Spectra of Toeplitz operators and compositions of Muckenhoupt weights with Blaschke products

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Abstract. We discuss the optimality of a sufficient condition from [12] for a point to belong to the essential spectrum of a Toeplitz operator with a bounded measurable coefficient. Our approach is based on a new sufficient condition for a composition of a Muckenhoupt weight with a Blaschke product to belong to the same Muckenhoupt class.

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1. Introduction and main results

Let $T = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ be the unit circle. A number $c \in \mathbb{C}$ is called a (left, right) cluster value of a measurable function $a : T \to \mathbb{C}$ at a point $\zeta \in T$ if $1/(a - c) \notin L^\infty(W)$ for every neighbourhood (left semi-neighbourhood, right semi-neighbourhood) $W \subset T$ of $\zeta$. Cluster values are invariant under changes of the function on measure zero sets. We denote the set of all left (right) cluster values of $a$ at $\zeta$ by $a(\zeta-0)$ (by $a(\zeta+0)$), and use also the following notation $a(\zeta) = a(\zeta-0) \cup a(\zeta+0)$, $a(T) = \cup_{\zeta \in T} a(\zeta)$. It is easy to see that $a(\zeta-0)$, $a(\zeta+0)$, $a(\zeta)$ and $a(T)$ are closed sets. Hence they are all compact if $a \in L^\infty(T)$.

Let $H^p(T)$, $1 \leq p \leq \infty$ denote the Hardy space, that is $H^p(T) := \{f \in L^p(T) : f_n = 0 \text{ for } n < 0\}$, where $f_n$ is the $n$th Fourier coefficient of $f$. Let $T(a) : H^p(T) \to H^p(T)$, $1 < p < \infty$ denote the Toeplitz operator generated by a function $a \in L^\infty(T)$, i.e. $T(a)f = P(af)$, $f \in H^p(T)$, where $P$ is the Riesz

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projection:
\[ Pg(\zeta) = \frac{1}{2} g(\zeta) + \frac{1}{2\pi i} \int_T \frac{g(w)}{w - \zeta} dw, \quad \zeta \in \mathbb{T}. \]

\[ P : L^p(\mathbb{T}) \to H^p(\mathbb{T}), \ 1 < p < \infty \]

is a bounded projection and
\[ P \left( \sum_{n=-\infty}^{+\infty} g_n \zeta^n \right) = \sum_{n=0}^{+\infty} g_n \zeta^n. \]

If \( a(\zeta) \) consists of at most two points for each \( \zeta \in \mathbb{T} \), in particular if \( a \) is continuous or piecewise continuous, then the spectrum of \( T(a) \) can be described in terms of \( a(\zeta \pm 0), \zeta \in \mathbb{T} \) (see \([3, 4, 13]\)). This is no longer possible if \( a(\zeta) \) is allowed to contain more than two points (see \([2, 4.71-4.78]\) and \([10]\)). It is no longer sufficient to know the values of \( a \) in this case, it is important to know “how these values are attained” by \( a \).

Since a complete description of the essential spectrum of \( T(a) \) in terms of the cluster values of \( a \in L^\infty(\mathbb{T}) \) is impossible, it is natural to try finding “optimal” sufficient conditions for a point \( \lambda \) to belong to the essential spectrum. Results of this sort were obtained in \([11, 12]\). In order to state them we need the following notation.

Let \( K \subset \mathbb{C} \) be an arbitrary compact set and \( \lambda \in \mathbb{C} \setminus K \). Then the set
\[ \sigma(K; \lambda) = \left\{ \frac{w-\lambda}{|w-\lambda|} \mid w \in K \right\} \subseteq \mathbb{T} \]
is compact as a continuous image of a compact set. Hence the set \( \Delta_\lambda(K) := \mathbb{T} \setminus \sigma(K; \lambda) \) is open in \( \mathbb{T} \). So, \( \Delta_\lambda(K) \) is the union of an at most countable family of open arcs.

We call an open arc of \( \mathbb{T} \) \( p \)-large if its length is greater than or equal to \( \frac{2\pi}{\max\{p,q\}} \), where \( q = \frac{p}{p+q}, \ 1 < p < \infty \).

The following result has been proved in \([12]\).

**Theorem 1.1.** Let \( 1 < p < \infty, \ a \in L^\infty(\mathbb{T}), \ \lambda \in \mathbb{C} \setminus a(\mathbb{T}) \) and suppose that, for some \( \zeta \in \mathbb{T} \),
(i) \( \Delta_\lambda(a(\zeta - 0)) \) (or \( \Delta_\lambda(a(\zeta + 0)) \)) contains at least two \( p \)-large arcs,
(ii) \( \Delta_\lambda(a(\zeta + 0)) \) (or \( \Delta_\lambda(a(\zeta - 0)) \) respectively) contains at least one \( p \)-large arc. Then \( \lambda \) belongs to the essential spectrum of \( T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \).

