \( \varepsilon > 0 \), we put
\[
\text{sp}^{B,C}_\varepsilon A = \bigcup_{\|K\| < \varepsilon} \text{sp} (A + BKC).
\]
(1)

The sets \( \text{sp}^{B,C}_\varepsilon A \) are called spectral value sets or structured pseudospectra. Such sets have been introduced and studied in connection with the problems of linear systems theory by Hinrichsen, Kelb, Pritchard, Gallestey, and others (see, e.g., [12,13,19,20]). For instance, the question whether the linear system \( \dot{x} = Ax + Bu, \ y = Cx \) has a feedback \( u = Ky \) such that \( \|K\| < \varepsilon \) and such that the resulting system operator \( A + BKC \) has \( \lambda \) in its spectrum is equivalent to the question whether \( \lambda \in \text{sp}^{B,C}_\varepsilon A \).

For fixed \( A, B, C \) the set \( \text{sp}^{B,C}_\varepsilon A \) increases as \( \varepsilon \) increases. This paper addresses the problem whether a small change of the parameter \( \varepsilon \) can lead to a dramatic change of \( \text{sp}^{B,C}_\varepsilon A \). This can never happen if \( A \) is a finite matrix. However, if \( A \) is an infinite matrix, then \( \text{sp}^{B,C}_\varepsilon A \) can jump—an example will be given in Section 2.

We here consider the case, where \( A \) is an infinite Toeplitz band matrix. Spectral value sets of Toeplitz band matrices are of interest in non-Hermitian quantum mechanics [8,10,11,15,18], population dynamics [22], or small world networks [24,25], for example. In these contexts, sufficiently intriguing questions arise when choosing \( B \) and \( C \) to be the identity operator \( I \) or one of the two projections \( E_j \) and \( P_m \) given by
\[
(E_j x)_k = \begin{cases} x_j & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases} \quad (P_m x)_k = \begin{cases} x_k & \text{for } k \in \{1, \ldots, m\}, \\ 0 & \text{for } k \notin \{1, \ldots, m\}. \end{cases}
\]

The first explicit application of (unstructured) pseudospectra to random bidiagonal Toeplitz matrices is given in [27]. In [3], we studied the spectral value sets (\( \text{sp}^{P_m,P_m}_\varepsilon A \)) of rationally generated Toeplitz matrices for small \( \varepsilon > 0 \). The result of [3] states that \( \text{sp}^{P_m,P_m}_\varepsilon A = \text{sp} A \) for all sufficiently small \( \varepsilon > 0 \) provided \( A \) is not a scalar multiple of the identity operator. In other words, a small feedback or a small impurity localized in a fixed finite set of sites of the matrix \( A \) cannot change the spectrum of \( A \).

Any attempt to understand the behavior of \( \text{sp}^{B,C}_\varepsilon A \) beyond the case of small \( \varepsilon \)’s eventually runs into the question whether these sets can jump. Our main results on this topic are precisely stated in Section 2. The remaining sections contain the proofs. The strategy may be outlined as follows. We first show that \( \text{sp}^{B,C}_\varepsilon A \) cannot jump if the function \( C(A - \lambda I)^{-1}B \) admits a special analytic continuation. In the case, where \( A \) is the Toeplitz band matrix generated by \( z^{-p}Q_n(z) \), where \( 1 \leq p \leq n \) and \( Q_n(z) = \sum_{j=0}^{n} r_n z^n \) with \( r_0 \neq 0 \) and \( r_n \neq 0 \), we are led to the problem whether the monodromy group \( G \) of the Riemann surface of \( Q_n(z) - \lambda z^p = 0 \) contains a special element. We show that \( G \) is sufficiently rich in several cases, which will imply that \( G \) possesses the desired element.
2. Main results

The usual (unstructured) pseudospectrum of a bounded Hilbert space operator \( A \) is defined by

\[
\text{sp}_\varepsilon A = \text{sp}_{I, I} A = \bigcup_{\|K\| < \varepsilon} \text{sp} (A + K).
\] (2)

It is well known that \( \text{sp}_\varepsilon A \) admits the alternative description

\[
\text{sp}_\varepsilon A = \text{sp} A \cup \{ \lambda \notin \text{sp} A : \| (A - \lambda I)^{-1} B \| > 1/\varepsilon \}.
\] (3)

Trefethen’s paper [26] is an excellent treatise of the properties of pseudospectra (see also the really exciting website [9]). Notice that Ref. [26] actually contains (2) and (3) with non-strict inequalities instead of strict inequalities. Other authors, for example Davies [7,8], find it more convenient to use (2) and (3) as they are stated here. We remark that the proof of Theorem 3.15 in [5] gives the equivalence of (2) and (3) both for non-strict and strict inequalities (see also [6]).

Equality (3) implies that \( \text{sp}_\varepsilon A \) has jumps as \( \varepsilon \) increases if and only if the resolvent norm \( \| (A - \lambda I)^{-1} B \| \) is a non-zero constant on some open subset of \( C \setminus \text{sp} A \). The question whether this may happen was posed by one of the authors (A.B.) in Warsaw in 1994. Subsequently, Andrzej Daniluk proved that \( \| (A - \lambda I)^{-1} B \| \) can never be locally constant and that hence \( \text{sp}_\varepsilon A \) cannot jump (see [2,4–6,16,17]).

As noticed in the introduction, things change when passing from pseudospectra to spectral value sets. The analog of (3) for spectral value sets was only recently proved by Gallestey, Hinrichsen, and Pritchard [12]. It reads

\[
\text{sp}_{\varepsilon}^{B,C} A = \text{sp} A \cup \{ \lambda \notin \text{sp} A : \| C (A - \lambda I)^{-1} B \| > 1/\varepsilon \}.
\] (4)

(In Appendix A we will prove (4) and will also show that the right-hand sides of (1) and (4) coincide with strict inequalities replaced by non-strict ones.) Thus, our problem amounts to the question whether \( \| C (A - \lambda I)^{-1} B \| \) may be a non-zero constant on some open subset of \( C \setminus \text{sp} A \). We here prove the following.

**Theorem 2.1.** Let \( A, B, C \) be bounded linear operators on \( l^2 \) and suppose \( B \) or \( C \) is compact. Let \( \Omega_{\infty} \) denote the unbounded component of \( C \setminus \text{sp} A \) and let \( \Omega \) be any component of \( C \setminus \text{sp} A \). Suppose further that there exists a path \( \Gamma \) in the plane that connects \( \Omega \) and \( \Omega_{\infty} \) such that \( C (A - \lambda I)^{-1} B \) can be analytically continued from \( \Omega \) along \( \Gamma \) to some function \( f(\lambda) \) defined in some open subset \( V \) of \( \Omega_{\infty} \) and that \( f(\lambda) = C (A - \lambda I)^{-1} B \) for \( \lambda \in V \). Then \( \| C (A - \lambda I)^{-1} B \| \) is either nowhere locally constant in \( \Omega \) or identically zero in \( \Omega \).

