CHAPTER 1

Simplicial sets

1.1. Simplices and simplicial complexes

1.1.1. Simplices. A set of points $x_0, x_1, \ldots, x_n$ in a euclidean space is \textit{affinely independent} if the vectors $x_1 - x_0, \ldots, x_n - x_0$ form a linearly independent set. A \textit{simplex}, or a \textit{geometric simplex}, is the convex hull of an affinely independent set of points, called \textit{vertices} of the simplex, together with an order on the vertices. In particular, two simplices which only differ by the ordering of their vertices are considered to be different. A simplex with $n + 1$ vertices is called an \textit{$n$-simplex} and its vertices are numbered from 0 to $n$.

Each subset of $k + 1$ vertices of a simplex $\tau$ is naturally ordered and gives rise to a $k$-simplex, referred to as a \textit{$k$-face} of $\tau$. Disregarding the order of the vertices, an $n$-simplex $\tau$ is a polyhedron of dimension $n$; the $k$-dimensional faces of this polyhedron correspond to the $k$-faces of $\tau$.

By convention, an $(-1)$-simplex is the empty set; each simplex has it as the face that corresponds to the empty set of vertices.

1.1.2. Simplicial complexes. A \textit{simplicial complex} is a set $\mathcal{K}$ of simplices in a euclidean space with the following properties:

- if a simplex is in $\mathcal{K}$, all of its faces are in $\mathcal{K}$;
- the intersection of any two simplices in $\mathcal{K}$ is a face of each (in particular, it may coincide with either of the simplices or be empty).

Often no distinction is made between a simplicial complex and the topological space formed by the union of its simplices; the latter can also be referred to as a simplicial complex.

1.1.3. The importance and the drawbacks of simplicial complexes. Simplicial complexes arise naturally in triangulation problems. Any surface in a euclidean space can be approximated by a simplicial complex whose vertices lie on this surface. More generally, for any compact smooth manifold there is a simplicial complex homeomorphic to it. Indeed, by the Whitney embedding theorem any compact smooth manifold can be realized as a smooth submanifold of a euclidean space of some finite dimension. Then, choosing sufficiently many points on this submanifold we can construct a simplicial complex, with vertices in those points, that projects onto the submanifold homeomorphically. We encourage the reader to fill in the details in this argument.

A homeomorphism between a simplicial complex and a topological space is called a \textit{triangulation} of this space. Once a topological space is triangulated, there are all sorts of invariants that can be calculated from the combinatorics of the
simplicial complex; moreover, a triangulation may be a convenient way of specifying a topological space. Nevertheless, working with simplicial complexes has its drawbacks.

The first inconvenience is that relatively simple spaces may require many simplices to represent them. For example, a simplicial complex homeomorphic to the sphere \( S^n \) has at least \( n + 1 \) simplices of dimension \( n \). The second, more serious, difficulty is that the cartesian product of two simplices of positive dimension is not a simplex. As a consequence, there is no natural operation of a cartesian product on simplicial complexes. Finally, there is no notion of a quotient simplicial complex, which would be necessary for many constructions in topology.

A very efficient solution to all these problems is to replace simplicial complexes with cell complexes. The building blocks, called cells, from which the cell complexes are assembled are not required to have any particular geometric shape; it is only assumed that they are homeomorphic to disks in euclidean spaces. In contrast to simplicial complexes, a cell complex does not need an ambient space for its definition: it is defined inductively as an abstract topological space by attaching cells of increasing dimensions to a set of vertices. (We refer to [?] for the precise definition of a cell complex.) Cell complexes arise naturally in geometry, for instance, in the context of Morse theory. Their efficiency is best illustrated by the fact that the sphere \( S^n \) for any \( n \) can be constructed as a cell complex with only two cells: one in dimension 0 and one in dimension \( n \). A product of two cell complexes is again a cell complex since a product of two cells is a cell. Quotient cell complexes exist and their definition is straightforward. However, cell complexes have their downside: it may be very difficult to describe the attaching of cells.