A weaker result (with \( \Delta_\lambda(a(\zeta)) \) in place of \( \Delta_\lambda(a(\zeta \pm 0)) \)) in condition (ii) was proved in \([11]\) where it was also shown that condition (i) is optimal in the following sense: for any compact \( K \subset \mathbb{C} \) and \( \lambda \in \mathbb{C} \setminus K \) such that \( \Delta_\lambda(K) \) contains at most one \( p \)-large arc there exists \( a \in L^\infty(\mathbb{T}) \) such that \( a(-1 \pm 0) = a(\mathbb{T}) = K \) and \( T(a) - \lambda I : H^r(\mathbb{T}) \to H^r(\mathbb{T}) \) is invertible for any \( r \in [\min\{p,q\}, \max\{p,q\}] \). A question that has been open since \([11]\) is whether or not condition (ii) can be dropped, i.e. whether condition (i) alone is sufficient for \( \lambda \) to belong to the essential spectrum of \( T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \). The following result gives a negative answer to this question.
Theorem 1.2. There exists \( a \in L^\infty(\mathbb{T}) \) such that \( a(1 - \overline{a}) = \{ \pm 1 \}, \ |a| \equiv 1, \ T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \) is invertible for any \( p \in (1, 2) \), and \( T(1/a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \) is invertible for any \( p \in (2, +\infty) \).

The proof of Theorem 1.2 relies on an argument which is related to the following question. Suppose \( v \) is an inner function, i.e., \( v \) is a nonconstant function in \( H^\infty(\mathbb{T}) \) such that \( |v| = 1 \) almost everywhere on \( \mathbb{T} \). If \( b \in L^\infty(\mathbb{T}) \), then \( b \circ v \in L^\infty(\mathbb{T}) \) and the question is whether or not the invertibility of \( T(b) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \) implies that of \( T(b \circ v) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \).

An equivalent form of this question is in terms \( A_p \) classes (see [1, Section 1]). We say that a measurable function \( \rho : \mathbb{T} \to [0, +\infty] \) satisfies the \( A_p \) condition if

\[
\sup_I \left( \frac{1}{|I|} \int_I \rho^p(\zeta) |d\zeta| \right) \geq \left( \frac{1}{|I|} \int_I \rho^{-q}(\zeta) |d\zeta| \right) = C_p < \infty, \tag{1.1}
\]

where \( I \subset \mathbb{T} \) is an arbitrary arc and \( |I| \) denotes its length. The question is whether or not \( p \in A_p \) implies \( \rho \circ v \in A_p \).

Although the answer is positive in the case \( p = 2 \) (see, e.g., [1, Section 2]), it turns out that for every \( p \in (1, +\infty) \setminus \{2\} \) there exist a Blaschke product \( B \) and \( \rho \in A_p \) such that \( \rho \circ B \notin A_p \) (see [1, Theorem 9]). Equivalently, there exists \( b \in L^\infty(\mathbb{T}) \) such that \( T(b) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \) is invertible, but \( T(b \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \) is not invertible (see [1, Theorem 12]).

We prove a result in the opposite direction, namely we describe a class of Blaschke products for which the implications

\[
\rho \in A_p \implies \rho \circ B \in A_p, \quad T(b) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \text{ is invertible} \implies T(b \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \text{ is invertible}
\]

do hold.

Consider the Blaschke product

\[
B(e^{i\theta}) = \prod_{k=1}^{\infty} \frac{r_k - e^{i\theta}}{1 - r_k e^{i\theta}}, \quad \theta \in [-\pi, \pi], \tag{1.2}
\]

where \( r_k \in (0, 1) \) and \( \sum_{k=1}^{\infty} (1 - r_k) < 1 \).

Theorem 1.3. Suppose \( r_1 \leq r_2 \leq \cdots \leq r_n \leq \cdots \), and

\[
\inf_{k \geq 1} \frac{1 - r_{k+1}}{1 - r_k} > 0. \tag{1.3}
\]

If \( \rho \) satisfies the \( A_p \) condition, then \( \rho \circ B \) also satisfies the \( A_p \) condition.

Corollary 1.4. Let \( 1 < p < \infty \), \( a \in L^\infty(\mathbb{T}) \), and let a Blaschke product \( B \) satisfy the conditions of Theorem 1.3. Then \( T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \) is invertible if and only if \( T(a \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T}) \) is invertible.