Of course, here “nowhere locally constant in \( \Omega \)” means that there is no open subset of \( \Omega \) on which the function is constant. Theorem 2.1 implies in particular that, provided \( B \) or \( C \) is compact, \( \text{sp}_{\varepsilon}^{B,C} A \) cannot jump if \( C \setminus \text{sp} A \) is connected (which is true for finite matrices \( A \) as well as for self-adjoint or compact operators \( A \)).
Example 2.2. Let $U$ be the forward shift on $l^2(\mathbb{Z})$, that is, $(Ux)_n = x_{n-1}$. Then $\text{sp } U$ is the unit circle $\mathbf{T}$ and the central $4 \times 4$ block of the resolvent operator $(U - \lambda I)^{-1}$ is

$$
\begin{pmatrix}
0 & 1 & \lambda & \lambda^2 \\
0 & 0 & 1 & \lambda \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
-1/\lambda & 0 & 0 & 0 \\
-1/\lambda^2 & -1/\lambda & 0 & 0 \\
-1/\lambda^3 & -1/\lambda^2 & -1/\lambda & 0 \\
-1/\lambda^4 & -1/\lambda^3 & -1/\lambda^2 & -1/\lambda
\end{pmatrix}
$$

for $|\lambda| < 1$ and $|\lambda| > 1$, respectively. Thus, $\|P_2(U - \lambda I)^{-1}P_2\| = 1$ for $|\lambda| < 1$. In this case, the crucial hypothesis of Theorem 2.1 is not satisfied: although each entry of $(U - \lambda I)^{-1}$ can be analytically continued from $|\lambda| < 1$ to all of $\mathbb{C}$, the result of this continuation is different from the corresponding entry of $(U - \lambda I)^{-1}$ for $|\lambda| > 1$.

Using (4) and the explicit expressions for $(U - \lambda I)^{-1}$ displayed above, it is easy to compute $\text{sp}_{P_2, P_2}^{P_2} U$. Put $\varepsilon_0 = \sqrt{2/(3 + \sqrt{5})} = 0.618 \ldots$. There is a continuous and strictly monotonically increasing function $h : [\varepsilon_0, \infty) \to [0, \infty)$ such that $h(\varepsilon_0) = 0$, $h(\infty) = \infty$, and

$$
\text{sp}_{P_2}^{P_2} U = \left\{ \begin{array}{l}
\{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0, \\
\{ \lambda \in \mathbb{C} : 1 \leq |\lambda| < 1 + h(\varepsilon) \} \quad \text{for } \varepsilon_0 < \varepsilon \leq 1, \\
\{ \lambda \in \mathbb{C} : 0 \leq |\lambda| < 1 + h(\varepsilon) \} \quad \text{for } 1 < \varepsilon.
\end{array} \right.
$$

Clearly, $\text{sp}_{P_2}^{P_2} U$ jumps at $\varepsilon = 1$.

Let $a(z)$ be a Laurent polynomial of the form

$$
a(z) = \sum_{j=-p}^{q} a_j z^j, \quad p \geq 0, \quad q \geq 0, \quad a_{-p} \neq 0, \quad a_q \neq 0; \quad (5)
$$

here $a_j \in \mathbb{C}$. The Toeplitz matrix associated with the Laurent polynomial (5) is $T(a) = (a_{j-k})_{j,k=1}^\infty$, where $a_{j-k} := 0$ if $j-k < -p$ or $j-k > q$. It is well known that $T(a)$ induces a bounded operator on $l^2(\mathbb{N})$ and that

$$
\text{sp } T(a) = a(\mathbf{T}) \cup \{ \lambda \notin a(\mathbf{T}) : \text{wind } (a - \lambda) \neq 0 \}, \quad (6)
$$

where wind $(a - \lambda)$ is the winding number of $a(\mathbf{T})$ about $\lambda$ (see, e.g., [5] or [14]). Obviously, $T(a)$ is a band matrix with at most $p+q+1$ non-zero diagonals. The matrix $T(a)$ is triangular if $p = 0$ or $q = 0$ and is called Hessenberg if $p = 1$ or $q = 1$. Since $T(a) - \lambda I = T(a - \lambda)$, we write $T^{-1}(a - \lambda)$ for $(T(a) - \lambda I)^{-1}$.

Conjecture 2.3. Let $a(z)$ be of form (5) and suppose that one of the operators $B$ and $C$ is compact. If $\Omega$ is any connected component of $\mathbb{C} \setminus \text{sp } T(a)$, then the norm $\|CT^{-1}(a - \lambda)B\|$ is either nowhere locally constant in $\Omega$ or identically zero in $\Omega$.

We will prove Conjecture 2.3 in several cases.

Let $T_n(a)$ be the $n \times n$ Toeplitz matrix $(a_{j-k})_{j,k=1}^n$. Schmidt and Spitzer [23] showed that the spectrum (= set of eigenvalues) of $T_n(a)$ converges in the Hausdorff
metric to some limiting set \( A(a) \) as \( n \to \infty \), that \( A(a) \) is either a singleton of a finite union of analytic arcs, and that
\[
A(a) = \bigcap_{\varrho \neq 0} \text{sp} \, T(a_{\varrho}),
\]
where \( a_{\varrho}(z) := a(\varrho z) \). Here is our main result.

**Theorem 2.4.** Suppose \( B \) and \( C \) are of the form \( B = P_m \tilde{B} \) and \( C = \tilde{C} P_k \) with bounded operators \( \tilde{B} \) and \( \tilde{C} \). Then Conjecture 2.3 is true in each of the following cases:

(a) \( C \setminus A(a) \) is connected;
(b) \( m = k = 1 \);
(c) \( T(a) \) is Hessenberg;
(d) \( p + q \) is a prime number and \( p \) or \( q \) equals 2;
(e) \( p + q \leq 5 \) or \( p + q = 7 \);
(f) \( B = C = P_m \) and \( p + q = 6 \).

We emphasize that \( C \setminus A(a) \) is connected in many cases in which \( C \setminus \text{sp} \, T(a) \) is not connected. We do not know any \( a(z) \) with \( p + q \leq 5 \) for which \( C \setminus A(a) \) is disconnected.

### 3. Analytic continuation

In this section, we prove Theorem 2.1. We abbreviate \( C(A - \lambda I)^{-1} B \) to \( f(\lambda) \) and assume that the hypotheses of Theorem 2.1 are satisfied.

**Lemma 3.1.** If \( \|f(\lambda)\| = M > 0 \) for all \( \lambda \) in some open subset \( U \) of \( \Omega \), then there exist \( x, y \in l^2 \) such that \( x/0 = 0 \), \( y/0 = 0 \), and \( f(\lambda)x = y \) for all \( \lambda \in U \).