Simplicial sets give a method of constructing spaces out of simplices by gluing them along their faces in a way compatible with cartesian products and their generalizations (fibre bundles). One can also take quotients and more general colimits of simplicial sets, and this allows to build the entire homotopy theory in the “simplicial language”. What may be considered as a disadvantage of simplicial sets is that the description of any simplicial set (even as simple as a point) requires an infinite number of so-called degenerate simplices. Nevertheless, as we shall see, simplicial sets can be more economical than simplicial complexes by requiring smaller numbers of non-degenerate simplices for constructing spaces.

1.2. Simplicial sets

1.2.1. The definitions. A simplicial set is a contravariant functor 
\[ \Delta \to \text{Sets}, \]
where \( \Delta \) is the category of non-empty finite ordered sets and non-decreasing maps. More generally, a simplicial object in a category \( \mathcal{C} \) is a contravariant functor 
\[ \Delta \to \mathcal{C}. \]
One can, therefore, speak of simplicial groups, simplicial topological spaces, simplicial categories and so on.

For a simplicial set \( K \) the set \( K(n) \) is called the set of \( n \)-simplices of \( K \). Note that, unless \( K \) assigns the empty set to each \( n \), the set of \( n \)-simplices of \( K \) cannot be empty.
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Non-decreasing maps \( n \to m \) are carried by \( K \) to the *structure maps*

\[ K(m) \to K(n). \]

The elements of \( K(n) \) that lie in the image of some \( K(m) \) for \( m < n \) under the structure maps are called the *degenerate simplices*; the rest are referred to as the *non-degenerate simplices*. Clearly, all the 0-simplices are non-degenerate.

A *map* of simplicial sets \( K \to K' \) is a natural transformation from \( K \) to \( K' \). In other words, it is a collection of maps \( K(n) \to K'(n) \) for all \( n \), which commute with the structure maps of \( K \) and \( K' \). With such maps as morphisms simplicial sets form a category.

Let us now give some examples.

1.2.2. Example: the simplicial set \( \Delta^n \). The most basic non-trivial simplicial set is the functor represented by the ordered set \( n \):

\[ \Delta^n : m \to \text{Hom}(m, n). \]

The non-degenerate \( m \)-simplices of \( \Delta^n \) are the injective non-decreasing maps \( m \to n \). If the elements of \( n \) are thought of as the vertices of a geometric \( n \)-simplex, such maps are in one-to-one correspondence with the \( m \)-faces of this simplex. For this reason \( \Delta^n \) is also known as the *\( n \)-simplex*. This is already the third meaning of the term "\( n \)-simplex" but nothing can be done about it. As we shall see below, the \( n \)-simplices of a simplicial set \( K \) are in one-to-one correspondence with simplicial maps \( \Delta^n \to K \).

The simplest possible non-empty simplicial set is \( \Delta^0 \): each set \( \Delta^0(n) \) consists of one element.

1.2.3. Example: simplicial sets coming from simplicial complexes. Any simplicial complex defines a simplicial set in the following fashion.

Choose a total order on the set of the vertices of a simplicial complex \( K \). Write \( \text{Map}(m, K) \) for the set of all non-decreasing maps from \( m \) to the set of vertices of \( K \) the convex hull of whose image is a geometric simplex belonging to \( K \). The functor

\[ K_S : m \to \text{Map}(m, K) \]

is a simplicial set. Its non-degenerate \( m \)-simplices are precisely the \( m \)-dimensional simplices in \( K \). Note that \( \Delta^n \) is a particular case of this construction.

1.2.4. Example: the singular simplicial set of a topological space. In fact, a simplicial set can be associated with any topological space, not just a simplicial complex.

For \( n > 0 \) the *standard \( n \)-simplex* \( |\Delta^n| \) is the subset of \( \mathbb{R}^n \) given by

\[ 0 \leq t_1 \leq \ldots \leq t_n \leq 1; \]

its vertices are numbered so that the \( k \)th vertex is the point whose first \( n - k \) coordinates are 0 and the last \( k \) coordinates are equal to 1. By definition, \( |\Delta^0| \) is a point.