Proof. The invertibility of \( T(a \circ B) \) implies that of \( T(a) \) according to [1, Theorem 12]. The opposite implication follows from Theorem 1.3 (see [1, Section 1]). \( \square \)
2. Auxiliary results on inner and outer functions

According to the canonical factorisation theorem (see, e.g., [5, Theorem 2.8]), any function from $H^p(\mathbb{T}) \setminus \{0\}$ has a unique, modulo a constant factor, representation as the product of an outer function from $H^p(\mathbb{T})$ and an inner function.

A function $F \in H^p(\mathbb{T})$ is called an **outer function** if

$$F(z) = e^{ie} \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \phi(t) \, dt \right), \quad |z| < 1,$$

where $e$ is a real number, $\phi \geq 0$, $\log \phi \in L^1([−\pi, \pi])$, and $\phi \in L^p([−\pi, \pi])$.

A function $v \in H^\infty(\mathbb{T})$ is called an **inner function** if $|v| = 1$ almost everywhere on $\mathbb{T}$. Any inner function $v$ admits a unique factorisation of the form

$$v(z) = e^{ie}B(z)S(z),$$

where $e$ is a real number, $B$ is a Blaschke product

$$B(z) = z^m \prod_{k} \frac{z_k - z}{1 - z_k z},$$

with $m \in \mathbb{N} \cup \{0\}$, $z_k = r_k \exp(it_k) \neq 0$, $t_k \in (−\pi, \pi]$, $r_k = |z_k| < 1$, and $S$ is a singular inner function

$$S(z) = \exp \left( -\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)$$

with a nonnegative measure $\mu$ which is singular with respect to the standard Lebesgue measure on $[−\pi, \pi]$.

We are particularly interested in the case where $v$ has a unique discontinuity at $z = 1$ and infinitely many zeros $z_k$. In this case, $\lim_{k \to \infty} z_k = 1$, the singular measure $\mu$ is supported by the point $t = 0$, and

$$S(z) = \exp \left( \kappa \frac{z + 1}{z - 1} \right), \quad \kappa = \text{const} > 0$$

(see [7, Ch. II, Theorems 6.1 and 6.2]). We will also assume that $B(0) \neq 0$. Then

$$B \left( e^{i\theta} \right) = \prod_{k=1}^{\infty} \frac{z_k - e^{i\theta}}{\bar{z}_k - e^{i\theta}}, \quad \theta \in [−\pi, \pi].$$

(2.2)

Theorem 2.1. ([6, Theorem 2.8]) Suppose $B$ has the form (2.2) and $\lim_{k \to \infty} z_k = 1$. Then one can choose a branch of $\arg B \left( e^{i\tau} \right)$ which is continuous and increasing on $(0, 2\pi)$, and which satisfies the following condition

$$\lim_{\tau \to 0^+} \arg B \left( e^{i\tau} \right) =: A_+ < 0, \quad \lim_{\tau \to 2\pi^-} \arg B \left( e^{i\tau} \right) =: A_- > 0.$$

Moreover, at least one of these limits is infinite and

$$\arg B \left( e^{i\theta} \right) = \begin{cases} -2 \left( \sum_{\theta_k \geq \theta} (\pi + \varphi_k(\theta)) + \sum_{\theta_k < \theta} \varphi_k(\theta) \right), & \theta \in (0, \pi], \\ 2 \left( \sum_{\theta_k \leq \theta} (\pi - \varphi_k(\theta)) - \sum_{\theta_k > \theta} \varphi_k(\theta) \right), & \theta \in [−\pi, 0), \end{cases}$$

(2.3)
where
\[ \varphi_k(\theta) = \arctan \left( \varepsilon_k \cot \frac{\theta - \theta_k}{2} \right), \quad \varepsilon_k = \frac{1 - r_k}{1 + r_k}. \tag{2.4} \]

**Theorem 2.2.** (See [6, Theorem 2.10 and the end of the proof of Theorem 5.9].) Suppose a real valued function \( \eta \) is continuous on \([-\pi, \pi] \setminus \{0\} \) and
\[ \lim_{t \to 0^{\pm}} (\eta(t) \mp \pi \log |t|) = 0. \]

Then the function \( e^{i\eta} \) admits the following representation
\[ e^{i\eta(t)} = B(e^{it}) g(B(e^{it})) d(e^{it}), \quad t \in [-\pi, \pi], \]
where \( g, d \in C(\mathbb{T}) \), the index of \( g \) is 0, and \( B \) is the infinite Blaschke product with the zeros
\[ z_k = \frac{2 - \exp(-k + 1/2)}{2 + \exp(-k + 1/2)}. \]

We finish this section with an example of an outer function which is used in the proof of Theorem 1.2.