**Proof.** Pick \( \lambda_0 \in U \). We have \( f(\lambda) = \sum_{n=0}^{\infty} f^{(n)}(\lambda_0)(\lambda_0 - \lambda_0)^n / n! \) for all \( \lambda \in U \) sufficiently close to \( \lambda_0 \). Hence, for every \( x \in l^2 \),
\[
\|f(\lambda)x\|^2 = \sum_{n,k=0}^{\infty} (f^{(n)}(\lambda_0)x, f^{(k)}(\lambda_0)x)(\lambda_0 - \lambda_0)^n(\lambda_0 - \lambda_0)^k / (n!k!).
\]
Integration of this equality along the circle \( |\lambda - \lambda_0| = r \) gives
\[
\frac{1}{2\pi} \int_0^{2\pi} \|f(\lambda)x\|^2 \, d\theta = \sum_{n=0}^{\infty} \|f^{(n)}(\lambda_0)x\|^2 r^{2n} / (n!)^2,
\]
and since \( \|f(\lambda)x\| \leq M\|x\| \), it follows that
\[
M^2\|x\|^2 \geq \|f(\lambda_0)x\|^2 + \sum_{n=1}^{\infty} \|f^{(n)}(\lambda_0)x\|^2 r^{2n} / (n!)^2.
\]
As \( f(\lambda_0) \) is compact, there is an \( x \in l^2 \) such that \( \|x\| = 1 \) and \( \|f(\lambda_0)x\| = M \). From (8) we now infer that \( f^{(n)}(\lambda_0)x = 0 \) for all \( n \geq 1 \). By analyticity, \( f^{(n)}(\lambda)x = 0 \) for all \( n \geq 1 \) and all \( \lambda \in \Omega \), whence \( f(\lambda)x = f(\lambda_0)x =: y \) for all \( \lambda \in \Omega \).

\[ \square \]

**Proof of Theorem 2.1.** If \( \|f(\lambda)\| = 0 \) for all \( \lambda \) in some open subset \( U \) of \( \Omega \), then \( f(\lambda) = 0 \) for all \( \lambda \in \Omega \), and we are done. So assume that \( \|f(\lambda)\| = M > 0 \) for all \( \lambda \in U \). Lemma 3.1 yields \( x \neq 0 \), \( y \neq 0 \) such that \( f(\lambda)x = y \) for all \( \lambda \in \Omega \), and analytic continuation along \( \Gamma \) delivers the equality \( f(\lambda)x = y \) for all \( \lambda \in \Omega_\infty \). Because \( \|C(A - \lambda I)^{-1}B\| \to 0 \) as \( \lambda \to \infty \), we arrive at the conclusion that \( y = 0 \), which is a contradiction. Thus, \( \|f(\lambda)\| \) cannot be a non-zero constant on some open subset of \( \Omega \).

Second proof of Theorem 2.1 (suggested by one of the referees). Pick \( \lambda_0 \in \Omega \). There exist \( x_0, y_0 \in l^2 \) of norm 1 such that \( (x_0, f(\lambda_0)y_0) = \|f(\lambda_0)\| \). Put \( h(\lambda) = (x_0, f(\lambda)y_0) \). The function \( h \) is analytic and hence either an open mapping or constant. In the former case, there is a \( \lambda_1 \in \Omega \) such that

\[ \|f(\lambda_0)\| = |h(\lambda_0)| < |h(\lambda_1)| = |(x_0, f(\lambda_1)y_0)| \leq \|f(\lambda_1)\|, \]

which shows that \( \|f(\lambda)\| \) cannot be locally constant in \( \Omega \). In the latter case, analytic continuation of \( h \) along \( \Gamma \) to \( \Omega_\infty \) and subsequently to infinity yields that the constant value assumed by \( h \) must be zero.

Once Theorem 2.1 is available, it is easy to establish a maximum modulus principle.

**Corollary 3.2.**

(a) If \( \|f(\lambda)\| \) is not identically zero, then \( \|f(\lambda)\| \) has no local maxima.

(b) If \( M > 0 \) and \( \|f(\lambda)\| \leq M \) for all \( \lambda \) in some open subset \( U \) of the component \( \Omega \), then \( \|f(\lambda)\| < M \) for all \( \lambda \in U \).

**Proof.** (a) Assume that \( \|f(\lambda_0)\| \geq \|f(\lambda)\| \) for all \( \lambda \) in some open neighborhood of \( \lambda_0 \). The formula \( f(\lambda_0) = (2\pi i)^{-1} \int_{\Gamma} f(\lambda) \, d\lambda/(\lambda - \lambda_0) \) implies that \( \|f(\lambda)\| = \|f(\lambda_0)\| \) for all \( \lambda \) sufficiently close to \( \lambda_0 \), and Theorem 2.1 now gives the assertion.

(b) This is trivial if \( f(\lambda) \) is identically zero and follows from part (a) in case \( f(\lambda) \) is not identically zero.

4. *The resolvent of Toeplitz band matrices*

Let \( a(z) \) be of form (5) and put

\[ n = p + q, \quad Q_n(z) = a_{-p} + a_{-p+1}z + \cdots + a_qz^{p+q}. \]  

(9)
Thus, \( a(z) = z^{-p} Q_n(z) \). For \( \lambda \in \mathbb{C} \), let \( z_1(\lambda), \ldots, z_n(\lambda) \) be the zeros of the polynomial \( Q_n(z) - \lambda z^p \), that is,

\[
Q_n(z) - \lambda z^p = a_q (z - z_1(\lambda)) \cdots (z - z_n(\lambda)).
\] (10)

Using (6) one can easily verify that \( T(a - \lambda) \) is invertible if and only if \( p \) of the zeros \( z_1(\lambda), \ldots, z_n(\lambda) \) have modulus less than 1 and the remaining \( q \) zeros are of modulus greater than 1. We denote the former zeros by \( \delta_1(\lambda), \ldots, \delta_p(\lambda) \) and the latter zeros by \( \mu_1(\lambda), \ldots, \mu_q(\lambda) \). We put

\[
\mu(\lambda) = \mu_1(\lambda) \cdots \mu_q(\lambda), \quad u_0(\lambda) = 1, \quad v_0(\lambda) = 1,
\]

\[
u_m(\lambda) = \sum_{\beta_j \geq 0} \delta_1(\lambda)^{\beta_1} \cdots \delta_p(\lambda)^{\beta_p} (m \geq 1),
\]

\[
U(\lambda) = \begin{pmatrix} u_0(\lambda) & u_1(\lambda) & u_2(\lambda) \cdots \end{pmatrix},
\]

\[
V(\lambda) = \begin{pmatrix} v_0(\lambda) & v_1(\lambda) & v_2(\lambda) \cdots \end{pmatrix}.
\]

It is well known that

\[
T^{-1}(a - \lambda) = \frac{(-1)^q}{a_q} \cdot \frac{1}{\mu(\lambda)} U(\lambda) V(\lambda)
\] (11)

(see, e.g., [5, Theorem 1.15] or [14, Section I.1.3]).

From (10) we see that each entry of \( T^{-1}(a - \lambda) \) is of the form

\[
[T^{-1}(a - \lambda)]_{jk} = R_{jk}(\delta_1(\lambda), \ldots, \delta_p(\lambda); \mu_1(\lambda), \ldots, \mu_q(\lambda)),
\] (12)

where \( R_{jk} \) is a rational function of \( p + q \) variables with coefficients in \( \mathbb{Z} \). Moreover, \( R_{jk} \) is symmetric in the first \( p \) variables and in the last \( q \) variables.