The standard simplices form a category \( |\Delta| \) whose objects are the \( |\Delta^n| \) for all \( n > 0 \) and whose morphisms are affine maps that send vertices to vertices and respect their order. Since every such map is determined uniquely by its value on the vertices, the category \( |\Delta| \) of standard simplices is isomorphic to the category \( \Delta \).
Now, for a topological space \( X \) define the \textit{singular simplicial set} of \( X \) by assigning to \( m \) the space of all continuous maps from \(|\Delta^m|\) to \( X \):

\[
S(X) : m \to \text{Map}(|\Delta^m|, X).
\]

Since the category \(|\Delta|\) is isomorphic to \( \Delta \), a morphism \( m \to n \) induces a continuous map

\[
\text{Map}(|\Delta^n|, X) \to \text{Map}(|\Delta^m|, X)
\]

and \( S(X) \) is, indeed, a simplicial set. In general, it has non-degenerate simplices in any dimension.

\section*{1.3. Example: simplicial sets coming from partially ordered sets.}

Let \( P \) be a partially ordered set. Then \( N(P) : m \to \text{Hom}(m, P) \) is a simplicial set, where \( \text{Hom}(m, P) \) consists of all order-preserving maps. Non-degenerate simplices correspond to strictly increasing maps. This construction is also a generalization of \( \Delta^n \) and is a particular case of the nerve of a category.

\section*{1.4. Example: the nerve of a category.}

For each \( m \geq 0 \) the ordered set \( m \) can be considered as a category. Then, for a small category \( C \) the \textit{nerve} of \( C \) is the simplicial set that assigns to \( m \) the set of all functors from \( m \) to \( C \):

\[
N(C) : m \to \text{Fun}(m, C).
\]

The \( m \)-simplices of \( N(C) \) are sequences consisting of \( m \) composable morphisms in \( C \). A simplex of \( N(C) \) is degenerate if and only if the corresponding sequence contains an identity morphism.

\section*{1.6. Simplicial sets as representable functors.}

All the examples above use essentially the same recipe to construct simplicial sets: first the category \( \Delta \) is realized as a subcategory of some other category \( C \), then the sets of morphisms from the objects of \( \Delta \) to some fixed object of \( C \) form a simplicial set. This turns out to be the general situation.

In a simplicial set \( K \) each \( n \)-simplex \( a \in K(m) \) defines a map

\[
\Delta^n \to K.
\]

This map sends an element of \( \text{Hom}(m, n) \) to the image of \( a \) under the corresponding structure map \( K(n) \to K(m) \). Conversely, any map \( \Delta^n \to K \) defines an \( n \)-simplex of \( K \), namely, the image of the identity in \( \text{Hom}(n, n) \). In particular, any map \( \Delta^n \to \Delta^k \) is induced by a non-decreasing map \( n \to k \). As a consequence, the \( n \)-simplices \( \Delta^n \) together with all the maps among them form a category which is naturally isomorphic to \( \Delta \).

We see that any simplicial set \( K \) can be represented as

\[
m \to \text{Hom}(\Delta^m, K),
\]

where \( \text{Hom} \) is the space of simplicial maps.

This point of view helps to understand the notion of the degeneracy of simplices. An \( n \)-simplex of \( K \) represented by \( f : \Delta^n \to K \) is degenerate if the map \( f \) factors as

\[
\Delta^n \to \Delta^m \to K
\]
with \( m < n \). For any \( f \) there exists a minimal factorization of this kind; in other words, each simplex of \( K \) is the image of a unique non-degenerate simplex under some structure map.

### 1.2.8. Operations on simplicial sets

Functorial operations defined on sets naturally give rise to operations of simplicial sets. The *disjoint union* of simplicial sets \( K \) and \( L \) is defined by

\[
K \sqcup L : m \mapsto K(m) \sqcup L(m),
\]

The *cartesian product* of \( K \) and \( L \) is

\[
K \times L : m \mapsto K(m) \times L(m).
\]

The disjoint union and the cartesian product are the simplest examples of colimits and limits, respectively. These exist in the category of simplicial sets since they exist in the category of sets. Indeed, a diagram of simplicial sets produces a diagram of sets for each \( m \). Assigning to \( m \) the colimit (limit) of the corresponding diagram we get the colimit (limit) of simplicial sets.