**Example 2.3.** Consider the function
\[ h(z) = \exp \left(-i c \log \left( i \frac{1 - z}{2} \right) \right), \]
where \( c > 0 \) and \( \log \) denotes the branch of logarithm which is analytic in the complex plane cut along \(( -\infty, 0 ] \) and real valued on \(( 0, +\infty ) \). It is clear that \( h \) is analytic inside the unit disk, and since
\[ \text{Im} \left( i \frac{1 - z}{2} \right) > 0, \quad |z| < 1, \]
h satisfies the following estimate
\[ 1 < |h(z)| < e^{c\pi}, \quad |z| < 1. \]
Hence, \( h, 1/h \in H^\infty(\mathbb{T}) \) and \( h \) is an outer function (see [7, Ch. II, Corollary 4.7]). It is also clear that \( h \in C^\infty(\mathbb{T} \setminus \{1\}) \), and since
\[ i \frac{1 - e^{i\theta}}{2} = e^{i \frac{\theta}{2}} \sin \frac{\theta}{2}, \]
we have
\[ |h(e^{i\theta})| = \begin{cases} \exp \left( c \frac{\theta}{2} \right), & \theta \in (0, \pi], \\ \exp \left( c \left( \frac{\theta}{2} + \pi \right) \right), & \theta \in [-\pi, 0), \end{cases} \]
\[ \arg h(e^{i\theta}) = -c \log \left| \sin \frac{\theta}{2} \right|. \tag{2.5} \]
### 3. Proof of Theorem 1.3

Suppose the conditions of Theorem 1.3 are satisfied and let

\[ A(\theta) := \arg B(e^{i\theta}), \quad A(\pm \pi) = 0. \]

The proof of Theorem 1.3 relies upon analysis of the properties of \( A \). The corresponding results are collected in the following two lemmas. Since \( A \) admits the representation (2.3), (2.4) (with \( \theta_k = 0 \) for all \( k = 1, 2, \ldots \)), it is convenient to rewrite (1.3) in the following equivalent form

\[
\inf_{k \geq 1} \frac{\varepsilon_{k+1}}{\varepsilon_k} := c_0 > 0. \tag{3.1}
\]

**Lemma 3.1.**

a) The derivative \( A' \) is increasing on \([-\pi, 0)\) and decreasing on \((0, \pi]\).

b) \[
\frac{c_1}{4|\sin \frac{\theta}{2}|} \leq A'(\theta) \leq \frac{|A(\theta)|}{|\sin \theta|}, \quad \forall \theta \in [-\pi, \pi] \setminus \{0\}, \quad c_1 := \min\{c_0, \varepsilon_1\}.
\]

c) \[
\frac{A'(\theta/c)}{A'(\theta)} < c^2, \quad \forall \theta \in [-\pi, \pi] \setminus \{0\}, \quad \forall c > 1.
\]

**Proof.** Let

\[ A_k(\theta):= \arctan \left( \varepsilon_k \cot \frac{\theta}{2} \right). \]

Then

\[ A(\theta) = -2 \sum_{k=1}^{\infty} A_k(\theta), \quad A'(\theta) = -2 \sum_{k=1}^{\infty} A_k'(\theta) \]

(see (2.3), (2.4)).

a) Since

\[
-A'_k(\theta) = \frac{\varepsilon_k}{2 \sin^2 \frac{\theta}{2}} \frac{1}{1 + (\varepsilon_k \cot \frac{\theta}{2})^2} = \frac{\varepsilon_k}{2 \left( \sin^2 \frac{\theta}{2} + (\varepsilon_k \cos \frac{\theta}{2})^2 \right)}
\]

\[ = \frac{\varepsilon_k}{2 \left( (1 - \varepsilon_k^2) \sin^2 \frac{\theta}{2} + \varepsilon_k^2 \right)}, \]

\( A' \) is increasing on \([-\pi, 0)\) and decreasing on \((0, \pi]\).

b) The equality

\[
-A'_k(\theta) = \frac{\varepsilon_k}{2 \sin^2 \frac{\theta}{2}} \frac{1}{1 + (\varepsilon_k \cot \frac{\theta}{2})^2} = \frac{1}{\sin \theta} \frac{\varepsilon_k \cot \frac{\theta}{2}}{1 + (\varepsilon_k \cot \frac{\theta}{2})^2}
\]

implies

\[
\left| \frac{A_k'(\theta)}{A_k(\theta)} \right| = \frac{1}{|\sin \theta|} \frac{u_k}{1 + u_k^2} \arctan u_k, \quad u_k = \varepsilon_k \cot \frac{|\theta|}{2}.
\]

\(^1\)We will not use the upper estimate for \( A'(\theta) \).
Since
\[
\sup_{u \in (0, +\infty)} \frac{u}{(1 + u^2) \arctan u} = \lim_{u \to 0^+} \frac{u}{(1 + u^2) \arctan u} = 1,
\]
we get the second inequality in b). Let us prove the first one.