5. The monodromy group

Throughout what follows we always assume that \( B \) and \( C \) are as in Theorem 2.4, that \( a \) and \( Q_n \) are given by (5) and (9), that \( f(\lambda) = C T^{-1}(a - \lambda) B \), and that \( \Omega \) is some bounded component of \( \mathbb{C} \setminus \text{sp} T(a) \).
We consider the Riemann surface of $Q_n(z) - \lambda z^p = 0$. The points $\lambda \in \mathbb{C}$ for which $Q_n(z) - \lambda z^p$ has a multiple zero are called the finite branch points. There exist at most $n$ finite branch points. We denote them by $\lambda_1, \ldots, \lambda_k$. The point at infinity is also a branch point. Thus, the set of all branch points is $W := \{\lambda_1, \ldots, \lambda_k, \infty\}$.

We join $\lambda_1$ to $\lambda_2$ by a cut $S_1$, $\lambda_2$ to $\lambda_3$ by a cut $S_2$, \ldots, $\lambda_k$ to $\infty$ by a cut $S_k$. Put $S = S_1 \cup \ldots \cup S_k \cup W$. We can draw $S_1$, $\ldots$, $S_k$ so that $\mathbb{C} \setminus S$ is connected. The zeros in (10) can be chosen so that $z_1(\lambda), \ldots, z_n(\lambda)$ are analytic functions in $\mathbb{C} \setminus S$.

Take $n$ copies $\Sigma_1, \ldots, \Sigma_n$ of $\mathbb{C} \setminus S$ and think of $z_j$ as a map of $\Sigma_j$ to $\mathbb{C}$. We glue $\Sigma_i$ and $\Sigma_j$ along the cut $S_l$ whenever the function $z_i(\lambda)$ can be continued analytically to the function $z_j(\lambda)$ across $S_l$. The resulting set $\Sigma$ is the Riemann surface of $Q_n(z) - \lambda z^p = 0$, and $\Sigma_j$ is referred to as the $j$th branch of $\Sigma$.

Each path in $\mathbb{C} \setminus W$ induces a permutation of the branches of $\Sigma$ in a natural way. The set of all these permutations is a group $G$, the monodromy group of $Q_n(z) - \lambda z^p = 0$. Let $\pi_j$ ($j = 1, \ldots, k$) be the permutation corresponding to a small counter-clockwise oriented circle around the branch point $\lambda_j$. Clearly, $G$ contains $\pi_1, \ldots, \pi_k$.

We put $\pi_\infty = \pi_1 \cdots \pi_k$. Thus, $\pi_\infty$ is the permutation of the branches resulting from a large counter-clockwise oriented circle containing all finite branch points in its interior. The group $G$ is generated by $\pi_1, \ldots, \pi_k$.

Let $\lambda \in \mathbb{C} \setminus \text{sp} T(\alpha)$. As mentioned in Section 4, we have $|z_j(\lambda)| < 1$ for exactly $p$ values of $j$. We call the branches $\Sigma_j$ corresponding to these values of $j$ the small branches at $\lambda$. The $q$ branches $\Sigma_j$ for which $|z_j(\lambda)| > 1$ will be called the large branches at $\lambda$.

**Theorem 5.1.** If there is a $\pi \in G$ that permutes the set of small branches at the points of $\Omega$ to the set of small branches at the points of the unbounded component $\Omega_\infty$, then $\|f(\lambda)\|$ is either nowhere locally constant in $\Omega$ or identically zero in $\Omega$.

**Proof.** Let $\Gamma$ be the path in $\mathbb{C} \setminus W$ that corresponds to the permutation $\pi$. From (11) and (12) we see that each entry of $P_k T^{-1}(a - \lambda) P_m B$ and hence also $\tilde{C} P_k T^{-1}(a - \lambda) P_m B$ can be analytically continued along $\Gamma$. Since $\pi$ permutes the small branches into themselves, the result of the analytic continuation coincides with the operator $\tilde{C} P_k T^{-1}(a - \lambda) P_m B$ for $\lambda \in \Omega_\infty$. It remains to apply Theorem 2.1. □

Schmidt and Spitzer [23] showed that the limiting set $A(\alpha)$ can be characterized as follows: when labelling the zeros $z_1(\lambda), \ldots, z_{p+q}(\lambda)$ of (10) so that

$$|z_1(\lambda)| \leq |z_2(\lambda)| \leq \cdots \leq |z_{p+q}(\lambda)|,$$

then

$$A(\alpha) = \{ \lambda \in \mathbb{C} : |z_p(\lambda)| = |z_{p+1}(\lambda)| \}.$$  \hspace{1cm} (14)

**Proof of Theorem 2.4(a).** Pick $\lambda_0 \in \Omega$. Since $\mathbb{C} \setminus A(\alpha)$ is connected, there is a path $\Gamma$ in $\mathbb{C} \setminus (W \cup A(\alpha))$ joining $\lambda_0$ to infinity. By (14), we have $|z_p(\lambda)| < |z_{p+1}(\lambda)|$ throughout this path. This means that when moving along this path we will eventu-
ally stay in the \( p \) small branches of \( \Sigma \) at infinity. In other words, there is a \( \pi \in G \) permuting the set of small branches at \( \lambda_0 \) to the set of small branches at infinity. The assertion is therefore immediate from Theorem 5.1. \( \square \)

As pointed out in [23], \( C \setminus \Lambda(a) \) is always connected if \( Q_n(z) \) is a trinomial,

\[
Q_n(z) = a_{-p} + a_0 z^p + a_q z^{p+q}.
\]

(15)

Toeplitz band matrices \( T(a) \) for which \( C \setminus \Lambda(a) \) is disconnected were detected numerically in [1,5]; the polynomial \( Q_n(z) \) is of degree 6 in these cases. Here is an “analytical” result in this direction. We let \([x] \) stand for the integral part of \( x \).

**Proposition 5.2.** Let \( Q_n(z) = \mu z^p + (z - \alpha)^p(z - \beta)^p \), where \( \alpha, \beta, \mu \) are complex numbers and \( \alpha \beta \neq 0 \). If \( p = 1 \) or \( p = 2 \), then the set \( C \setminus \Lambda(a) \) is connected. If \( p \geq 3 \), then \( C \setminus \Lambda(a) \) has at most \([(p + 1)/2]\) components and for each natural number \( j \) between 1 and \([(p + 1)/2]\) there exist \( \alpha \) and \( \beta \) such that \( C \setminus \Lambda(a) \) has exactly \( j \) components.