A fundamental example of a colimit is an arbitrary simplicial set \( K \) as the colimit of its simplices. More precisely, given a simplicial set \( K \), the maps \( \Delta^n \to K \) form a category \( \Delta \downarrow K \) whose morphisms are commutative diagrams of the form

\[
\begin{array}{ccc}
\Delta^m & \to & K \\
\downarrow & \parallel & \\
\Delta^n & \to & K 
\end{array}
\]

Consider the functor

\[
\Delta \downarrow K \to \Delta
\]

which forgets the map to \( K \); its colimit over \( \Delta \downarrow K \) is precisely \( K \).

Indeed, \( K \) has the universal property of the colimit. Take a family of morphisms

\[
f_x : \Delta^n \to L,
\]

where \( x \) varies over all the maps \( \Delta^n \to K \), for all \( n \), and assume that for all \( x : \Delta^n \to K \) and \( h : \Delta^m \to \Delta^n \) we have

\[
f_{x \circ h} = f_x \circ h.
\]

The \( n \)-simplices of \( K \) are in one-to one correspondence with the maps \( \Delta^n \to K \); we define the map \( K \to L \) by sending an \( n \)-simplex \( a \in K(n) \) to the image of \( \Delta^n \) under \( f_a \).

In fact, a simplicial set is a colimit of its *non-degenerate* simplices. If we identify, as above, the simplices of \( K \) with the objects of \( \Delta \downarrow K \), non-degenerate simplices form a full subcategory; the morphisms between non-degenerates simplices are the structure maps induced by injective maps \( m \to n \). For each degenerate simplex there is precisely one morphism to a non-degenerate simplex, and, therefore, the colimit of any functor from \( \Delta \downarrow K \) coincides with its colimit on the subcategory of non-degenerate simplices.
1.3. The geometric realization of a simplicial set

1.3.1. The geometric realization functor. One can manufacture a topological space out of a simplicial set by replacing each $k$-simplex of this set by an actual standard simplex.

For a simplicial set $K$ its geometric realization $|K|$ is defined as the colimit, over the category $\Delta \downarrow K$, of the functor

$$\Delta \downarrow K \to |\Delta|,$$

which consists of forgetting the map to $K$ and then replacing $\Delta$ by the isomorphic category $|\Delta|$ of the standard simplices.

Geometric realization is, actually, a functor. Indeed, a map of simplicial sets $K \to K'$ induces a functor

$$\Delta \downarrow K \to \Delta \downarrow K',$$

which, in turn, induces a map of colimits $|K| \to |K'|$.

It is easy to see that the geometric realization of a disjoint union of two simplicial sets is naturally homeomorphic to the disjoint union of the realizations:

$$|K \sqcup L| = |K| \sqcup |L|.$$

More generally, the geometric realization functor commutes with all colimits. This amounts to saying that colimits commute with colimits: if $K$ is the colimit of a functor $F$ from a category $C$ to simplicial sets, the colimit over the category $\Delta \downarrow K$ is the same thing as the colimit over $C$ of the colimits over $\Delta \downarrow F(x)$ for $x \in C$.

1.3.2. Example: the geometric realization of $\Delta^n$. As suggested by the notation, the geometric realization of $\Delta^n$ is the standard simplex $|\Delta^n|$. Indeed, any map $\Delta^n \to \Delta^n$ factors through the identity map of $\Delta^n$. In particular, the colimit of any functor over the category $\Delta \downarrow \Delta^n$ coincides with the value of this functor on the identity map of $\Delta^n$, which, in the case of the geometric realization, is $|\Delta^n|$.

1.3.3. Example: simplicial complexes. Taking colimits in the category of sets amounts to taking disjoint unions and quotients by equivalence relations. In particular, for any simplicial set $K$ the space $|K|$ is the quotient of the disjoint union

$$\bigsqcup_n K(n) \times |\Delta^n|$$

of standard simplices, one for each simplex of $K$, by the equivalence relations

$$(a, f(x)) \sim (K(f)(a), x)$$

for all the morphisms $f$ in the category $|\Delta|$ of standard simplices; here we think of $K$ as a functor on this category.

As noted in the end of the previous section, any colimit over $\Delta \downarrow K$ can be taken over the subcategory of non-degenerate simplices. Given a simplicial complex $\mathcal{K}$, its $n$-simplices are in one-to-one correspondence with the non-degenerate $n$-simplices of the corresponding simplicial set $\mathcal{K}_S$; the inclusion maps between the simplices of $\mathcal{K}$ correspond to the morphisms in $\Delta \downarrow \mathcal{K}_S$. It follows that $|\mathcal{K}_S|$ is homeomorphic to the original simplicial complex $\mathcal{K}$.