It is clear that
\[
A'(\theta) \geq \frac{1}{\sin \theta} \frac{\varepsilon_{k_0} \cot \frac{\theta}{2}}{1 + (\varepsilon_{k_0} \cot \frac{\theta}{2})^2} = \frac{1}{|\sin \theta|} \frac{u_{k_0}}{1 + u_{k_0}^2}, \quad u_{k_0} = \varepsilon_{k_0} \cot \frac{\theta}{2}
\]
for any \( k_0 \in \mathbb{N} \). Let \( k_0 \) be the smallest natural number such that \( u_{k_0} \leq 1 \). If \( k_0 > 1 \), then (3.1) implies
\[
c_0 \leq \varepsilon_{k_0} = \frac{u_{k_0}}{u_{k_0} - 1} \leq u_{k_0} \leq 1.
\]
Hence
\[
\frac{u_{k_0}}{1 + u_{k_0}^2} \geq \frac{c_0}{2}
\]
and
\[
A' (\theta) \geq \frac{c_0}{2|\sin \theta|} \geq \frac{c_0}{4|\sin \frac{\theta}{2}|}.
\]
If \( k_0 = 1 \), then
\[
A' (\theta) \geq \frac{\varepsilon_1}{2\sin^2 \frac{\theta}{2}} \frac{1}{1 + (\varepsilon_1 \cot \frac{\theta}{2})^2} \geq \frac{\varepsilon_1}{4\sin^2 \frac{\theta}{2}} \geq \frac{\varepsilon_1}{4|\sin \frac{\theta}{2}|}.
\]
This proves the first inequality in b).

c) Since \( \sin \vartheta \leq c \sin \frac{\vartheta}{c} \) and \( \cot \frac{\vartheta}{c} > \cot \vartheta \), \( \forall \vartheta \in (0, \pi/2] \), we have
\[
\frac{A'_{k_0}(\vartheta/c)}{A'_{k_0}(\vartheta)} = \frac{\sin^2 \frac{\vartheta}{c}}{\sin^2 \frac{\vartheta}{2c} \left( 1 + (\varepsilon_{k_0} \cot \frac{\vartheta}{2c})^2 \right)} < c^2.
\]
\[
\square
\]

**Lemma 3.2.** Suppose \( \vartheta_0, \vartheta_1, \vartheta_2 \in [-\pi, \pi] \setminus \{0\} \), \( \text{sign}\vartheta_0 = \text{sign}\vartheta_1 = \text{sign}\vartheta_2 \), \( |\vartheta_0| > |\vartheta_1| > |\vartheta_2| \), and
\[
|A(\vartheta_1) - A(\vartheta_0)| = 2\pi = |A(\vartheta_2) - A(\vartheta_1)|.
\]
Then
a) \( |\vartheta_0 - \vartheta_1| \leq c_2|\vartheta_0| \), where the constant \( c_2 \in (0, 1) \) depends only on \( c_1 \) from Lemma 3.1-b);
b) \( 1 \leq \frac{|\vartheta_0 - \vartheta_1|}{|\vartheta_1 - \vartheta_2|} \leq c_3 \),
where \( c_3 \) depends only on \( c_1 \).
Proof. a) Let $\tilde{\vartheta} \in (\vartheta_1, \vartheta_0)$ be such that
\[
|A(\tilde{\vartheta}) - A(\vartheta_0)| = \frac{c_1}{4}.
\]
Then, according to the mean value theorem, there exists $\vartheta^* \in (\tilde{\vartheta}, \vartheta_0)$ such that
\[
|A'(\vartheta^*)| |\tilde{\vartheta} - \vartheta_0| = \frac{c_1}{4}.
\]
It follows from Lemma 3.1-b) that
\[
\frac{c_1}{4} |\sin \frac{\vartheta^*}{2}| \leq \frac{c_1}{4} \Rightarrow |\sin \frac{\vartheta_0}{2}| \leq |\frac{\vartheta_0}{2}|.
\]
Since $|\vartheta_0 - \tilde{\vartheta}| \leq |\vartheta_0|/2$, the monotonicity of $A$ implies
\[
|A(\vartheta_0/2) - A(\vartheta_0)| \geq \frac{c_1}{4}.
\]
Similarly
\[
|A(\vartheta_0/2^j) - A(\vartheta_0/2^{j-1})| \geq \frac{c_1}{4}, \quad j \in \mathbb{N}.
\]
Let $M = [8\pi/c_1] + 1$. Then
\[
|A(\vartheta_0/2^M) - A(\vartheta_0)| = \sum_{j=1}^M |A(\vartheta_0/2^j) - A(\vartheta_0/2^{j-1})| \geq M \frac{c_1}{4} > \frac{8\pi}{c_1} \frac{c_1}{4} = 2\pi.
\]
Hence $\vartheta_1 \in (\vartheta_0/2^M, \vartheta_0)$ and
\[
|\vartheta_0 - \vartheta_1| < |\vartheta_0 - \vartheta_0/2^M| = (1 - 2^{-M}) |\vartheta_0|.
\]
This proves a) with $c_2 = 1 - 2^{-M} = 1 - 2^{-([8\pi/c_1]+1)}$.