**Proof.** Put \( b(z) = z - (\alpha + \beta) + \alpha \beta z^{-1} \). The curve \( b(T) \) is an ellipse with the foci \(- (\alpha + \beta) \pm 2\sqrt{\alpha \beta} \) and \( \Lambda(b) \) is known to be the line segment between these foci. Obviously, \( a(z) = z^{-p} Q_n(z) = \mu + (b(z))^p \). We claim that

\[
\Lambda(a) = \mu + (\Lambda(b))^p.
\]

(16)

To prove our claim, we may without loss of generality assume that \( b(t) \) traces out \( b(T) \) counter-clockwise as \( t \) moves around \( T \) counter-clockwise. Also notice that \( a_\varrho = \mu + b_\varrho^p \) for all \( \varrho \neq 0 \) (recall (7)). Pick \( \lambda \in \Lambda(b) \). Then \( \lambda \in \text{sp} T(b_\varrho) \) for all \( \varrho \neq 0 \) by (7). If \( \lambda \in b_\varrho(T) \), then \( \mu + \lambda p \in a_\varrho(T) \) and hence \( \mu + \lambda p \in \text{sp} T(a_\varrho) \). If \( \lambda \notin b_\varrho(T) \), then \( \text{wind} (b_\varrho - \lambda) = 1 \). As \( a_\varrho - \mu - \lambda p = b_\varrho - \lambda p \), we have

\[
\text{wind} (a_\varrho - \mu - \lambda p) = \text{wind} \prod_{k=1}^p (b_\varrho - \omega^k \lambda) = \sum_{k=1}^p \text{wind} (b_\varrho - \omega^k \lambda),
\]

where \( \omega = \exp(2\pi i/p) \), and since \( \text{wind} (b_\varrho - \omega^k \lambda) \) is either 0 or 1, it follows that \( \text{wind} (a_\varrho - \mu - \lambda p) \geq 1 \). Consequently, \( \mu + \lambda p \in \text{sp} T(a_\varrho) \). From (7) we now obtain the inclusion “\( \supset \)” in (16). To verify the reverse inclusion, let \( \varrho \neq 0 \) be any number such that \( \varrho^2 = \alpha \beta \). Then \( b_\varrho(T) = \Lambda(b) \). It results that \( a_\varrho(T) = \mu + (\Lambda(b))^p \), and as \( \text{wind} (a_\varrho - \xi) = 0 \) for all \( \xi \notin a_\varrho(T) \), we see that \( \text{sp} T(a_\varrho) = \mu + (\Lambda(b))^p \). Thus, by (7), \( \Lambda(a) \subset \mu + (\Lambda(b))^p \), which is the inclusion “\( \subset \)” of (16).

Since neither a line segment nor the square of a line segment does separate the plane, we obtain from (16) the assertion for \( p = 1 \) and \( p = 2 \). Combining (16) with the fact that \( \Lambda(b) \) is a line segment, one can easily see that \( C \setminus \Lambda(a) \) has at most \([(p + 1)/2]\) components and that each number of components between 1 and \([(p + 1)/2]\)
1)/2] can indeed be realized (for example, we get exactly \([(p + 1)/2]\) components if \(|\mu - (\alpha + \beta)| > 0\) is sufficiently small and \(|\alpha\beta|\) is sufficiently large). □

Let \(Q_n(z)\) be as in Proposition 5.2 and let \(\Omega\) be any component of \(\mathbb{C}\backslash \Lambda(a)\). Then, by (14), \(|z_p(\lambda)| < |z_{p+1}(\lambda)|\) for all \(\lambda \in \Omega\). If \(|z_p(\lambda)| < \varrho < |z_{p+1}(\lambda)|\), then \(\Lambda(a_\varrho) = \Lambda(a)\) and \(T(a_\varrho - \lambda)\) is invertible. This reveals that by appropriately choosing \(\alpha\) and \(\beta\) we can achieve that \(\Lambda(a)\) has any prescribed number of components between 1 and \([(p + 1)/2]\) and that any prescribed single component of \(\mathbb{C}\backslash \Lambda(a)\) contains points in \(\mathbb{C}\backslash \text{sp} T(a)\).

The group \(G\) is said to be \(p\)-transitive if for every two \(p\)-tuples \((\Sigma_{i_1}, \ldots, \Sigma_{i_p})\) and \((\Sigma_{j_1}, \ldots, \Sigma_{j_p})\) of \(p\) distinct branches of \(\Sigma\) there is a \(\pi \in G\) such that \(\pi(\Sigma_{i_1}) = \Sigma_{j_1}, \ldots, \pi(\Sigma_{i_p}) = \Sigma_{j_p}\). We call \(G\) weakly \(p\)-transitive if for every two sets \(\{\Sigma_{i_1}, \ldots, \Sigma_{i_p}\}\) and \(\{\Sigma_{j_1}, \ldots, \Sigma_{j_p}\}\) of \(p\) distinct branches of \(\Sigma\) there exists a \(\pi \in G\) such that \(\{\pi(\Sigma_{i_1}), \ldots, \pi(\Sigma_{i_p})\} = \{\Sigma_{j_1}, \ldots, \Sigma_{j_p}\}\), i.e., such that \(\pi(\Sigma_{i_1}), \ldots, \pi(\Sigma_{i_p})\) coincide with \(\Sigma_{j_1}, \ldots, \Sigma_{j_p}\) up to the arrangement. Clearly, weak 1-transitivity and 1-transitivity are equivalent.

**Theorem 5.3.** If the monodromy group of \(Q_n(z) - \lambda z^p = 0\) is weakly \(p\)-transitive, then \(\|f(\lambda)\|\) is either nowhere locally constant in \(\Omega\) or identically zero in \(\Omega\).

**Proof.** Weak \(p\)-transitivity means that we can permute any prescribed set of \(p\) branches into any prescribed set of \(p\) branches. We can in particular permute the \(p\) small branches at the points of \(\Omega\) into the \(p\) small branches at the points of \(W\). The assertion is therefore a direct consequence of Theorem 5.1. □

**Theorem 5.4.** The monodromy group of the polynomial \(Q_n(z) - \lambda z^p = 0\) is always 1-transitive.

**Proof.** It is well known (see, e.g., [21, Section 4.14]) that \(G\) is 1-transitive if and only if \(Q_n(z) - \lambda z^p\) is irreducible in \(\mathbb{C}[z, \lambda]\). But the irreducibility of \(Q_n(z) - \lambda z^p\) in \(\mathbb{C}[z, \lambda]\) can be readily verified. □

**Proof of Theorem 2.4(b).** Let \(\lambda\) be a point in \(\mathbb{C}\backslash \text{sp} T(a)\). We label the zeros \(z_1(\lambda), \ldots, z_{p+q}(\lambda)\) so that (13) holds. Thus, the small branches at \(\lambda\) are \(\Sigma_1, \ldots, \Sigma_p\) and the large branches at \(\lambda\) are \(\Sigma_{p+1}, \ldots, \Sigma_{p+q}\). By formula (11), the function \(g(\lambda) := \{T^{-1}(a - \lambda)\}_{11}\) equals

\[
(-1)^q a_q^{-1} \mu_1(\lambda)^{-1} \cdots \mu_q(\lambda)^{-1} = (-1)^q a_q^{-1} z_{p+1}(\lambda)^{-1} \cdots z_{p+q}(\lambda)^{-1}.
\]

This shows that \(g(\lambda) \neq 0\). Since \(G\) is 1-transitive (Theorem 5.4), there is a path in \(\mathbb{C}\backslash W\) starting and terminating at \(\lambda\) such that the \(q\) large branches at \(\lambda\) are permuted into \(q\) branches \(\Sigma_{i_1}, \ldots, \Sigma_{i_q}\) containing at least one small branch. Consequently, after analytic continuation along this curve, \(g(\lambda)\) becomes \(z_{i_1}(\lambda)^{-1} \cdots z_{i_q}(\lambda)^{-1}\), and because, obviously,
In this case we can have recourse to Proposition 5.2, which shows that it follows that \(|g(\lambda)|\) cannot be locally constant. Since \(f(\lambda) = \tilde{C}P_1g(\lambda)P_1\tilde{B} = g(\lambda)\tilde{C}P_1\tilde{B}\), we arrive at the conclusion that \(\|f(\lambda)\| = |g(\lambda)|\|\tilde{C}P_1\tilde{B}\|\) is either identically zero or nowhere locally constant in \(\Omega\). \(\square\)