Explicitly, take the map sending each standard simplex of the disjoint union above to the corresponding simplex of $\mathcal{K}$. It factors through the quotient by the equivalence relations and defines a continuous and one-to-one map from $|\mathcal{K}_S|$ to $\mathcal{K}$.
As \( \mathcal{K}_S \) only has a finite number of simplices, the space \(|\mathcal{K}_S|\) is compact; since \( \mathcal{K} \) is Hausdorff, this is a homomorphism.

### 1.3.4. Example: partially ordered sets.

For a finite partially ordered set \( P \) the geometric realization of the corresponding simplicial set \(|\mathcal{N}(P)|\) can be interpreted in terms of non-decreasing functions from the interval \([0, 1]\) to \( P \). Let us call two such functions equivalent if their values coincide away from a finite number of points and let \( \text{Hom}([0, 1], P) \) be the set of equivalence classes of such functions. If \( P \) is given the discrete metric \( d \) by setting the distance between any two distinct elements of \( P \) to be 1, the set \( \text{Hom}([0, 1], P) \) becomes a metric space with the metric

\[
d(f, g) = \int_0^1 d(f(x), g(x)) \, dx.
\]

Then \(|\mathcal{N}(P)|\), as a functor of \( P \), coincides with \( \text{Hom}([0, 1], P) \).

Indeed, a point in \(|\mathcal{N}(P)|\) can be specified by a pair \((k, t)\) where \( k \) is an \( n \)-simplex of \( \mathcal{N}(P) \) and \( t \) is a point in the standard \( n \)-simplex \(|\Delta^n|\). Define a map

\[
|\mathcal{N}(P)| \to \text{Hom}([0, 1], P)
\]

by sending the pair

\[
(a_0 \leq \ldots \leq a_n, 0 \leq t_1 \leq \ldots \leq t_n \leq 1)
\]

to a function whose value on the interval between \( t_i \) and \( t_{i+1} \) is equal to \( a_i \). (Here we take \( t_0 = 0 \) and \( t_{n+1} = 1 \).) This map is a continuous bijection. \(|\mathcal{N}(P)|\) is compact, being a finite colimit of compact spaces, and \( \text{Hom}([0, 1], P) \) is Hausdorff; therefore it is a homeomorphism. It is clear that also commutes with the maps induced by morphisms of partially ordered sets.

### 1.3.5. Geometric realization and cartesian products.

This description of a geometric realization of a partially ordered set in terms of the non-decreasing maps of an interval has an important application: it shows that the geometric realization functor commutes with finite cartesian products, at least on countable simplicial sets.

For two finite partially ordered sets \( P_1 \) and \( P_2 \) we have

\[
|\mathcal{N}(P_1 \times P_2)| = \text{Hom}([0, 1], P_1 \times P_2) = \text{Hom}([0, 1], P_1) \times \text{Hom}([0, 1], P_2) = |\mathcal{N}(P_1)| \times |\mathcal{N}(P_2)|.
\]

In particular,

\[
|\Delta^n \times \Delta^m| = |\Delta^n| \times |\Delta^m|.
\]

In general,

\[
|K \times L| = \lim_{\Delta \to K} \Delta^n \times \lim_{\Delta \to L} \Delta^m = \lim_{\Delta \to K \times L} \Delta^n \times \Delta^m
\]

The last expression is equal to the product of the corresponding colimits, that is, to \(|K| \times |L|\), if at least one of these spaces is locally compact. This happens, for instance, if \( K \) or \( L \) has a countable number of simplices.

Note that this property of geometric realizations provides us with a method of triangulating a product of simplicial complexes \( \mathcal{K} \) and \( \mathcal{L} \) since the space \(|\mathcal{K}_S \times \mathcal{L}_S|\) is naturally subdivided into simplices. This may seem strange at the first sight: the \( n \)-simplices of \( \mathcal{K}_S \times \mathcal{L}_S \) are simply pairs of \( n \)-simplices of \( \mathcal{K}_S \) and \( \mathcal{L}_S \) with the same
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The trick, however, is that we are only interested in non-degenerate simplices and the interaction of products and the degeneracy property is not straightforward: for instance, a product of two degenerate simplices can be non-degenerate.