b) According to the mean value theorem, there exist $\varphi_1 \in (\vartheta_1, \vartheta_0)$ and $\varphi_2 \in (\vartheta_2, \vartheta_1)$ such that
\[
\frac{|\vartheta_0 - \vartheta_1|}{|\vartheta_1 - \vartheta_2|} = \frac{|A'(\varphi_2)|}{|A'(\varphi_1)|}.
\]
It follows from part a) that
\[
1 \geq \frac{\varphi_2}{\varphi_1} > \frac{\vartheta_2}{\vartheta_0} = \frac{\vartheta_1}{\vartheta_0} (1 - c_2)^2 = 2^{-2([8\pi/c_1]+1)}.
\]
It is now left to use Lemma 3.1-a), c). One can take $c_3 = 2^{4([8\pi/c_1]+1)}$. \qed

Proof of Theorem 1.3. Let $\theta_j \in (-\pi, \pi]$ be the points such that
\[
A(\theta_j) = -2\pi j, \quad j = 0, \pm 1, \pm 2, \ldots
\]\nand let
\[
I_j = \gamma \left(\exp(i\theta_{j+1}), \exp(i\theta_j)\right), \quad j = 0, \pm 1, \pm 2, \ldots,
\]
where $\gamma(\zeta, \zeta') \subset T$ is the arc described by a point moving from $\zeta$ to $\zeta'$ in the counterclockwise direction.
Any arc $I \subset \mathbb{T}$ admits the representation:

$$I = \left( \bigcup_{j \in \mathcal{J}} I_j \right) \cup \left( \bigcup_{j \in \tilde{\mathcal{J}}} \tilde{I}_j \right),$$

where the set $\mathcal{J}$ is finite or infinite, the set $\tilde{\mathcal{J}}$ contains at most two elements, and the arcs $\tilde{I}_j$ have one of the following forms:

a) if $\mathcal{J} \neq \emptyset$, then

$$\tilde{I}_j = \gamma \left( \exp(i\theta_j), \exp(i\tilde{\theta}_j) \right) \quad \text{or} \quad \gamma \left( \exp(i\tilde{\theta}_j), \exp(i\theta_j) \right)$$

and

$$|A(\theta_j) - A(\tilde{\theta}_j)| < 2\pi;$$

b) if $\mathcal{J} = \emptyset$, then $\tilde{\mathcal{J}}$ contains one element and

$$\tilde{I}_j = \gamma \left( \exp(i\tilde{\theta}_{j+1}), \exp(i\tilde{\theta}_j) \right),$$

where

$$|A(\tilde{\theta}_{j+1}) - A(\tilde{\theta}_j)| < 4\pi.$$

Case b). Suppose $\mathcal{J} = \emptyset$.

$$I = \tilde{I}_j = \gamma \left( \exp(i\tilde{\theta}_{j+1}), \exp(i\tilde{\theta}_j) \right), \quad |A(\tilde{\theta}_{j+1}) - A(\tilde{\theta}_j)| < 4\pi.$$

Since $I$ may contain the point $-1$, but does not contain in our case the point $1$, it is convenient to switch from the function $A$ defined on $[-\pi, \pi] \setminus \{0\}$ to the following function defined on $(0, 2\pi)$:

$$A(\psi) = \begin{cases} 
A(\psi), & \text{if } \psi \in (0, \pi], \\
A(\psi - 2\pi), & \text{if } \psi \in (\pi, 2\pi). 
\end{cases} \quad (3.3)$$

Let $\psi_0 < \psi_1$ be such that $A(\psi_0) = A(\tilde{\theta}_{j+1})$ and $A(\psi_1) = A(\tilde{\theta}_j)$.