Proof of Theorem 2.4(c). Without loss of generality assume \(p = 1\); the case \(q = 1\) can be reduced to the case \(p = 1\) by passage to adjoint operators. Combining Theorems 5.3 and 5.4 we arrive at the assertion. \(\square\)

6. Degree 4

We now turn to the case \(n = p + q = 4\). By what was already proved, we are left with the constellation \(p = q = 2\). Everything would be fine if the monodromy group of \(Q_4(z) - \lambda z^2 = 0\) were always weakly 2-transitive (Theorem 5.3). Unfortunately, this need not be the case. Indeed, let \(Q_4(z)\) be the polynomial \(\mu z^2 + (z - \alpha)^2(z - \beta)^2\), where \(\alpha, \beta, \mu \in \mathbb{C}\), \(\alpha \neq \beta\), and \(\alpha \beta \neq 0\). We have exactly two finite branch points, \(\lambda_1 (= \mu)\) and \(\lambda_2\), and the monodromy group is generated by \(\pi_1 = (12)(43)\) and \(\pi_2 = (13)(24)\). Here and in what follows we identify \(\Sigma_1, \Sigma_2, \ldots\) with the numbers \(1, 2, \ldots\). Clearly, we cannot permute \(\{1, 2\}\) into \(\{1, 3\}\). Fortunately, in this case we can have recourse to Proposition 5.2, which shows that \(\mathbb{C}\setminus A(a)\) is connected.

Let us return for a moment to the case of general \(n\) and \(p, q\). The permutation \(\pi\) associated with a branch point \(\lambda\) can be written as a product of cycles. We say that \(\lambda\) is of the type \((L_1, L_2, \ldots)\) if the cycle lengths of \(\pi\) are \(L_1, L_2, \ldots\). In the previous paragraph, we encountered two finite branch points of the type \((2, 2)\).

Theorem 6.1. If all finite branch points of the Riemann surface of the polynomial \(Q_n(z) - \lambda z^2 = 0\) are of the type \((2, 1, \ldots, 1)\), then the monodromy group of \(Q_n(z) - \lambda z^2 = 0\) is weakly 2-transitive.

Proof. It suffices to show that we can permute \(\{1, 2\}\) to \(\{1, m\}\) for arbitrary \(m \neq 1\). Let \(\lambda_1, \ldots, \lambda_k\) be the finite branch points. Without loss of generality suppose that the permutations \((12), (13), \ldots, (1 r)\) are among \(\pi_1, \ldots, \pi_k\) and that the permutations \((1 r + 1), \ldots, (1 n)\) are not among \(\pi_1, \ldots, \pi_k\). It is easily seen that we can permute \(\{1, 2\}\) to \(\{1, m\}\) for every \(m \in \{2, \ldots, r\}\). Thus, let \(m \geq r + 1\). Since \(G\) is 1-transitive (Theorem 5.4), there is a path on \(\Sigma\) joining \(\Sigma_1\) to \(\Sigma_m\). This path goes through the branches \(x_1, x_2, \ldots, x_l\) and we may assume that \(x_j \neq 1\) and \(x_j \neq m\) for all \(j\) and that \(x_1 \in \{2, \ldots, r\}\). As each branch point is of the type \((2, 1, \ldots, 1)\), it follows that the path goes from the branch \(x_1\) to the branch 1 and stays there. Thus, we can permute \(\{1, x_1\}\) to \(\{1, m\}\). \(\square\)
Theorem 6.2. If \( p = q = 2 \), then \( \| f(\lambda) \| \) is either nowhere locally constant in \( \Omega \) or identically zero in \( \Omega \).

Proof. If there is a \( \lambda \) such that \( Q_4(z) - \lambda z^2 \) has two distinct zeros of multiplicity 2 or one zero of multiplicity 4, then the polynomial \( Q_4(z) \) is of the form \( \lambda z^2 + (z - \alpha)^2(z - \beta)^2 \), and the assertion follows from Proposition 5.2 and Theorem 2.4(a). Thus, we may restrict ourselves to the case where all of the (at most four) finite branch points are of the types \((3, 1)\) or \((2, 1, 1)\). It is easily seen that the branch point at infinity is of the type \((2, 2)\). Our goal is to show that \( G \) is weakly 2-transitive, so that the present theorem follows from Theorem 5.3.

If all finite branch points have the type \((2, 1, 1)\), then Theorem 6.1 implies weak 2-transitivity. Thus, assume there is at least one finite branch point, \( \lambda_1 \), of the type \((3, 1)\) and that \( \pi_1 = (123) \). Since \( G \) is 1-transitive, there must exist at least one more finite branch point \( \lambda_2 \).

We first consider the case where all finite branch points different from \( \lambda_1 \) are of the type \((2, 1, 1)\). Because \( G \) is 1-transitive, branch 4 must be in the cycle of length 2 of one of these branch points, say \( \lambda_2 \). By symmetry, we may assume that the permutation \( \pi_2 \) is \((14)\). But a group of permutations of 1, 2, 3, 4 containing \( \pi_1 = (123) \) and \( \pi_2 = (14) \) is easily seen to be weakly 2-transitive.

We are left with the case where \( \lambda_2 \) is of the type \((3, 1)\). If branch 4 is in the cycle of length 3 of \( \pi_2 \), then \( G \) is readily checked to be weakly 2-transitive. So assume branch 4 is left fixed by \( \pi_2 \), then there must exist a third finite branch point \( \lambda_3 \) of the type \((2, 1, 1)\) such that branch 4 is contained in a cycle of length 2. By direct inspection of the few possible cases one sees that the group \( G \) is always weakly 2-transitive. \( \Box \)

7. Prime number degree

In this section, we complete the proofs of parts (d) and (e) of Theorem 2.4.

Lemma 7.1. The order of the monodromy group \( G \) of \( Q_n(z) - \lambda z^p = 0 \) is divisible by \( n \).