1.3.6. Geometric realization and the singular set functor. The construction of the singular set functor $S$ that assigns a simplicial set to any topological space is inspired by the procedure of making a simplicial set out of a simplicial complex. While the geometric realization is not an inverse for it, in any sense, it turns out to be the left adjoint functor for $S$: there is an isomorphism

$$\text{Hom}([|X|], Y) = \text{Hom}(X, S(Y))$$

for any simplicial set $X$ and any topological space $Y$ which is natural both in $X$ and in $Y$, and the sets of morphisms on each side are taken in the respective categories.

Indeed, when $X = \Delta^n$ this isomorphism expresses the fact that the $n$-simplices of $S(Y)$ are in one-to-one correspondence with the simplicial maps $\Delta^n \to S(Y)$.

For an arbitrary $X$ we have

$$\text{Hom}([|X|], Y) = \text{Hom}(\text{colim}_{\Delta \downarrow X} \Delta^n, Y) = \lim_{\Delta \downarrow X} \text{Hom}(\Delta^n, Y) = \text{Hom}(\text{colim}_{\Delta \downarrow X} \Delta^n, S(Y)) = \text{Hom}(X, S(Y)).$$

The naturality in $X$ and $Y$ is obvious.

1.4. Further examples

1.4.1. Spheres. An important case of a colimit is the quotient of two simplicial sets.

An injective map $A \hookrightarrow K$ of simplicial sets is a map such that $A(m) \to K(m)$ is injective for all $m$. A simplicial set $A$ together with an injective map $A \hookrightarrow K$ is referred to as a simplicial subset of $K$. The quotient $K/A$ of a simplicial set by its simplicial subset is defined by

$$K/A : m \to K(m)/A(m).$$

A good illustration of this construction is the simplicial $n$-sphere. Write $\partial \Delta^n$ for the simplicial set that assigns to $m$ the subset of $\text{Hom}(m, n)$ consisting of non-surjective maps. Then $\Sigma^n = \Delta^n/\partial \Delta^n$.

Note that $\Sigma^n$ has only two non-degenerate simplices: one 0-simplex and one $n$-simplex. Its geometric realization is obtained by identifying the boundary of the standard $n$-simplex to one point:

$$|\Sigma^n| = |\Delta^n|/|\partial \Delta^n| = S^n,$$

an $n$-dimensional sphere.

1.4.2. One-point unions and smash products. A simplicial set $K$ is called pointed if a map $\ast : \Delta^0 \to K$ (basepoint) is specified. In this case the sets $K(m)$ are all pointed and the geometric realization of $K$ is a pointed topological space. We shall write $\ast(m)$ or $\ast_K(m)$ for the $m$-simplex of the basepoint in $K$.

Any quotient $K/A$ is naturally a pointed simplicial set with $\ast(m)$ being the image of $A(m)$ in $K(m)/A(m)$. A cartesian product of two pointed simplicial sets $K$ and $L$ is also pointed, with

$$\ast_{K \times L}(m) = (\ast_K(m), \ast_L(m)).$$
The one-point union $K \vee L$ of two pointed simplicial sets $K$ and $L$ is defined as
the one-point union of the corresponding sets of $n$-simplices:
\[ K \vee L : m \to K(m) \vee L(m) , \]
or, in other words, as the quotient of the disjoint union $K \sqcup L$ by the union of the
two basepoints. There is a map
\[ K \vee L \to K \times L \]
defined as $id \times *_L$ on $K$ and as $*_K \times id$ on $L$. The quotient of $K \times L$ by $K \vee L$ is
the smash product of $K$ and $L$; it is denoted by $K \wedge L$.

\[ \ldots \]
\[ 1.4.3. \text{Cones, suspensions and joins.} \]
\[ \ldots \]
\[ 1.4.4. \text{Quotients by group actions.} \]
\[ \ldots \]
\[ 1.4.5. \text{Symmetric products and finite subset spaces.} \]
\[ \ldots \]
\[ 1.4.6. \text{Barycentric subdivisions.} \]
\[ \ldots \]
\[ 1.5. \text{Simplicial maps} \]