Using the change of variable $u = A(\psi)$ we get

$$\Delta_p := \frac{1}{|I|} \int_I \rho^p(B(\zeta))|d\zeta| = \frac{1}{\psi_1 - \psi_0} \int_{\psi_1}^{\psi_0} \rho^p \left( \exp(iA(\psi)) \right) d\psi$$

$$= \frac{1}{\psi_1 - \psi_0} \int_{A(\psi_0)}^{A(\psi_1)} \rho^p(\exp(iu)) \frac{du}{A'(\psi(u))} \leq \frac{\max_{\psi \in [\psi_0, \psi_1]} (A'(\psi))^{-1}}{\psi_1 - \psi_0} \int_{A(\psi_0)}^{A(\psi_1)} \rho^p(\exp(iu)) du.$$

According to the mean value theorem there exists $\psi^* \in (\psi_0, \psi_1)$ such that

$$A'(\psi^*)(\psi_1 - \psi_0) = A(\psi_1) - A(\psi_0).$$
It is now easy to derive from Lemmas 3.1 and 3.2 that
\[
\Delta_p \leq \frac{A'(\psi^*)}{\min_{\psi \in [\psi_0, \psi_1]} A'(\psi)} \left( \frac{1}{A'(\psi_1) - A'(\psi_0)} \int_{A'(\psi_0)}^{A'(\psi_1)} \rho^p(\exp(iu))du \right)
\leq \frac{c_4}{A'(\psi_1) - A'(\psi_0)} \int_{A'(\psi_0)}^{A'(\psi_1)} \rho^p(\exp(iu))du,
\]
where the constant \(c_4\) depends only on \(c_1\) from Lemma 3.1-b). Similarly,
\[
\frac{1}{|I|} \int_I \rho^{-q}(B(\zeta))d\zeta \leq \frac{c_4}{A'(\psi_1) - A'(\psi_0)} \int_{A'(\psi_0)}^{A'(\psi_1)} \rho^{-q}(\exp(iu))du.
\]
Hence
\[
\left( \frac{1}{|I|} \int_I \rho^p(B(\zeta))d\zeta \right)^{\frac{1}{p}} \left( \frac{1}{|I|} \int_I \rho^{-q}(B(\zeta))d\zeta \right)^{\frac{1}{q}} \leq c_4 \left( \frac{1}{A'(\psi_1) - A'(\psi_0)} \int_{A'(\psi_0)}^{A'(\psi_1)} \rho^p(\exp(iu))du \right)^{\frac{1}{p}} \times \left( \frac{1}{A'(\psi_1) - A'(\psi_0)} \int_{A'(\psi_0)}^{A'(\psi_1)} \rho^{-q}(\exp(iu))du \right)^{\frac{1}{q}} \leq 2c_4 C_p
\]
(see (1.1)). The factor 2 appears in the right-hand side because \(A(\psi_1) - A(\psi_0)\) may be larger than \(2\pi\) but is less than \(2 \times 2\pi\).

**Case a).** Let \(J_0 \subset \mathbb{Z}\) be the smallest set such that
\[
I \subseteq \bigcup_{j \in J_0} I_j.
\]
It follows from Lemma 3.2-b) that
\[
\sum_{j \in J_0} |I_j| \leq c_5 \sum_{j \in J} |I_j| \leq c_5 |I|,
\]
where the constant \(c_5\) depends only on \(c_1\) from Lemma 3.1-b).

Let us estimate
\[
\Lambda_{j,p} = \int_{I_j} \rho^p(B(\zeta))d\zeta.
\]
This is similar but easier than the estimate for \(\Delta_p\) in the case b), because we do not need to deal with the function (3.3) now. Since \(A(\theta_j) - A(\theta_{j+1}) = 2\pi\), we have
\[
\Lambda_{j,p} \leq \frac{c_4 |I_j|}{2\pi} \int_{-2\pi(j+1)}^{-2\pi j} \rho^p(\exp(iu))du = \frac{c_4 |I_j|}{2\pi} \|\rho\|_{L^p(T)}^p.
\]
Hence
\[
\int_I \rho^p(B(\zeta))|d\zeta| \leq \sum_{j \in J_0} \frac{c_4|I_j|}{2\pi} \|\rho\|^p_{L^p(T)} \sum_{j \in J_0} |I_j| \leq \frac{c_4c_5}{2\pi} \|\rho\|^p_{L^p(T)} |I|
\]
(see (3.4)). Similarly
\[
\int_I \rho^{-q}(B(\zeta))|d\zeta| \leq \frac{c_4c_5}{2\pi} \|\rho^{-1}\|^q_{L^q(T)} |I|.
\]
Hence
\[
\left(\frac{1}{|T|} \int_I \rho^p(B(\zeta))|d\zeta|\right)^{\frac{1}{p}} \left(\frac{1}{|T|} \int_I \rho^{-q}(B(\zeta))|d\zeta|\right)^{\frac{1}{q}} \leq \frac{c_4c_5}{2\pi} \|\rho\|_{L^p(T)} \|\rho^{-1}\|_{L^q(T)} \leq c_4c_5C_p.
\]
\[\square\]

**Remark 3.3.** The proof of Theorem 1.3 can be easily extended to any inner function \(v\) such that \(\arg v(e^{i\tau})\) has a continuous and increasing branch on \((0, 2\pi)\), and \(A(\theta) := \arg v(e^{i\theta})\) has the following property
\[
\frac{\max_{\theta \in [\theta_{j+1}, \theta_j]} A'(\theta)}{\min_{\theta \in [\theta_{j+1}, \theta_j]} A'(\theta)} \leq m < +\infty, \quad \forall j \in \mathbb{Z},
\]
(3.5)
where \(\theta_j\)'s are defined by (3.2). Indeed, (3.5) is exactly what is needed for the case b) in the proof of Theorem 1.3. The case a) relies also on Lemma 3.2-b) which in turn follows from (3.5).