Proof. For \( 1 \leq j \leq n \), let \( G_j = \{ \pi \in G : \pi(1) = j \} \). Clearly, \( G = G_1 \cup \cdots \cup G_n \) and \( G_i \cap G_j = \emptyset \) for \( i \neq j \). Given two distinct numbers \( i, j \) in \( \{1, \ldots, n\} \), we can find a \( \sigma \in G \) such that \( \sigma(i) = j \) (Theorem 5.4). The map \( G_i \to G_j, \pi \mapsto \sigma \pi \) is obviously bijective. This implies that all the sets \( G_j \) have the same number of elements, say \( l \). Consequently, the order of \( G \) is \( nl \). \( \Box \)

Lemma 7.2. If \( n \) is a prime number, then the monodromy group \( G \) of the polynomial \( Q_n(z) - \lambda z^p = 0 \) contains an \( n \)-cycle, that is, after appropriately labelling the branches, \( (1, 2, \ldots, n) \) \( \in G \).
**Proof.** This follows from Lemma 7.1 and a well-known theorem by Cauchy, which says that if $n$ is a prime and the order of a finite group is divisible by $n$, then the group contains an element of the order $n$. □

**Lemma 7.3** (well known). If a subgroup $G$ of the full symmetric group $S_n$ contains an $n$-cycle and a transposition, then $G = S_n$. □

**Proof of Theorem 2.4(d).** If $p = 2$ or $q = 2$, then the branch point at infinity of the Riemann surface of $Q_n(z) - \lambda z^p = 0$ is of the type $(n - 2, 2)$. Hence $\pi_n^{n-2}$ is a transposition (notice that $n - 2$ is odd). Combining Theorem 5.3 with Lemmas 7.2 and 7.3 we arrive at the assertion. □

**Proof of Theorem 2.4(e).** Without loss of generality, assume $p \leq q$. Parts (a) and (c) of Theorem 2.4 imply the desired result for $p = 0$ and $p = 1$. Theorem 6.2 disposes the case $p = q = 2$, and Theorem 2.4(d) gives the assertion for $p = 2$, $q = 3$ and $p = 2$, $q = 5$. We are left with the case where $p = 3$ and $q = 4$. In this case, we deduce from Lemma 7.2 that $G$ contains a 7-cycle, say $\pi = (1, 2, \ldots, 7)$. The branch point at infinity provides us with a permutation of the type $\sigma = (1xyz)(abc)$. By checking all possible cases, one sees that the group generated by $\pi$ and $\sigma$ is always weakly 3-transitive. It remains to make use of Theorem 5.3. □

**Remark.** The arguments of this section are standard in the Galois theory, and in fact the monodromy group $G$ of $Q_n(z) - \lambda z^p = 0$ is known to be isomorphic to the Galois group $G_0$ of $Q_n(z) - \lambda z^p = 0$. Let $\mathcal{R} = \mathbb{C}(\lambda)$ be the field of the rational functions (of $\lambda$) with complex coefficients. We think of $Q_n(z) - \lambda z^p$ as an element of $\mathcal{R}[z]$. The splitting field of this polynomial is $\mathcal{R}(z_1(\lambda), \ldots, z_n(\lambda))$, where $z_1(\lambda), \ldots, z_n(\lambda)$ are given by (10). The Galois group $G_0$ may be identified with the group of all permutations of $z_1(\lambda), \ldots, z_n(\lambda)$ that can be extended to an automorphism of the field $\mathcal{R}(z_1(\lambda), \ldots, z_n(\lambda))$ that leaves the elements of $\mathcal{R}$ fixed. When working with the monodromy group, we transform valid equalities into new equalities by analytic continuation, while from the algebraic point of view, valid equalities are transformed into new equalities by the action of the Galois group.

The polynomial $Q_n(z) - \lambda z^p$ can be shown to be the minimal polynomial of each of its zeros $z_1(\lambda), \ldots, z_n(\lambda)$. This implies that the dimension of $\mathcal{R}(z_1(\lambda))$ over $\mathcal{R}$ is $n$. Lemma 7.1 so amounts to saying that this dimension divides the order of $G_0$, and this is one of the conclusions of the so-called fundamental theorem of the Galois theory.

**8. Degree 6**

In this section, we prove part (f) of Theorem 2.4. By virtue of parts (a) and (c) of Theorem 2.4, we can restrict ourselves to the cases $p = 2$, $q = 4$ and $p = 3$, $q = 3$. 
Unfortunately, in these cases the simple monodromy group arguments employed in the previous sections do not work. We therefore proceed in a different way.

Thus, let \( f(\lambda) = P_m T^{-1}(a - \lambda) P_m \), where the function \( a(z) \) is of form (5) with \( p + q = 6 \). We may assume that \( a_6 = 1 \). Suppose the norm \( \| f(\lambda) \| \) is a non-zero constant on some open subset of \( \Omega \). By Lemma 3.1, there exist non-zero vectors

\[
x = (x_0 x_1 \cdots x_{m-1})^T \in \mathbb{C}^m \quad \text{and} \quad y = (y_0 y_1 \cdots y_{m-1})^T \in \mathbb{C}^m
\]

such that \( f(\lambda) x = y \) for all \( \lambda \) in \( \Omega \).

We consider only the case \( p = q = 3 \); the case \( p = 2, q = 4 \) can be treated similarly. By (11), the first of the \( m \) equalities \( f(\lambda)x = y \) can be written in the form

\[
x_0 + \sum_{j=1}^{m-1} x_j f_j(\delta_1(\lambda), \delta_2(\lambda), \delta_3(\lambda)) = y_0 \mu_1(\lambda) \mu_2(\lambda) \mu_3(\lambda),
\]

where \( f_1, \ldots, f_{m-1} \) are symmetric polynomials of three variables with coefficients in \( \mathbb{Z} \). We continue this equality analytically to an equality in \( \Omega_\infty \). For the sake of definiteness, suppose that the resulting equality is

\[
x_0 + \sum_{j=1}^{m-1} x_j f_j(\mu_1(\lambda), \mu_2(\lambda), \delta_3(\lambda)) = y_0 \mu_3(\lambda) \delta_1(\lambda) \delta_2(\lambda);
\]

(17) the other possibilities can be tackled analogously (also notice that, by virtue of the 1-transitivity of \( G \), we may always assume that one small branch goes over into a small branch).

Uniformization at infinity shows that if we write \( \lambda = \tau^3 \), then

\[
\mu_j(\lambda) = \omega^j \tau + b + c/(\omega^j \tau) + O(1/\tau^2) \quad \text{as} \quad \tau \to \infty,
\]

(18)

\[
\delta_j(\tau) = \gamma/(\omega^j \tau) + O(1/\tau^2) \quad \text{as} \quad \tau \to \infty,
\]

(19)

where \( \omega = \exp(2\pi i/3) \) and

\[
\gamma^3 = a_{-3}, \quad b = -a_2/3, \quad c = a_2^2/9 - a_1,
\]

(20)

Analytic continuation along a large counter-clockwise oriented circle transforms \( \mu_j(\lambda) \) and \( \delta_j(\lambda) \) to \( \omega \mu_j(\lambda) \) and \( \omega^2 \delta_j(\lambda) \), respectively. Thus, analytically continuing (17) once and twice along a large circle gives the two equalities

\[
x_0 + \sum_{j=1}^{m-1} x_j f_j(\omega \mu_1(\lambda), \omega \mu_2(\lambda), \omega^2 \delta_1(\lambda)) = y_0 \omega^2 \mu_3(\lambda) \delta_1(\lambda) \delta_2(\lambda),
\]

(21)

\[
x_0 + \sum_{j=1}^{m-1} x_j f_j(\omega^2 \mu_1(\lambda), \omega^2 \mu_2(\lambda), \omega \delta_1(\lambda)) = y_0 \omega \mu_3(\lambda) \delta_1(\lambda) \delta_2(\lambda).
\]

(22)

Equalities (17), (21), and (22) can be written in the form

\[
F_1(\lambda) + F_2(\lambda) + F_3(\lambda) = y_0 \mu_3(\lambda) \delta_1(\lambda) \delta_2(\lambda),
\]
\( F_1(\lambda) + \omega F_2(\lambda) + \omega^2 F_3(\lambda) = \omega^3 y_0 \mu_3(\lambda) \delta_1(\lambda) \delta_2(\lambda), \)

\( F_1(\lambda) + \omega^2 F_2(\lambda) + \omega F_3(\lambda) = \omega y_0 \mu_3(\lambda) \delta_1(\lambda) \delta_2(\lambda), \)

where \( F_1(\lambda), F_2(\lambda), F_3(\lambda) \) are independent of \( \omega \). It follows that

\[
F_1(\lambda) = 0, \quad F_2(\lambda) = 0, \quad F_3(\lambda) = y_0 \mu_3(\lambda) \delta_1(\lambda) \delta_2(\lambda). \tag{23}
\]

Without loss of generality assume that \( m = 3k \) (extend \( x \) by zeros if necessary). We can then write the left-hand sides of the equalities (23) as

\[
x_{3k-1}(\sigma_{3k-2} \delta_1 + \cdots) + x_{3k-2}(\sigma_{3k-4} \delta_1^2 + \cdots) + x_{3k-3}(\sigma_{3k-3} + \cdots) + \cdots,
\]

\[
x_{3k-1}(\sigma_{3k-3} \delta_1^2 + \cdots) + x_{3k-2}(\sigma_{3k-2} + \cdots) + x_{3k-3}(\sigma_{3k-4} \delta_1 + \cdots) + \cdots,
\]

\[
x_{3k-1}(\sigma_{3k-1} + \cdots) + x_{3k-2}(\sigma_{3k-3} \delta_1 + \cdots) + x_{3k-3}(\sigma_{3k-5} \delta_1^2 + \cdots) + \cdots,
\]

where \( \delta_1 \) stands for \( \delta_1(\lambda) \) and \( \sigma_n = \sigma_n(\lambda) \) is defined by

\[
\sigma_n = \sum_{\alpha+\beta=n} \mu_1(\lambda)^\alpha \mu_2(\lambda)^\beta.
\]

Using (18) one can show that

\[
\sigma_n = \frac{1 - \omega^{n+1}}{1 - \omega} \tau^n + (n + 1)b \frac{1 - \omega^n}{1 - \omega} \tau^{n-1} + K \tau^{n-2} + O(\tau^{n-3}), \tag{27}
\]

\[
K = \frac{n + 1}{1 - \omega} \left[ c(1 - \omega^{n+2}) + \frac{n}{2} b^2 (1 - \omega^{n-1}) - \frac{(1 - \omega^{n+1})(1 + \omega)}{n + 1} c \right]. \tag{28}
\]

With the help of (19), (27) and (28) we can identify the highest powers of \( \tau \) in (24) to (26). In this way we obtain from (25) that \( x_{3k-2} = 0 \). Thus, put \( x_{3k-2} = 0 \) in (24) and (26). If \( b \neq 0 \), then (27), (19), and (26) imply that \( x_{3k-1} = 0 \), while (27), (19), and (24) give \( x_{3k-3} = 0 \). Repeating this argument we arrive at the conclusion that \( x = 0 \), which is impossible. Hence, \( b = 0 \). If \( c \neq 0 \), we can combine (27), (28), (19) first with (26) and then with (24) to conclude that \( x_{3k-1} = 0 \) and \( x_{3k-1} = 0 \), which after repetition shows that \( x = 0 \). As this is a contradiction, we must have \( b = c = 0 \). By (20), this means that \( a_1 = a_2 = 0 \). Consideration of the adjoint operator \( f(\zeta)^* \) yields in the same way that \( a_{-1} = a_{-2} = 0 \). Consequently, \( Q_6(z) \) is of the form

\[
Q_6(z) = a_{-3} + a_0 \zeta^3 + \zeta^6.
\]

But as mentioned in Section 5 (recall (15)), in this case \( \mathbb{C} \setminus A(a) \) is connected (actually it is easily seen that \( A(a) \) is a line segment). This completes the proof.
Appendix A

For the reader’s convenience, we here give a proof of the following result of [12]: if $A$, $B$, $C$ are bounded linear Hilbert space operators and $\varepsilon > 0$, then

$$
\bigcup_{\|K\|<\varepsilon} \text{sp} (A + BKC) = \text{sp} A \cup \{ \lambda \notin \text{sp} A : \|C(A - \lambda I)^{-1}B\| > 1/\varepsilon \}, \quad \text{(A.1)}
$$

$$
\bigcup_{\|K\|\leq \varepsilon} \text{sp} (A + BKC) = \text{sp} A \cup \{ \lambda \notin \text{sp} A : \|C(A - \lambda I)^{-1}B\| \geq 1/\varepsilon \}. \quad \text{(A.2)}
$$

Our proof is based on the ideas of [12] and in particular on the following well-known lemma.

**Lemma A.1.** If $M$ and $N$ are linear operators, then $I + MN$ is invertible if and only if $I + NM$ is invertible, in which case

$$(I + MN)^{-1} = I - M(I + NM)^{-1}N.$$ 

That the left-hand sides of (A.1) and (A.2) are contained in the corresponding right-hand sides follows from the following observation.

**Proposition A.2.** If $A$ is invertible and $\|CA^{-1}B\| < 1/\delta$, then there exists an operator $K$ such that $\|K\| = \delta$ and $A + BKC$ is not invertible.

**Proof.** In either case, $\|CA^{-1}B\| < 1$. Hence $I + CA^{-1}B$ is invertible. Using Lemma A.1 with $N = C$ and $M = A^{-1}B$, we see that $I + A^{-1}BKC$ and thus also $A + BKC = A(I + A^{-1}BKC)$ are invertible. $\square$

The following result shows that the right-hand sides of (A.1) and (A.2) are subsets of the corresponding left-hand sides.

**Proposition A.3.** If $A$ is invertible and $\|CA^{-1}B\| = 1/\delta$, then there exists an operator $K$ such that $\|K\| = \delta$ and $A + BKC$ is not invertible.

**Proof.** Clearly, $A + BKC = A(I + A^{-1}BKC)$ is invertible if and only if $I + A^{-1}BKC$ is invertible, which, by Lemma A.1 with $M = A^{-1}B$ and $N = KC$, is equivalent to the invertibility of $I + KCA^{-1}B$. Abbreviate $CA^{-1}B$ to $X$ and put $K = -\delta^2 X^*$. Then $\|K\| = \delta$ and $I + KCA^{-1}B = I - \delta^2 X^*X$. The spectral radius of the positively semi-definite operator $X^*X$ coincides with its norm, that is, with $\|X^*X\| = \|X\|^2 = 1/\delta^2$. It follows that $1/\delta^2 \in \text{sp} X^*X$. The spectral mapping theorem therefore implies that $0 \in \text{sp} (I - \delta^2 X^*X)$. $\square$
We finally remark that [12] contains a modification of Proposition A.3 (and its proof) that yields (A.1) for Banach space operators and (A.2) in the Banach space case under the additional assumption that $B$ or $C$ is compact. We do not know whether (8) is always true in Banach space.

Acknowledgement

We thank both referees for their valuable remarks. We are especially indebted to one of the referees for pointing out an oversight in the original version of this paper.

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