The above applies for example to the singular inner function
\[
S(\zeta) = \exp \left(\kappa \frac{\zeta + 1}{\zeta - 1}\right), \quad \kappa = \text{const} > 0.
\]
Indeed,
\[
A(\theta) = \arg S(e^{i\theta}) = -\kappa \cot \frac{\theta}{2}
\]
and it is not difficult to see that (3.5) holds in this case. This corresponds to the case of the so-called periodic discontinuity which was considered in [9].

**4. Proof of Theorem 1.2**

**Proof.** Let \(a_0 \in L^\infty(T)\) be defined by
\[
a_0(e^{i\tau}) = \exp \left(\frac{i \tau}{2}\right), \quad \tau \in (0, 2\pi).
\]
Then \(a_0\) is continuous on \(T \setminus \{1\}\), \(a_0(1 \pm 0) = \pm 1\), \(T(a_0) : H^p(T) \to H^p(T)\) is invertible for any \(p \in (1, 2)\), and \(T(1/a_0) : H^p(T) \to H^p(T)\) is invertible for any \(p \in (2, +\infty)\) (see [8, 9.3, 9.8] or [2, 5.39]).
Let $h_0 = h \exp \left(-i \frac{c}{2} \log 2 \right)$, where $h$ is the function from Example 2.3 with $c = \frac{\pi}{2}$. Then

$$h_0(e^{it}) = |h(e^{it})| e^{i \varphi(t)}, \quad t \in [-\pi, \pi],$$

where

$$\varphi(t) = -\frac{\pi}{2} \log \left|2 \sin \frac{t}{2}\right|$$

(see (2.5)).

Let $f$ be a $2\pi$-periodic function such that $f \in C^\infty([-\pi, \pi] \setminus \{0\})$, $f(t) = \varphi(t)$ if $-\pi/2 \leq t < 0$, and $f(t) = -f(-t)$ if $0 < t \leq \pi/2$. Then

$$e^{2i f(t)} = B(e^{it}) g\left(B(e^{it})\right) d\left(e^{it}\right), \quad t \in [-\pi, \pi],$$

(4.1)

where $g, d \in C(T)$, the index of $g$ is 0, and $B$ is the infinite Blaschke product with the zeros

$$z_k = \frac{2 - \exp(-k + 1/2)}{2 + \exp(-k + 1/2)}$$

(see Theorem 2.2). Since the index of $g$ is 0, there exists $g_0 \in C(T)$ such that $g_0^2 = g$. Let $d_0 \in C(T)$ be such that $d_0^2(e^{it}) = d(e^{it})$ for $t \in [-\pi/2, \pi/2]$, $d_0(e^{it}) \neq 0$ for $t \in [-\pi, \pi]$ and the index of $d_0$ is 0.

Consider the function $a \in L^\infty(T)$ defined by

$$a(e^{it}) = a_0\left(B\left(e^{it}\right)\right) \left(\frac{g_0\left(B\left(e^{it}\right)\right) d_0\left(e^{it}\right) |h_0(e^{it})|}{h_0(e^{it})}\right).$$

(4.2)

It follows from (4.1) and from the definition of the function $f$ that $a^2(e^{it}) = 1$ if $-\pi/2 \leq t < 0$. It is clear that the second factor in the right-hand side of (4.2) is continuous on $\{e^{it} | -\pi/2 \leq t < 0\}$, whereas the first one has infinitely many discontinuities in any left semi-neighbourhood of 1. Hence $a$ takes values 1 and $-1$ in any left semi-neighbourhood of 1. So, $a(1 - 0) = \{\pm 1\}$.

The operator $T(a^{\pm 1}) : H^p(T) \to H^p(T)$ is invertible if and only if $T(a_0^{\pm 1} \circ B) : H^p(T) \to H^p(T)$ is invertible (see, e.g., [6, Theorem 2.1, Propositions 2.3, 4.1 and 5.4]). The latter operator is indeed invertible because $T(a_0^{\pm 1}) : H^p(T) \to H^p(T)$ is invertible and $B$ satisfies (1.3) (see Corollary 1.4). \hfill \Box

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